Journal of
Mathematical
Inequalities
Volume 9, Number 4 (2015), 961–981 doi:10.7153/jmi-09-79

A THOUSAND PAPERS OF JOSIP PEČARIĆ

SANJA VAROŠANEC

(*Communicated by I. Perić*)

Abstract. In the last four decades academician Josip Pečarić has produced an enormous number of scientific papers and books. He has published more than one thousand mathematical papers in scientific journals and books. Furthermore, he is an author of more than 20 mathematical monographs, books and textbooks. Besides the great mathematical work, he has actively taken part in social life which has resulted in 30 books, more than 300 articles, notes and interviews in books, newspapers and other sources. In this paper we present some of ideas of his mathematical work which have a great impact on further investigations.

1. Introduction

During the last 40 years J. Pečarić published a great number of mathematical, physical and hystorical articles, books, textbooks etc. On May 5, 2015 a simple search of the MathSciNet base gave 1028 published mathematical items under his name. Another 60 papers have recently been published or accepted for publication. In this article we travel chronologicaly through the last four decades and try to describe some significant moments of J. Pečarić's work. That period can be split into three smaller time intervals in which his work had different accents: early years (70's and 80's), the nineties and the new century.

1.1. 70's and 80's.

As a student J. Pečarić attended the Faculty of Electrical Engineering at University of Belgrade where a famous mathematician D.S. Mitrinović worked as a professor. That circumstance had a great influence to his professional life. After earning his Master's degree in Physics, he took a job at the Institute for Geomagnetism and soon after that at the Faculty of Civil Engineering at the same University as an research assistant in physics. So, in the late 70's papers connected with his actual work on the problems in theory of Earth magnetic field were published. But at the same time he started to work with D.S. Mitrinović and other mathematicians from his group. J. Pečarić's first mathematical paper was published in 1976 and till the end of eighties he wrote more than 130 papers, mostly devoted to convex functions, various types of convexity and Jensen type inequalities. The topic of his dissertation which was defended in 1982

Mathematics subject classification (2010): 26D15, 26D20, 41A10, 47A63, 65D32.
Keywords and phrases: convex function, harmonic polynomials, the Jensen inequality, the Mond-Pečarić method, positive linear functional, quadrature formulae, the Steffensen inequality.

Paper JMI-09-79

was also about the Jensen and related inequalities. He was extremely interested in considering conditions under which the Jensen inequality holds with particular interest to the case when negative weights appear. Very often his ability to give a new proof of a certain inequality was a starting point for a new direction of investigation, for giving weaker conditions, for obtaining related inverse and companion inequalities or even generalization of it. Titles of his papers from that period give us a closer insight to his main interests. For example, in the list of publication we find the following titles: On the Jensen inequality ([41]), On the Jensen-Steffensen inequality ([17]), A new proof of the Jensen-Steffensen inequality ([19]), Inverse of the Jensen-Steffensen inequality ([20]), A short proof of a variant of Jensen's inequality ([22]), An inequality for 3-convex function ([23]), On variants of Jensen's inequality ([24]), A simple proof of the Jensen-Steffensen inequality ([25]), The inductive proof of the Jensen-Steffensen inequality ([27]), On Jessen's inequality for convex functions I, II ([4]) and ([31]), On the Popoviciu conversion of Jensen's inequality ([28]) etc.

A natural consequence of such hard work on convexity was a publication of his first book *Konveksne funkcije - nejednakosti* (eng: Convex functions - inequalities), ([29]), in 1987. In the following five years he wrote ten books, mostly in cooperation with D.S. Mitrinović. Focus of his work in the 80's was not only on convex functions. Numerous papers about Chebyshev type inequalities, the Ostrowski and the Grüss inequalities were published. Also, in the late 80's he investigated geometric inequalities and that research resulted in the monograph *Recent Advances in Geometric Inequalities* ([14]). After that he abandoned this area of inequalities. In this period he also developed an idea of using positive linear functionals in the theory of inequalities, see for example [4]), ([32]) and ([6]. Application of linear functionals has been a constant theme and inspiration throughout Pečarić's work and he devoted his book [34] to development of this concept.

1.2. The nineties.

In 1987 J. Pečarić moved with his family to Zagreb and got a position at the Faculty of Textile Technology, University of Zagreb. He has begun to cooperate with mathematicians from the University of Zagreb such as M. Alić, N. Elezović, B. Guljaš, C. Jardas, N. Sarapa, D. Svrtan etc. The first half of the nineties was marked with the war in Croatia and J. Pečarić actively took part in public life, intensively writing and publishing articles, books and notes connected with history and recent situation in Croatia.

In his mathematical life he had continued to work with D. S. Mitrinović, with Romanian mathematicians S. S. Dragomir, I. Rąsa, J. Sandor, Gh. Toader and others in problems involving positive linear functionals, n-convexity and the Chebyshev inequality. In 1992 the first paper with Australian mathematician B. Mond appeared in Houston J. Math. It was a beginning of a very fruitful cooperation. They wrote 87 papers, mainly about matrix inequalities, inequalities for operators, convexity and means involving matrices and operators, inequalities in inner and normed spaces. Cooperation with mathematicians from Australia was expanded to a joint work with C. E. M. Pearce with whom J. Pečarić worked on various analytical inequalities which very often have applications in probability. In that period he visited Australian universities in Adelaide and Melbourne several times and served as a co-advisor for two dissertations.

Since the nineties, we have witnessed a significant cooperation with the Swedish group of mathematicians lead by L. E. Persson and L. Maligranda. They have worked together on generalizations of Stolarsky's and Bergh's inequality, on Grüss-Barnes, Borell and Hardy type inequalities, on inequalities for monotone functions, on inequalities in weighted L_p spaces etc. Also, another fruitfull cooperation has been begun in the middle of nineties with Israelian mathematician S. Abramovich and it has continued till today developing various refinements and improvements of the classical inequalities. We also have to mention joint work with Italian mathematicians G. Allasia and C. Giordano on topics such as the Gamma function, the Stirling formula, means etc.

After he moved to Zagreb, J. Pečarić took an active role in the postgraduate studies of Mathematics at the University of Zagreb. The first dissertation under his supervision was completed in 1994 (S. Varošanec: *Gauss' type inequalities*) and since than more than 30 students finished their doctorial studies under J. Pečarić's supervision. The titles of theses reflects nicely current interests and the field of research. For example, the first several theses were: *Opial and related inequalties* (I. Brnetić, 1997); *Inequalities inverse to inequalities of Hölder type* (I. Perić, 1997); *Jensen type inequalities with applications in information theory* (M. Matić, 1998); *Mixed means, Hardy and Carleman type inequalities* (A. Čižmešija, 1999); *Hadamard's, Jensen's and related inequalities* (V. Čuljak, 1999).

Last years of the nineties were a period of preparation for the foundation of the first Croatian mathematical journal devoted to inequalities. In 1998 the first number of journal Mathematical Inequalities and Applications appeared. The editoral board had almost 60 well-known mathematicians from different fields of mathematics. Managing editor has been N. Elezović and Editor-in-Chief has been J. Pečarić. As it is written on the cover page of the journal: *MIA brings together original research papers in all areas of mathematics, provided they are concerned with inequalities or their role.* This journal became the first Croatian mathematical journal which is indexed in SCI-expanded base. In the next 15 years new scientific journals were founded under J. Pečarić's managing. He serves as an Editor-in-Chief in *Journal of Mathematical Inequalities* (from 2007) and in *Fractional differential calculus* (from 2011) and as a founding editor-in-chief in *Operators and Matrices*.

1.3. New century

In 2000, J. Pečarić became a full member of the Croatian Academy of Sciences and Arts. In the framework of the doctorial studies of mathematics at University of Zagreb, J. Pečarić has established a seminar *Inequalities and Applications* - a place where new results and ideas are reported by members of his school of inequalities and their guests. New doctorial students came and under the guidance of Professor Pečarić they have investigated some new directions and finished their theses. In summary, since 1994 twenty one dissertations and two master degree theses have been completed in Croatia under the direct supervision of J. Pečarić and further 17 dissertations were done under his close advisorship.

At the same time, J. Pečarić has continued to work with previously mentioned mathematicians, but he has also created new connections with group of Japanese mathematicians i.e. with Y. Seo, S.E. Takahashi, M. Fujii, T. Furuta, S. Izumino, M. Tominaga etc. They have applied known operator inequalities previously obtained by B. Mond and J. Pečarić and have created new results involving matrices and operators. Also, J. Pečarić and Korean colleague Y.J. Cho gave a series of nice results involving inequalities in n-inner and semi-inner spaces.

In 2005, J. Pečarić accepted an invitation of the Abdus Salam School of Mathematical Sciences in the University of Lahore to become a visiting professor at that institution. This cooperation has lasted untill today. Students of this School have worked together with J. Pečarić and under his supervision 12 dissertations were completed.

In 2005, J. Pečarić has founded the new mathematical book series *Monographs in Inequalitites* in publishing house Element from Zagreb. Titles of books reflect the main directions of investigation on which J. Pečarić worked with his colleagues. The first book in the series is *Mond-Pečarić Method in Operator Inequalities - Inequalities for Bounded Selfadjoint Operators on a Hilbert Space* (with T. Furuta, J. Mićić Hot and Y. Seo) while its continuation is published in 2012. The other books are: *Euler Integral Identity, Quadrature Formulae and Error Estimations - From the Point of View of Inequality Theory* (with I. Franjić, I. Perić and A. Vukelić, 2011); *Recent Advances in Hilbert-type Inequalities - A Unified treatment of Hilbert-type Inequalities* (with M. Krnić, I. Perić and P. Vuković, 2012); *Inequalities of Hardy and Jensen - New Hardy Type Inequalities with General Kernels* (with K. Krulić Himmelreich and D. Pokaz, 2013); *General Integral Identities and Related Inequalities - Arising from Weighted Montgomery Identity* (with A. Aglić Aljinović, A. Čivljak, S. Kovač and M. Ribičić Penava, 2013); *Steffensen's and Related Inequalities - A Comprehensive Survey and Recent Advances* (with K. Smoljak Kalamir and S. Varošanec, 2014) and *Combinatorial Improvements of Jensen's Inequality - Classical and New Refinements of Jensen's Inequality with Applications* (with L. Horváth and K. Ali Khan, 2014).

Besides topics described in these books J. Pečarić are recently working on several topics, including the concept of exponential convexity, Cauchy type means, quasilinearity of certain functionals, applications in time scales and fractional calculus etc.

In the following sections we describe some ideas from three parts of Pečarić's research. For more details on some other topics we refer the reader to the article [5].

2. Integral identities and applications in quadrature formulae

Some of the constant themes throughout Pečarić's work are integral identities, quadrature rules and related inequalities. His very first paper [11] contains a generalization of Ostrowski's inequality for n-times differentiable functions and it is obtained by considering the well-known Taylor formula. But the real progress happened at the beginning of this millenium, when several young Croatian mathematicians begin working with Professor J. Pečarić.

The following elegant results involving the so-called harmonic polynomials are good illustrations of methods in this topic.

Let $\sigma = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ be a subdivision of the interval

$[a,b]$. Set

$$S_m(t,\sigma)=\begin{cases} P_{1,m}(t), & t\in[a,x_1]\\ P_{2,m}(t), & t\in(x_1,x_2]\\ \quad\vdots & \\ P_{n,m}(t), & t\in(x_{n-1},x_n],\end{cases}$$

where $\{P_{j,m}\}_m$ are sequences of harmonic polynomials, i.e. polynomials $P_{j,m}$ satisfy

$$P'_{j,m}=P_{j,m-1},\ \text{ for } m,j\in\mathbb{N},\ P_{j,0}=1.$$

Using integration by parts the following integral identity for $(m-1)$-times differentiable function $f:[a,b]\to\mathbb{R}$ such that $f^{(m-1)}$ is bounded is obtained:

$$\begin{aligned} I_m(\sigma) &:= (-1)^m\int_a^b S_m(t,\sigma)f^{(m-1)}(t)\\ &= \int_a^b f(t)dt+\sum_{k=1}^{m}(-1)^k\Bigg[P_{n,k}(b)f^{(k-1)}(b) \\ &+\sum_{j=1}^{n-1}(P_{j,k}(x_j)-P_{j+1,k}(x_j))f^{(k-1)}(x_j)-P_{1,k}(a)f^{(k-1)}(a)\Bigg]\end{aligned} \tag{1}$$

whenever integrals exist, ([37]).

With an appropriate choice of polynomials $P_{j,k}$ and nodes x_j generalizations of the well-known quadrature formulae are obtained. In the following theorems an error estimation for identity (1) is given.

THEOREM 1. *Suppose $f:[a,b]\to\mathbb{R}$ is such that $f^{(m)}$ is integrable with $\gamma_m\leqslant f^{(m)}(t)\leqslant\Gamma_m$, for all $t\in[a,b]$. Put*

$$U_k(\sigma)=\frac{1}{b-a}\left(P_{n,k}(b)+\sum_{j=1}^{n-1}(P_{j,k}(x_j)-P_{j+1,k}(x_j))-P_{1,k}(a)\right). \tag{2}$$

Then

$$|I_m(\sigma)-(-1)^m U_{m+1}(\sigma)(f^{(m-1)}(b)-f^{(m-1)}(a))|\leqslant\frac{1}{2}K_m(\Gamma_m-\gamma_m)(b-a) \tag{3}$$

where

$$K_m=\sqrt{\frac{1}{b-a}\left(\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}P_{j,m}^2(t)dt\right)-U_{m+1}^2(\sigma)}.$$

Proof. Use of definition of S_m, relation (2) and identity (1) implies

$$\begin{aligned} &|I_m(\sigma)-(-1)^m U_{m+1}(\sigma)(f^{(m-1)}(b)-f^{(m-1)}(a))|\\ &=(b-a)\left|\frac{1}{b-a}\int_a^b S_m(t,\sigma)f^{(m)}(t)dt-\frac{1}{(b-a)^2}\int_a^b S_m(t,\sigma)dt\int_a^b f^{(m)}(t)dt\right|\\ &=(b-a)|T(S_m,f^{(m)})|\end{aligned} \tag{4}$$

where T is the so-called Chebyshev functional defined as

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$$

where f,g,fg are integrable functions on $[a,b]$. The well-known classical result regarding the Chebyshev functional is the Grüss inequality which states that

$$|T(f,g)|\leqslant\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma)$$

where $\Phi,\varphi,\Gamma,\gamma$ are constants such that $\varphi\leqslant f(x)\leqslant\Phi$, $\gamma\leqslant g(x)\leqslant\Gamma$ for $x\in[a,b]$. In [10] the following pre-Grüss inequality is given:

$$|T(f,g)|\leqslant\frac{1}{2}(\Gamma-\gamma)\sqrt{T(f,f)}.$$

Using the pre-Grüss inequality in (4) it follows

$$(b-a)|T(S_m,f^{(m)})|\leqslant\frac{1}{2}(b-a)(\Gamma_m-\gamma_m)\sqrt{T(S_m,S_m)}.$$

An easy calculation gives that

$$\sqrt{T(S_m,S_m)}=\sqrt{\frac{1}{b-a}\left(\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}P_{j,m}^2(t)dt\right)-U_{m+1}^2(\sigma)}$$

and desired inequality (3) is proved.

□

The following results also give error estimations for $I_m(\sigma)$ but for different function classes. Their proofs are similar to proofs given in papers [38] and [39].

THEOREM 2. *(i) Let a function $f:[a,b]\to\mathbb{R}$ be $(m-1)$-times differentiable such that $f^{(m-1)}$ is the L_m–Lipschitz function on $[a,b]$. Then*

$$|I_m(\sigma)|\leqslant L_m\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}|P_{j,m}(t)|dt,\qquad m\in\mathbb{N}.$$

(ii) Let a function $f:[a,b]\to\mathbb{R}$ be such that $f^{(m-1)}$ is of bounded variation on $[a,b]$. Then

$$|I_m(\sigma)|\leqslant max\left\{\sup_{t\in[a,x_1]}|P_{1,m}(t)|,\dots,\sup_{t\in(x_{n-1},b]}|P_{n,m}(t)|\right\}V_a^b(f^{(m-1)}).$$

(iii) Let f be m-times differentiable with $f^{(m)}\in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. Then

$$|I_m(\sigma)|\leqslant\left(\sum_{j=1}^{n}\int_{x_{j=1}}^{x_j}|P_{j,m}(t)|^q dt\right)^{\frac{1}{q}}||f^{(m)}||_p.$$

Let us apply previous general results on the three-point subdivision $\sigma = \{a, \frac{a+b}{2}, b\}$ of an interval $[a,b]$. In this case S_m is defined as

$$S_m(t,\sigma) = \begin{cases} P_{1,m}(t) = \frac{(t-a)^m}{m!} - \frac{b-a}{6}\frac{(t-a)^{m-1}}{(m-1)!}, & t \in [a, \frac{a+b}{2}] \\ P_{2,m}(t) = \frac{(t-b)^m}{m!} + \frac{b-a}{6}\frac{(t-b)^{m-1}}{(m-1)!}, & t \in (\frac{a+b}{2}, b]. \end{cases}$$

The term $I_m(\sigma)$ has the following form

$$I_m(\sigma) = \int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$+ \sum_{k=5, k \text{ odd}}^{m} (-1)^k \frac{2}{(k-1)!}\left(\frac{b-a}{2}\right)^k \left(\frac{1}{k} - \frac{1}{3}\right) f^{(k-1)}\left(\frac{a+b}{2}\right).$$

As we can see, for $m = 1,2,3,4$

$$I_m(\sigma) = \int_a^b f(t)dt - \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

and it is, in fact, the error term of the Simpson formula. Applying Theorem 1 for this particular case the following estimation of $I_m(\sigma)$ is obtained.

THEOREM 3. *Suppose $f : [a,b] \to \mathbb{R}$ is such that $f^{(m)}$ is integrable and such that $\gamma_m \leqslant f^{(m)}(t) \leqslant \Gamma_m$, $t \in [a,b]$. Then for $m \in \{1,2,3\}$ we have*

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leqslant C_m(\Gamma_m - \gamma_m)(b-a)^{m+1}$$

where

$$C_1 = \frac{1}{12}, \ C_2 = \frac{1}{24\sqrt{30}}, \ C_3 = \frac{1}{96\sqrt{105}}.$$

If $m \geqslant 4$ then we have

$$|I_m(\sigma) - (-1)^m U_{m+1}(\sigma)(f^{(m-1)}(b) - f^{(m-1)}(a))| \leqslant \frac{1}{3m!}\left(\frac{b-a}{2}\right)^{m+1} K_m(\Gamma_m - \gamma_m),$$

where

$$U_{m+1}(\sigma) = \begin{cases} 0, & m \in 2\mathbb{N}+1 \\ \frac{2-m}{3(m+1)!}\left(\frac{b-a}{2}\right)^m, & m \in 2\mathbb{N}, \end{cases}$$

and

$$K_m = \begin{cases} \sqrt{\frac{2m^3-11m^2+18m-6}{4m^2-1}}, & m \in 2\mathbb{N}+1 \\ \frac{1}{m+1}\sqrt{\frac{2m^5-11m^4+14m^3+4m^2+2m-2}{4m^2-1}} & m \in 2\mathbb{N}. \end{cases}$$

Estimations for $I_m(\sigma)$ when $f^{(m-1)}$ is a L_m–Lipschitz function or $f^{(m-1)}$ is a function of bounded variation or $f^{(m)} \in L_p[a,b]$ are simple consequences of the corresponding Theorem 2 (see [37]).

During the last 15 years Pečarić and his coauthors published more than 120 papers in which they investigated various kinds of integral quadrature formulae based on different integral identities. For example, they studied the Euler integral identity, the general Euler-Ostrowski formula, 2,3,4,5-point quadrature formulae of Euler type, Radau-type quadrature formulae, different generalizations of weighted Montgomery identity etc. Recent related results connected with the above-mentioned topics are collected in books [1] and [7].

3. Relating "unrelated" results. The Steffensen inequality

A standard method in Pečarić's work is unifying results which seem unrelated at the first glance and putting them in the same framework. A nice pearl among such results is a result which covers three famous inequalities which were independently obtained: the Gauss, the Steffensen and the Ostrowski inequality.

In [9] C.F. Gauss mentioned the following inequality:

If f is a nonnegative and non-increasing function and $a > 0$ then

$$\int_a^\infty f(x)dx \leqslant \frac{4}{9a^2}\int_0^\infty x^2 f(x)dx. \tag{5}$$

One century later A. Ostrowski proved the following result (see [16]).

Let f be a non-increasing function on $[0,a]$ and g be a non-decreasing continuous function with continuous derivative and $g(t) \leqslant t$ for $0 \leqslant t \leqslant a$ with $g(0) = 0$. Then

$$\int_0^a f(t)g'(t)dt \leqslant \int_0^{g(a)} f(t)dt. \tag{6}$$

Independently, in the twenties J.F. Steffensen (see [40]) obtained that for two integrable functions f and g defined on (a,b), where f is non-increasing and $0 \leqslant g \leqslant 1$ we have

$$\int_{b-\lambda}^b f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt, \tag{7}$$

where $\lambda = \int_a^b g(t)dt$.

Nowadays, inequality (7) is known as the Steffensen inequality. In 1989 J. Pečarić proved the following theorem which merges all three mentioned inequalities ([30]).

THEOREM 4. *Let $G : [a,b] \to \mathbb{R}$ be an increasing differentiable function and let $f : I \to \mathbb{R}$ be a non-increasing function (I is an interval in $\mathbb{R}$ such that $a,b,G(a),G(b) \in I$). If $G(x) \geqslant x$ then*

$$\int_{G(a)}^{G(b)} f(x)dx \leqslant \int_a^b f(x)G'(x)dx. \tag{8}$$

If $G(x) \leqslant x$, the reverse inequality in (8) is valid.

As most of Pečarić's proofs the proof is short and very elegant.

Proof. After substituting $G(x) = z$ it follows

$$\int_a^b f(x)G'(x)dx = \int_a^b f(x)dG(x) = \int_{G(a)}^{G(b)} f(G^{-1}(z))dz.$$

If $G(z) \geqslant z$, then $G^{-1}(z) \leqslant z$ and $f(G^{-1}(z)) \geqslant f(z)$. So it holds

$$\int_{G(a)}^{G(b)} f(G^{-1}(z))dz \geqslant \int_{G(a)}^{G(b)} f(z)dz,$$

and (8) holds. Of course, if $G(z) \leqslant z$, we get the reverse inequality. □

It is obvious that (8) is a generalization of (6). Taking $G(x) = \frac{4x^3}{27a^2} + a$, $a > 0$, $x \in [0,\infty)$ inequality (8) reduces to the Gauss inequality (5). And finally, if we let $G(x) = a + \int_a^x g(t)dt$ in Theorem 4, where g is a non-negative function, then in the case $G(x) \leqslant x$, i.e. if

$$\int_a^x g(t)dt \leqslant x - a,$$

we get the second inequality in (7). For the first inequality take $G(x) = b - \int_x^b g(t)dt$ in Theorem 4.

In the last 40 years J. Pečarić published more than 30 papers related to the Steffensen inequality. Here we describe some significant places in his investigation of that inequality.

At the beginning of his mathematical professional activity, J. Pečarić proved the Steffensen inequality under weaker conditions. Namely, the condition "$0 \leqslant g \leqslant 1$" can be changed to be as it is given in the following theorem from [42].

THEOREM 5. *Let f and g be integrable functions on $[a,b]$ and let $\lambda = \int_a^b g(t)dt$.*

a) The second inequality in (7) holds for every non-increasing function f if and only if

$$\int_a^x g(t)dt \leqslant x - a \ \text{ and } \ \int_x^b g(t)dt \geqslant 0 \ \text{ for every } x \in [a,b].$$

b) The first inequality in (7) holds for every non-increasing function f if and only if

$$\int_x^b g(t)dt \leqslant b - x \ \text{ and } \ \int_a^x g(t)dt \geqslant 0 \ \text{ for every } x \in [a,b].$$

The proof is based on the following identities:

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)dt - \int_a^b f(t)g(t)dt \\
&= \int_a^{a+\lambda} [f(t) - f(a+\lambda)][1-g(t)]dt + \int_{a+\lambda}^b [f(a+\lambda) - f(t)]g(t)dt,
\end{aligned}$$

and

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda}^b f(t)dt$$
$$= \int_a^{b-\lambda} [f(t) - f(b-\lambda)]g(t)dt + \int_{b-\lambda}^b [f(b-\lambda) - f(t)][1-g(t)]dt.$$

The same identities are basis for the another theorem under weaker conditions which appeared in [36].

THEOREM 6. *Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions and $\lambda = \int_a^b g(t)dt$. If f satisfies*

$$\begin{cases} f(t) \geqslant f(c) & \text{for} \quad t\in[a,c] \\ f(d) \leqslant f(t) \leqslant f(c) & \text{for} \quad t\in[c,d] \\ f(t)\leqslant f(d) & \text{for} \quad t\in[d,b], \end{cases}$$

where $c = \min\{a+\lambda, b-\lambda\}$, $d=\max\{a+\lambda,b-\lambda\}$, $c,d\in[a,b]$, and if g satisfies

$$\begin{cases} 0\leqslant g(t)\leqslant 1 & \text{for} \quad t\in[a,c]\cup[d,b] \\ g(t)\geqslant 0 & \text{for} \quad t\in[c,d] \text{ when } c=a+\lambda \\ g(t)\leqslant 1 & \text{for} \quad t\in[c,d] \text{ when } c=b-\lambda, \end{cases}$$

then (7) holds.

Also, in the same paper the discrete Steffensen inequality was given.

In the statement of the original Steffensen inequality a function f is non-increasing which can be considered as a convex function of order 1. So, a natural generalization is to consider a function which is convex of order n. This problem is solved in one of his first papers published in 1979 (see [12]) where the following result was presented.

THEOREM 7. *Let functions f and g satisfy conditions:*

(1) *f is convex of order n $(n\in\mathbb{N})$;*

(2) $f^{(k)}(a)=0, \quad k=0,1,\ldots,n-1$;

(3) *for all $x\in[a,b]$*

$$\int_a^x (t-a)^n g(t)dt \leqslant \frac{(x-a)^{n+1}}{n+1} \text{ and } \int_x^b (t-a)^n g(t)dt \geqslant 0.$$

Then

$$\int_a^{a+\lambda_1} f(x)dx \leqslant \int_a^b f(x)g(x)dx,$$

where

$$\lambda_1 = \left[(n+1)\int_a^b (x-a)^n g(x)dx\right]^{\frac{1}{n+1}}.$$

Answer to the question ”Under which conditions do reverse inequalities in (7) hold?” was given by J. Pečarić in [21]. He proved that a reverse inequality on the left-hand side of Steffensen inequality holds for every non-increasing function f if and only if

$$\int_a^x g(t)dt \geqslant x-a \text{ for } x \in [a,a+\lambda] \text{ and } \int_x^b g(t)dt \leqslant 0 \text{ for } x \in (a+\lambda,b]$$

with $0 \leqslant \lambda \leqslant b-a$; or

$$\int_a^x g(t)dt \geqslant x-a \quad \text{ for } x \in [a,b], \text{ when } \lambda \geqslant b-a;$$

or

$$\int_x^b g(t)dt \leqslant 0 \quad \text{ for } x \in [a,b] \text{ when } \lambda \leqslant 0.$$

In the same paper, conditions under which the reverse inequality on the right-hand side of the Steffensen inequality are given. Two years later he obtained power version of the Steffensen inequality (see [26]). Namely, if $G:[a,b] \to \mathbb{R}$ is an integrable function and $p \geqslant 1$ such that

$$0 \leqslant G(x)\left(\int_a^b G(t)dt\right)^{p-1} \leqslant 1 \quad (x \in [a,b]),$$

then for a non-increasing function f we have

$$\left(\int_a^b G(x)f(x)dx\right)^p \leqslant \int_a^{a+\lambda} f^p(t)dt,$$

where

$$\lambda = \left(\int_a^b G(t)dt\right)^p.$$

J. Pečarić also investigated a multidimensional version of the Steffensen inequality and its analogue in a general measure space. In the last decade a focus of his investigation has been on an application of different identities and interpolation formulae, (see [35]). Several generalizations of the Steffensen inequality obtained via the weighted Montgomery, Euler's and Fink's identities, via Taylor's formula, by Lidstone's, Hermitte's and Abel-Gontscharoff's polynomials have appeared. Here we mention only one such result which illustrates the method (see [3]).

In the next theorem two integral means of the same function are compared.

THEOREM 8. *Let $f:[a,b]\cup[c,d] \to \mathbb{R}$ be a continuous function of bounded variation on $[a,b]\cup[c,d]$, $w:[a,b] \to \mathbb{R}$ and $u:[c,d] \to \mathbb{R}$ weight functions such that $\int_a^b w(t)\,dt \neq 0$, $\int_c^d u(t)\,dt \neq 0$ and $[a,b]\cap[c,d] \neq \emptyset$. Then*

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt \leqslant \frac{1}{\int_c^d u(t)\,dt}\int_c^d u(t)f(t)\,dt$$

holds for every non-increasing function f if and only if $[c,d] \subseteq [a,b]$ and

$$\frac{W(x)}{W(b)} \leqslant 0 \text{ for } x \in [a,c\rangle, \quad \frac{W(x)}{W(b)} \leqslant \frac{U(x)}{U(d)} \text{ for } x \in [c,d], \ \frac{W(x)}{W(b)} \leqslant 1 \text{ for } x \in \langle d,b],$$

or $[a,b] \cap [c,d] = [c,b]$ and

$$\frac{W(x)}{W(b)} \leqslant 0 \text{ for } x \in [a,c\rangle, \quad \frac{W(x)}{W(b)} \leqslant \frac{U(x)}{U(d)} \text{ for } x \in [c,b\rangle, \ 1 \leqslant \frac{U(x)}{U(d)} \text{ for } x \in [b,d].$$

Functions W and U are defined by

$$W(x) = \begin{cases} 0, & x < a \\ \int_a^x w(t)\,dt, & a \leqslant x \leqslant b \\ \int_a^b w(t)\,dt, & x > b, \end{cases} \quad U(x) = \begin{cases} 0, & x < c \\ \int_c^x u(t)\,dt & c \leqslant x \leqslant d \\ \int_c^d u(t)\,dt, & x > d. \end{cases}$$

The proof is based on the weighted Montgomery identity given by J. Pečarić in [18], which states

$$f(x) - \frac{1}{\int_a^b w(t)\,dt} \int_a^b f(t)w(t)\,dt = \int_a^b P_w(x,t)\,df(t)$$

where $P_w(x,t)$ is the weighted Peano kernel, defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \leqslant t \leqslant x \\ \\ \frac{W(t)}{W(b)} - 1, & x < t \leqslant b \end{cases}$$

and on the following formula given in [2]:

$$\frac{\int_a^b w(t)f(t)dt}{\int_a^b w(t)dt} - \frac{\int_c^d u(t)f(t)\,dt}{\int_c^d u(t)\,dt} = \int_{\min\{a,c\}}^{\max\{b,d\}} (P_u(x,t) - P_w(x,t))\,df(t).$$

For the weight functions $w(t) = g(t)$ for $t \in [a,b]$ and $u(t) = 1$ for $t \in [a, a+\lambda]$ we get the left-hand side Steffensen inequality.

At the end of this section we have to mention a wide range of results involving various kinds of fractional integrals and derivatives. J. Pečarić and his coauthors obtained Lagrange type and Cauchy type theorems which lead them to obtaining new means based on linear functionals arisen from inequalities involving fractional integrals and derivatives. Most of these results have been collected in monograph [35].

4. Convexity property as a starting point for converse inequalities

Attempts of obtaining converses of the Jensen and other classical inequalities have a long history. For example, in 1914 P. Schweitzer proved that

$$\left(\frac{1}{n}\sum_{k=1}^n a_k\right)\left(\frac{1}{n}\sum_{k=1}^n \frac{1}{a_k}\right) \leqslant \frac{(M+m)^2}{4Mm},$$

holds for an n-tuple $(a_1,\dots,a_n)$ with $0<m\leqslant a_k\leqslant M$ for $k=1,2,\dots,n$. It is a converse of inequality $\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{-1}\leqslant\left(\frac{1}{n}\sum_{k=1}^{n}(a_k)^{-1}\right)$ which can be interpreted as the Jensen inequality for the function $f(x)=x^{-1}$. Thirty years later L.B. Kantorovich gave a weighted version of the above-mentioned inequality. Meanwhile, similar results due to G. Pólya and G. Szegö, and to W. Greub and W. Rheinboldt have appeared (see [13, pp.121-122]). In 1935 K. Knopp published a result for the difference $\int_0^1 f(g(t))dt - f\left(\int_0^1 g(t)dt\right)$ for a convex function f, i.e. he gave an upper bound for the difference between the left-hand side and the right-hand side of the Jensen inequality. As we can see, difference and ratio type converses of the classical inequalities attracted the attention of many mathematicians. In [33] J. Pečarić gave a nice generalization of the above-mentioned results considering not just the difference or the ratio of the left-hand side and the right-hand side of the Jensen inequality. It is based on one fundamental property of a convex function.

As we know, a real function f is convex on $[m,M]$ if for all $\alpha\in[0,1]$ and all $x,y\in[m,M]$ the following inequality holds:

$$f(\alpha x+(1-\alpha)y)\leqslant\alpha f(x)+(1-\alpha)f(y). \tag{9}$$

Taking

$$x=m,\ \ y=M,\ \ \alpha=\frac{M-t}{M-m},\ \ 1-\alpha=\frac{t-m}{M-m}$$

in (9) we get

$$f(t)\leqslant\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M),\ \ t\in[m,M]. \tag{10}$$

Inequality (10) is the origin for several converses of the Jensen inequality in different settings. This idea appeared for the first time in the already mentioned paper [33] where converses of the Jensen inequality involving positive (isotonic) linear functional is given. Later, J. Pečarić developed this method and in collaboration with B. Mond investigated converses of the Jensen inequality for matrices and operators. Nowadays, this method is called the Mond-Pečarić method.

Here we describe results for a positive linear functional. Let L be a set of functions defined on a set E with properties that for any $\alpha,\beta\in\mathbb{R}$, $f,g\in L$, a function $\alpha f+\beta g\in L$ and the function 1 belongs to L where 1 is defined as $1(t)=1$ for all $t\in E$. We say that a functional $A:L\to\mathbb{R}$ is positive linear if $A(\alpha f+\beta g)=\alpha A(f)+\beta A(g)$ and if $f\geqslant 0$, then $A(f)\geqslant 0$. Furthermore, if $A(1)=1$, then we have a normalized positive linear functional A.

Classical examples of positive linear functionals are summation and integration, but there are also some other examples which have many applications such as integration on time scales, fractional integral operators etc. In 1931 B. Jessen gave the following generalization of the Jensen inequality.

If A is a normalized positive functional on L and if f is a continuous convex function on an interval I, then for all $g \in L$ such that $f \circ g \in L$ we have

$$f(A(g)) \leqslant A(f \circ g).$$

J. Pečarić had investigated intensively inequalities involving positive linear functionals and a great number of such results have been given in the book [34]. Advantages of this approach are that statements and proofs are usually very clear and elegant and they cover a lot of particular cases as we will see in the further text.

THEOREM 9. *Let f be convex on $I = [m, M]$, $m, M \in \mathbb{R}$ and let J be an interval such that $f(I) \subseteq J$. If $F(u, v)$ is a real function defined on $J \times J$, non-decreasing in the first variable, then we have*

$$\begin{aligned} F(A(f \circ g), f(A(g))] &\leqslant \max_{x \in [m,M]} F\left[\frac{M-x}{M-m} f(m) + \frac{x-m}{M-m} f(M), f(x)\right] \\ &= \max_{t \in [0,1]} F\left[t f(m) + (1-t) f(M), f(tm + (1-t)M)\right] \end{aligned} \tag{11}$$

where $g, f \circ g \in L$, $g(E) \subseteq I$.

Proof. Since a function f is convex on $[m, M]$

$$f(\alpha m + (1-\alpha)M) \leqslant \alpha f(m) + (1-\alpha) f(M)$$

holds for all $\alpha \in [0, 1]$. Since $g(E) \subseteq I$ we get $g(t) \in [m, M]$ for $t \in E$ and taking

$$\alpha = \frac{M - g(t)}{M - m}, \quad 1 - \alpha = \frac{g(t) - m}{M - m}$$

in the above inequality it follows

$$f(g(t)) \leqslant \frac{M - g(t)}{M - m} f(m) + \frac{g(t) - m}{M - m} f(M).$$

Applying a linear functional A gives that

$$A(f \circ g) \leqslant \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M).$$

Since F is non-decreasing in the first variable, it follows

$$\begin{aligned} F(A(f \circ g), f(A(g))] &\leqslant F\left[\frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M), f(A(g))\right] \\ &\leqslant \max_{x \in [m,M]} F\left[\frac{M-x}{M-m} f(m) + \frac{x-m}{M-m} f(M), f(x)\right] \\ &= \max_{t \in [0,1]} F\left[t f(m) + (1-t) f(M), f(tm + (1-t)M)\right], \end{aligned}$$

where in the last inequality transformation $t = \dfrac{M-x}{M-m}$ is used.

□

Using the same arguments as in the previous proof, if assumptions of Theorem 9 hold with function F being non-increasing in its first variable, then

$$\begin{aligned} F(A(f\circ g), f(A(g))] &\geqslant \min_{x\in[m,M]} F\left[\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M), f(x)\right] \\ &= \min_{t\in[0,1]} F\left[tf(m)+(1-t)f(M), f(tm+(1-t)M)\right]. \end{aligned}$$

The most researched examples for function F are $F(u,v) = u-v$ and $F(u,v) = \dfrac{u}{v}$. Firstly, let us consider the case $F(u,v) = \dfrac{u}{v}$. If f is convex on $I = [m,M]$ such that $f''(x) \geqslant 0$ with equality for at most isolated points of I, and if either (i) $f > 0$ or (ii) $f < 0$, then

$$A(f\circ g) \leqslant \lambda f(A(g)),$$

holds for some $\lambda > 1$ in case (i) or $\lambda \in (0,1)$ in case (ii). Furthermore, the value of λ may be determined. If $f(m) = f(M)$, and if $\overline{x} \in (m,M)$ is the solution of the equation $f'(x) = 0$, then $\lambda = f(m)/\overline{x}$. If $f(m) \neq f(M)$ and if $\overline{x} \in (m,M)$ is the solution of the equation $\dfrac{f(M)-f(m)}{M-m}f(x) - f'(x)\left[f(m)+\dfrac{f(M)-f(m)}{M-m}(x-m)\right] = 0$, then $\lambda = \frac{f(M)-f(m)}{(M-m)f'(\overline{x})}$.

Let us consider the second case: $F(u,v) = u-v$. If f is differentiable and f' is increasing on I, then

$$A(f\circ g) - f(A(g)) \leqslant \lambda,$$

where $\lambda = f(m) - f(\overline{x}) + \dfrac{f(M)-f(m)}{M-m}(\overline{x}-m)$ and $\overline{x}$ is the unique solution of the equation $f'(x) = \dfrac{f(M)-f(m)}{M-m}$.

In the previous results some very well-known inequalities are hidden.

EXAMPLE 1. Let $F(u,v) = u-v$ and $f(x) = x^2$. Then the right-hand side of inequality (11) has the form:

$$y(t) = tm^2 + (1-t)M^2 - (tm+(1-t)M)^2.$$

Easy calculation gives that the maximum of $y(t)$ is $\frac{(m-M)^2}{4}$ and applying Theorem 9 it follows

$$A(g^2) - A^2(g) \leqslant \frac{1}{4}(m-M)^2.$$

If $A(f) = \int_a^b f(x)dx$, this is the classical Grüss inequality with $f = g$ (see [13, p.296]).

EXAMPLE 2. Let $F(u,v)=\frac{u}{v}$, $f(x)=\frac{1}{x}$. Then the maximum of

$$y(t)=\left(\frac{t}{m}+\frac{1-t}{M}\right)(tm+(1-t)M)$$

is equal to $\dfrac{(m+M)^2}{4mM}$ and applying Theorem 9 it follows

$$A(g^{-1})A(g)\leqslant\frac{(m+M)^2}{4mM}.$$

If the linear functional A is defined as $A(a)=\dfrac{\sum_{i=1}^n p_ia_i}{\sum_{i=1}^n p_i}$, where $a(i)=a_i$, the for n-tuples $(a_1,\dots,a_n)$ and $(p_1,\dots,p_n)$ such that $a_i\in[m,M]$, $(m>0)$, $p_i\geqslant 0$ for all i, the above inequality becomes

$$\sum_{i=1}^n p_ia_i\sum_{i=1}^n\frac{p_i}{a_i}\leqslant\frac{(m+M)^2}{4mM}\left(\sum_{i=1}^n p_i\right)^2.$$

It is the well-known Kantorovich inequality ([13, p.684]).

The following result gives an upper bound for the function F whose arguments are generalized means. A generalized mean with respect to an operator A and a function f is defined by

$$M_f(h,A)=f^{-1}(A(f\circ h))$$

where f is a continuous strictly monotone function on I, $h\in L$.

THEOREM 10. *Let f and g be continuous strictly monotone real functions on $I=[m,M]$ and such that $f\circ h, g\circ h\in L$. If F satisfies the assumptions of Theorem 9, f is increasing, and $f\circ g^{-1}$ is convex, then*

$$\begin{aligned}&F(M_f(h,A),M_g(h,A))\\&\qquad\leqslant\max_{t\in[0,1]}F[f^{-1}(tf(m)+(1-t)f(M)),g^{-1}(tg(m)+(1-t)g(M))].\end{aligned}\tag{12}$$

Proof. Let us suppose that f is increasing. We consider the functions F_1,f_1,g_1 defined by

$$F_1(x,y)=F(f^{-1}(x),f^{-1}(y)),\ f_1=f\circ g^{-1},\ g_1=g\circ h.$$

The function F_1 is non-decreasing in the first variable, f_1 is convex. Applying Theorem 9 to the functions F_1,f_1,g_1 with $m_1=g(m)$, $M_1=g(M)$ we get

$$F_1(A(f_1\circ g_1),f_1(A(g_1))]\leqslant\max_{t\in[0,1]}F_1\left[tf_1(m)+(1-t)f_1(M),f_1(tm+(1-t)M)\right].\tag{13}$$

After substitutions, the left-hand side of the previous inequality becomes equal to

$$\begin{aligned}F_1(A(f_1\circ g_1),f_1(A(g_1))]&=F_1(A((f\circ g^{-1})\circ(g\circ h),(f\circ g^{-1})(A(g\circ h)))\\&=F_1(A(f\circ h),f(g^{-1}(A(g\circ h))))=F(M_f(h,A),M_g(h,A)).\end{aligned}$$

Similarly, after substitutions the right-hand side of (13) becomes equal to the right-hand side of (12).

□

It is interesting to see how this idea is applied in operator theory [15]. Let us denote by $B_h(H)$ a set of all bounded linear operators on the Hilbert space H where the identity operator is denoted by 1_H. An analogue of the Jensen inequality for operator convex functions is called the Davis-Choi-Jensen inequality and it states the following:

If $\Phi : B_h(H) \to B_h(K)$ is a normalized positive linear map and if f is an operator convex function on an interval I, then

$$\Phi(f(A)) \geqslant f(\Phi(A))$$

for every selfadjoint operator A on H whose spectrum is contained in I.

Moreover, if $A_j \in B_h(H)$, $j = 1, \ldots, k$ are selfadjoint operators with $Sp(A_j) \subseteq [m,M]$ for scalars $m < M$, $\Phi_j : B_h(H) \to B_h(K)$ are normalized positive linear maps, $w_1, \ldots, w_k$ are positive real numbers such that $\sum_{j=1}^{k} w_j = 1$, and if $f \in C(m,M)$ is operator convex on $[m,M]$, then

$$f\left(\sum_{j=1}^{k} w_j \Phi_j(A_j)\right) \leqslant \sum_{j=1}^{k} w_j \Phi_j(f(A_j)).$$

Furthermore, if $g \in C(m,M)$ and f is operator convex on $[m,M]$ such that $f \leqslant g$, then we get

$$f\left(\sum_{j=1}^{k} w_j \Phi_j(A_j)\right) \leqslant \sum_{j=1}^{k} w_j \Phi_j(g(A_j)).$$

The following theorem gives a companion inequality which includes the difference and the ratio converses of the Davis-Choi-Jensen inequality.

THEOREM 11. *Let $A_j \in B_h(H)$, $j = 1, \ldots, k$ be selfadjoint operators with $Sp(A_j) \subseteq [m,M]$ for some scalars $m < M$ and let $\Phi_j : B_h(H) \to B_h(K)$, $j = 1, \ldots, k$, be normalized positive linear maps. Let $w_1, \ldots, w_k$ be positive real numbers such that $\sum_{j=1}^{k} w_j = 1$. Let f and g be continuous functions on $[m,M]$ and $F(u,v)$ be a real continuous function on $U \times V$, $f([m,M]) \subseteq U$, $g([m,M]) \subseteq V$. If $F(u,v)$ is operator monotone in the first variable u and f is convex on $[m,M]$, then*

$$F\left[\sum_{j=1}^{k} w_j \Phi_j(f(A_j)), g\left(\sum_{j=1}^{k} w_j \Phi_j(A_j)\right)\right]$$
$$\leqslant \left[\max_{x\in[m,M]} F\left(\frac{M-x}{M-m} f(m) + \frac{x-m}{M-m} f(M), g(x)\right)\right] 1_K.$$

If f is concave, then we have the reverse inequality with min instead of max.

Proof. Let us assume that f is convex on $[m,M]$. Since property (10) holds for every $t \in [m,M]$ we get

$$f(A_j) \leqslant \frac{M1_H - A_j}{M-m} f(m) + \frac{A_j - m1_H}{M-m} f(M).$$

for all $j = 1, \dots, k$. Since Φ_j is a normalized positive linear map, we have

$$\begin{aligned}\Phi_j(f(A_j)) &\leqslant \Phi_j\left(\frac{M1_H - A_j}{M-m}f(m) + \frac{A_j - m1_H}{M-m}f(M)\right)\\ &= \frac{M1_K - \Phi_j(A_j)}{M-m}f(m) + \frac{\Phi_j(A_j) - m1_K}{M-m}f(M)\end{aligned}$$

for $j = 1, \dots, k$. Multiplying the above inequalities with w_j and summing for all $j = 1, \dots, k$ we have

$$\sum_{j=1}^{k} w_j\Phi_j(f(A_j)) \leqslant \frac{M1_K - \sum_{j=1}^{k} w_j\Phi_j(A_j)}{M-m}f(m) + \frac{\sum_{j=1}^{k} w_j\Phi_j(A_j) - m1_K}{M-m}f(M).$$

Since $m1_H \leqslant A_j \leqslant M1_H$ we have $m1_K \leqslant \sum_{j=1}^{k} w_j\Phi_j(A_j) \leqslant M1_K$ i.e. $Sp(\sum_{j=1}^{k} w_j\Phi_j(A_j)) \subseteq [m,M]$. Using operator monotonicity of F in the first variable we obtain

$$\begin{aligned}&F\left[\sum_{j=1}^{k} w_j\Phi_j(f(A_j)), g\left(\sum_{j=1}^{k} w_j\Phi_j(A_j)\right)\right]\\ &\leqslant F\left[\frac{M1_K - \sum_{j=1}^{k} w_j\Phi_j(A_j)}{M-m}f(m) + \frac{\sum_{j=1}^{k} w_j\Phi_j(A_j) - m1_K}{M-m}f(M), g\left(\sum_{j=1}^{k} w_j\Phi_j(A_j)\right)\right]\\ &\leqslant \left[\max_{x\in[m,M]} F\left(\frac{M-x}{M-m}f(m) + \frac{x-m}{M-m}f(M), g(x)\right)\right]1_K\end{aligned}$$

which is the desired inequality.

□

As applications of the above general theorem we give the following ratio type reverse inequality [8].

THEOREM 12. *Let $A_j \in B_h(H)$, $j = 1, \dots, k$ be selfadjoint operators with $Sp(A_j) \subseteq [m,M]$ for some scalars $m < M$ and let $\Phi_j : B_h(H) \to B_h(K)$, $j = 1, \dots, k$, be normalized positive linear maps. Let $w_1, \dots, w_k$ be positive real numbers such that $\sum_{j=1}^{k} w_j = 1$. Let f and g be continuous functions on $[m,M]$ and suppose that either of the following conditions holds:*

(i) $g(t) > 0$ for all $t \in [m,M]$,

(ii) $g(t) < 0$ for all $t \in [m,M]$.

If f is a convex function on $[m,M]$, then

$$\sum_{j=1}^{k} w_j\Phi_j(f(A_j)) \leqslant \alpha_0 g\left(\sum_{j=1}^{k} w_j\Phi_j(A_j)\right),$$

where

$$\alpha_0 = \max_{t\in[m,M]} \left\{ \frac{1}{g(t)}(\mu_f t + \nu_f) \right\} \ \textit{in the case } (i)$$
$$\textit{or } \alpha_0 = \min_{t\in[m,M]} \left\{ \frac{1}{g(t)}(\mu_f t + \nu_f) \right\} \ \textit{in the case } (ii)$$

and

$$\mu_f = \frac{f(M)-f(m)}{M-m} \quad \textit{and} \quad \nu_f = \frac{Mf(m)-mf(M)}{M-m}.$$

If $f(m) > 0, f(M) > 0$ *and* g *is a strictly concave twice differentiable function in the case of (i), then*

$$\alpha_0 = \frac{\mu_f t_0 + \nu_f}{g(t_0)},$$

where

$$t_0 = \begin{cases} M & \textit{if } \frac{\mu_f}{\mu_g}\nu_g \geqslant \nu_f, \\ m & \textit{if } \frac{\mu_f}{\mu_g}\nu_g < \nu_f. \end{cases}$$

If $f(m) < 0, f(M) < 0$ *and* g *is a strictly convex twice differentiable function in the case of (ii), then* $\alpha_0 = \dfrac{\mu_f t_0 + \nu_f}{g(t_0)}$, *where*

$$t_0 = \begin{cases} \textit{the solution of} & & \\ \mu_f g(t) = (\mu_f t + \nu_f) g'(t) & \textit{if} & f(m)\frac{g'(m)}{g(m)} < \mu_f < f(M)\frac{g'(M)}{g(M)}, \\ M & \textit{if} & \mu_f \geqslant f(M)\frac{g'(M)}{g(M)}, \\ m & \textit{if} & \mu_f \leqslant f(m)\frac{g'(m)}{g(m)}. \end{cases}$$

A simple consequence of the above theorem is the following Kantorovich-type inequality for operators:

$$\sum_{j=1}^{k} w_j \Phi_j(A_j^{-1}) \leqslant \frac{(M+m)^2}{4Mm} \left(\sum_{j=1}^{k} w_j \Phi_j(A_j) \right)^{-1},$$

where A_j, Φ_j and w_j, $j = 1, \ldots, k$, satisfy assumptions of the above theorem.

More about applications of the above theorems in determining upper estimates for $\displaystyle\sum_{j=1}^{k} w_j \Phi_j(f(A_j)) - g\left(\sum_{j=1}^{k} w_j \Phi_j(A_j)\right)$ and $\dfrac{\sum_{j=1}^{k} w_j \Phi_j(f(A_j))}{g\left(\sum_{j=1}^{k} w_j \Phi_j(A_j)\right)}$ by means of scalar multiples of the identity operator can be found in book [8].

Acknowledgements. The research of the author was supported by Croatian Science Foundation under the project 5435.

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ, A. ČIVLJAK, S. KOVAČ, J. PEČARIĆ, M. RIBIČIĆ PENAVA, *General integral identities and related inequalities*, Element, Zagreb, 2013.

[2] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, *Error estimates for approximations of the Fourier transform of functions in L_p spaces*, Aust. J. Math. Anal. Appl. **1**, 2 (2004), Article 10.

[3] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, A. PERUŠIĆ, *Generalizations of Steffensen's inequality via weighted Montgomery identity*, Math. Inequal. Appl., **17** (2014), 779–799.

[4] P. R. BEESACK, J. PEČARIĆ, *On Jessen's inequality for convex functions*, J. Math. Anal. Appl. **110** (1985), 536-552.

[5] P. BULLEN, *A look at the work of Josip Pečarić*, Math. Inequal. Appl., **12** (2014), 693–692.

[6] S. S. DRAGOMIR, J. PEČARIĆ, *Refinements of some inequalities for isotonic functionals*, Rev. Anal. Numer. Theor. Approx. **18** (1989) 61–65.

[7] I. FRANJIĆ, J. PEČARIĆ, I. PERIĆ, A. VUKELIĆ, *Euler integral identity, quadrature formulae and error estimations*, Element, Zagreb, 2011.

[8] T. FURUTA, J. MIĆIĆ HOT, J. PEČARIĆ, Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.

[9] C. F. GAUSS, Theoria combinationis observationum, 1821., German transl. in *Abhandlungen zur Methode der kleinsten Quadrate.* Neudruck. Würzburg 1964, pp. 9 and 12.

[10] M. MATIĆ, J. PEČARIĆ, N. UJEVIĆ, *Improvement and further generalization of some inequalities of Ostrowski-Grüss type*, Comput. Math. Appl. **39** (2000), 161–175.

[11] G. V. MILOVANOVIĆ, J. PEČARIĆ, *On generalizations of the inequality of A. Ostrowski and some related applications*, Univ. Beograd Publ. Elektrotehn. Fak., Ser.Mat.Fiz., **544-576** (1976), 155–158.

[12] G. MILOVANOVIĆ, J. PEČARIĆ, *The Steffensen inequality for convex function of order n*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **634–677** (1979), 97–100.

[13] D. S. MITRINOVIĆ, J. PEČARIĆ, A. M. FINK, *Classical and new inequalities in analysis*, Kluwer Acad. Publ. Dordrecht-Boston-London, 1993.

[14] D. S. MITRINOVIĆ, J. PEČARIĆ, V. VOLENEC, *Recent Advances in Geometric Inequalities*, Kluwer Acad. Publ. Dordrecht-Boston-London, 1989.

[15] B. MOND, J. PEČARIĆ, *Bounds for Jensen's inequality for several operators*, Houston J. Math., **20** (1994), 645–651.

[16] A. OSTROWSKI, *Aufgabensammlung zur Infinitesimalrechnung*, Erster Band: Funktionen einer Variablen, Birkhäuser Verlag, Basel-Stuttgart, 1964.

[17] J. PEČARIĆ, *On the Jensen-Steffensen inequality*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **634-677** (1979), 101–107.

[18] J. E. PEČARIĆ, *On the Čebyšev inequality*, Bul. Stiint. Tehn. Inst. Politehn. Timisoara **25**(39) (1980), 10–11.

[19] J. PEČARIĆ, *A new proof of the Jensen-Steffensen inequality*, Mathematica (Cluj) **23** (46) (1981), 73–77.

[20] J. PEČARIĆ, *Inverse of Jensen-Steffensen inequality*, Glasnik Matematički **16** (36) (1981), 229–233.

[21] J. E. PEČARIĆ, *Inverse of Steffensen's inequality*, Glasnik Matematički Ser. III **17**(37) (1982), 265–270.

[22] J. PEČARIĆ, *A short proof of a variant of Jensen's inequality*, J. Math. Anal. Appl. **87** (1982), 278–280.

[23] J. PEČARIĆ, *An inequality for 3-convex function*, J. Math. Anal. Appl. **90** (1982), 213–218.

[24] J. PEČARIĆ, *On variants of Jessen's inequality*, Glasnik Matematički **18** (38) (1983), 281–289.

[25] J. PEČARIĆ, *A simple proof of the Jensen-Steffensen inequality*, Amer. Math. Monthly **91** (1984), 195-196.

[26] J. E. PEČARIĆ, *On the Bellman generalization of Steffensen's inequality II*, J. Math. Anal. Appl. **104**(2) (1984), 432–434.

[27] J. PEČARIĆ, *The inductive proof of the Jensen-Steffensen inequality*, Anal. Univ. Timisoara **23** (1985), 33–34.

[28] J. PEČARIĆ, *On the Popoviciu conversion of Jensen's inequality*, Revue Anal. Numer. Theor. Approx. **16** (1987) 65–68.

[29] J. PEČARIĆ, *Konveksne funkcije: nejednakosti*, Naučna knjiga, Beograd, 1987.

[30] J. E. PEČARIĆ, *Connections among some inequalities of Gauss, Steffensen and Ostrowski*, Southeast Asian Bull. Math. **13**, 2 (1989), 89–91.

[31] J. PEČARIĆ, P. R. BEESACK, *On Jessen's inequality for convex functions II*, J. Math. Anal. Appl. **118** (1986), 125–144.

[32] J. PEČARIĆ, P. R. BEESACK, *Some Jessen-Beckenbach Inequalities*, Rocky Mountain J. Math. **17** (1987), 629–641.

[33] J. E. PEČARIĆ, P. R. BEESACK, On Knopp's inequality for convex functions, Canad. Math. Bull. **30**(3) (1987), 267–272.

[34] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex functions, partial orderings and statistical applications*, Mathematics in Science and Engineering, 187, Academic Press Inc., Boston, 1992.

[35] J. PEČARIĆ, K. SMOLJAK KALAMIR, S. VAROŠANEC, *Steffensen's and related inequalities*, Element, Zagreb, 2014.

[36] J. PEČARIĆ, S. VAROŠANEC, *Multidimensional and discrete Steffensen inequality*, Southeast Asian Bull. Math. **23**(2) (1999), 277–284.

[37] J. PEČARIĆ, S. VAROŠANEC, *Harmonic polynomials and generalization of Ostrowski inequality with applications in numerical integration*, Nonlinear Analysis **47** (2001), 2365–2374.

[38] J. PEČARIĆ, S. VAROŠANEC, *A note on Simpson's inequality for Lipschitzian functions*, Soochow J. Math. **27** (2001), 53–57.

[39] J. PEČARIĆ, S. VAROŠANEC, *On Simpson's formula for functions whose derivatives belong to L_p spaces*, Math. Appl. Letters **14** (2001), 131–135.

[40] J. F. STEFFENSEN, *On certain inequalities between mean values and their application to actuarial problems*, Skand. Aktuarietids. (1918), 82–97.

[41] P. M. VASIĆ, J. PEČARIĆ, *On the Jensen inequality*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **634-677** (1979), 50–54.

[42] P. M. VASIĆ, J. PEČARIĆ, *Note on the Steffensen inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **716-734** (1981), 80–82.

(Received April 15, 2015)

Department of Mathematics
University of Zagreb
Zagreb, Croatia
e-mail: varosans@math.hr