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ON BOUNDARY DOMINATION IN THE JENSEN–MERCER INEQUALITY

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Abstract. The main purpose of this, mainly expository, paper is to give various arguments that the boundary domination is a crucial property for the Jensen-Mercer inequality. Although this is an obvious property of convex functions and it is already expressed in the Jensen inequality it seems that the Jensen-Mercer inequality contains this information in a more vivid, explicit sense. This domination is presented using Steffensen-Popoviciu measures, The Majorization Theorem and a crude domination of weights of vertices in the multidimensional case (polytopes, simplices).

1. Some basic ideas

An interesting Jensen-type inequality was proved in [16].

THEOREM 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function, $x_i\in[a,b]$, $w_i\geqslant 0$, $i=1,\dots,n$, such that $W_n=\sum_{i=1}^n w_i>0$. Then*

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^n w_ix_i\right)\leqslant f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^n w_if(x_i)\ldots\text{theJensen}-\text{Mercer inequality.} \tag{1.1}$$

Proof. Set $W_n=1$. Then

$$\begin{aligned}f\left(a+b-\sum_{i=1}^n w_ix_i\right)&=f\left(\sum_{i=1}^n w_i(a+b-x_i)\right)\\&\leqslant\sum_{i=1}^n w_if(a+b-x_i)\leqslant\sum_{i=1}^n w_i(f(a)+f(b)-f(x_i))\\&=f(a)+f(b)-\sum_{i=1}^n w_if(x_i).\end{aligned}$$

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The weighted Jensen inequality and the positivity of weights are crucial for this proof. The inequality

$$f(a+b-x) \leqslant f(a)+f(b)-f(x) \tag{1.2}$$

is also used in the proof. It is instructive to see possible proofs of inequality (1.2). We briefly overview some of the possible proofs:

1. The Wright convexity:

$$\begin{aligned} & f(a+b-x)+f(x) \\ & = f\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}b\right)+f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right) \\ & \leqslant \frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)+\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b) \\ & = f(a)+f(b) \end{aligned}$$

2. The increasing increments property: Obvious, writing inequality (1.2) in the form $f(a+b-x)-f(a) \leqslant f(b)-f(x)$.

3. The Majorization Theorem: Obvious, writing inequality (1.2) in the form

$$f(a+b-x)+f(x) \leqslant f(a)+f(b). \tag{1.3}$$

4. The Jensen-Steffensen inequality: Obvious, writing inequality (1.2) in the form $f(a-x+b) \leqslant f(a)-f(x)+f(b)$ and using weights $w_1=1$, $w_2=-1$, $w_3=1$.

It is apparent from the proof of inequality (1.2), using the Wright convexity, that actually the secant inequality

$$f(x) \leqslant \frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b), \text{ for all } x \in [a,b],$$

is applied twice. An analogous method is applied to prove functional forms of Jensen-Mercer inequality (see [4]). A general form is given in [3] as follows:

THEOREM 1.2. *Let L be a vector space of real functions on a non-empty set E which contains constant functions and let $A : L \to \mathbb{R}$ be a positive linear functional such that $A(1)=1$. If $\phi : [m,M] \to \mathbb{R}$ is a continuous convex function, then*

$$\begin{aligned} & \phi\left(m+M-A(g)\right) \leqslant A\left(\phi\left(m+M-g\right)\right) \\ & \leqslant \frac{M-A(g)}{M-m}\phi(M)+\frac{A(g)-m}{M-m}\phi(m) \\ & \leqslant \phi(m)+\phi(M)-A\left(\phi(g)\right) \ldots \text{the Jessen} - \text{Mercer inequality}, \end{aligned} \tag{1.4}$$

where g, $\phi(g)$, $\phi\left(m+M-g\right) \in L$.

A similar arguing with application of functional calculus gives Jensen type operator inequality without operator convexity assumptions. The following theorem is proved in [14].

THEOREM 1.3. *Let $A_1,\dots,A_k$ be self-adjoint operators whose spectra are in $[m,M]\subset\mathbb{R}$ and let $\Phi_1,\dots,\Phi_k$ be positive linear mappings such that $\sum_{i=1}^k \Phi_k(I)=I$. If $f:[m,M]\to\mathbb{R}$ is a convex function , then*

$$\begin{aligned} f\left(m+M-\sum_{j=1}^k \Phi_j(A_j)\right) &\leqslant \frac{M-\sum_{j=1}^k \Phi_j(A_j)}{M-m} f(M)+\frac{\sum_{j=1}^k \Phi_j(A_j)-m}{M-m} f(m) \\ &\leqslant f(m)+f(M)-\sum_{j=1}^k \Phi_j(f(A_j)). \end{aligned}$$

Theorem 1.3 enables to give a comparison of operator power Mercer means for broader range of exponents than in usual operator convexity case.

As application of Theorem 1.3 and interesting characterization of operator convexity obtained in [25] the following theorem is given in [9].

THEOREM 1.4. *Let A_i, $i=1,\dots,n$ be positive operators acting on a finite dimensional Hilbert space with $\sum_{i=1}^n A_i=I$. If f is convex on an interval $[m,M]$ containing 0, then*

$$f\left(m+M-\sum_{i=1}^n x_i A_i\right)\leqslant f(m)+f(M)-\sum_{i=1}^n f(x_i)A_i,$$

where $x_1,\dots,x_n\in[m,M]$.

2. The Jensen-Mercer inequality as the Jensen inequality for Steffensen-Popoviciu measures

In the previous section the Jensen-Mercer inequality was obtained by using the Jensen inequality and some specific properties of convex functions (the Wright convexity, the increasing increment property) or applying the Lah-Ribarič inequality twice or some characteristic method for convex functions (Jensen-Steffensen's inequality or The Majorization Theorem). Each of the mentioned methods reveals a different inner property of the Jensen-Mercer inequality. The purpose of this section is to show that boundary weights domination is crucial property of the Jensen-Mercer inequality. From this point of view the Jensen-Mercer inequality (in the form presented in the introduction) is just a special case of Jensen's inequality for Steffensen-Popoviciu measures (see [19]).

The first inequality in the following theorem is the classical Jensen-Steffensen inequality (see [24]).

THEOREM 2.1. *Let* $x_i \in I$, $I \subseteq \mathbb{R}$ *an interval,* $w_i \in \mathbb{R}$, $i = 1, \ldots, n$ *be such that*

$$0 \leqslant W_k \leqslant W_n,\ k = 1, \ldots, n-1,\ W_k = \sum_{i=1}^{k} w_i,\ W_n > 0. \tag{2.1}$$

If $f : I \to \mathbb{R}$ *is a convex function and* $x_1, \ldots, x_n$ *a monotonic sequence, then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i),$$

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqslant f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i),$$

where $\{x_1, \ldots, x_n\} \subset [a,b] \subseteq I$.

Proof. (see [1]) Suppose that $a \leqslant x_1 \leqslant \cdots \leqslant x_n$. The weights

$$v_0 = 1, v_1 = -\frac{w_1}{W_n}, \cdots, v_n = -\frac{w_n}{W_n}, v_{n+1} = 1$$

obviously satisfy (2.1).

An interesting feature of the Jensen-Mercer inequality, a kind of robust property with the respect to weights, is given in the following theorem (see [11]). Again, the first part is the classical Reverse Jensen's inequality. Although the proof is a simple one, the form of the inequality is, in some sense, unexpected, since it has the same direction as the "usual" Jensen-Mercer inequality.

THEOREM 2.2. *Let* $f : [a,b] \to \mathbb{R}$ *be a convex function,* $x_i \in [a,b]$, $w_i \in \mathbb{R}$, $i = 1, \ldots, n$, *be such that* $w_1 > 0$, $w_i \leqslant 0$, $i = 2, \ldots, n$, $W_n = \sum_{i=1}^{n} w_i > 0$ *and* $\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \in [a,b]$. *Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \geqslant \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i), \tag{2.2}$$

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leqslant f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \tag{2.3}$$

Proof. Using $f(a+b-x) \leqslant f(a)+f(b)-f(x)$ (the increasing increments property, the Wright convexity) (2.3) immediately follows from (2.2).

REMARK 2.3. If $a \leqslant x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)} \leqslant b$ is the increasing rearrangement of the given sequence in the previous theorem, then the weights

$$v_0 = 1, v_1 = -\frac{w_{(1)}}{W_n}, \cdots, v_n = -\frac{w_{(n)}}{W_n}, v_{n+1} = 1$$

"never" satisfy the Jensen-Steffensen conditions (2.1). This can be seen directly. Of course this is obvious since the Jensen-Steffensen conditions imply Jensen's inequality and the conditions in Theorem 2.2 give reverse Jensen's inequality.

A natural question arises from the previous remark: Is it possible to give a unified approach to both of the weights conditions? It is obvious from the either cases that a boundary weights domination is crucial for the Jensen-Mercer inequality. This domination can be also expressed using Steffensen-Popoviciu measures (see [19]).

DEFINITION 2.4. *A Steffensen-Popoviciu measure is any signed Borel measure μ on K (compact convex subset of a locally convex Hausdorff real vector space) such that*

$$\mu(K) > 0,\ \int_K f^+(x)d\mu(x) \geqslant 0$$

for any continuous convex function f on K.

The following theorem is proved in [19].

THEOREM 2.5. *(Jensen's inequality for Steffensen-Popoviciu measures) Suppose that μ is a signed Borel measure on K with $\mu(K) > 0$. Then the following assertions are equivalent:*

(i) μ is a Steffensen-Popoviciu measure;

(ii) μ admits a barycenter $x_\mu \in K$ and

$$f\left(x_\mu\right) \leqslant \frac{1}{\mu\left(K\right)} \int_K f(x)d\mu(x)$$

for all continuous convex functions f on K.

The following characterization of discrete Steffensen-Popoviciu measures is given in the same book.

THEOREM 2.6. *Suppose that $x_i, p_i \in \mathbb{R}$, $i = 1,\dots,n$, such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. Then the discrete measure $\mu = \sum_{k=1}^n p_k \delta_{x_k}$ is a Steffensen-Popoviciu measure if and only if*

$$\sum_{k=1}^n p_k > 0,\ \sum_{k=1}^m p_k\left(x_m - x_k\right) \geqslant 0,\ \sum_{k=m}^n p_k\left(x_k - x_m\right) \geqslant 0 \tag{2.4}$$

for all $m = 1,\dots,n$.
If

$$\sum_{k=1}^n p_k > 0,\ \ 0 \leqslant \sum_{k=1}^m p_k \leqslant \sum_{k=1}^n p_k \tag{2.5}$$

for every $m = 1,\dots,n$, then (2.4) holds.

We use this characterization of discrete Steffensen-Popoviciu measure to prove that all previous conditions on weights (positive weights, Jensen-Steffensen weights, Reverse Jensen weights) generate Steffensen-Popoviciu measures.

THEOREM 2.7. *Let $x_i \in [a,b]$, $i=1,\dots,n$ be such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. Suppose that weights w_i, $i=1,\dots n$, are either non-negative with $W_n > 0$ or satisfy Jensen-Steffensen conditions (2.1) or satisfy conditions for the reverse Jensen's inequality (see Theorem 2.2). Then*

$$\mu = \delta_a + \delta_b - \sum_{i=1}^{n} w_i \delta_{x_i}$$

is a Steffensen-Popoviciu measure.

Proof. Set $W_n = 1$. Set $y_0 = a$, $y_i = x_i$, $i=1,\dots,n$, $y_{n+1} = b$ and $v_0 = v_{n+1}$, $v_i = -w_i$, $i=1,\dots,n$. Then

$$\mu = \sum_{i=0}^{n+1} v_i \delta_{y_i}.$$

In the cases where $w_i \geqslant 0$ or $0 \leqslant W_k \leqslant W_n = 1$ obviously $V_k = 1 - W_k$, $V_0 = V_{n+1} = 1$ satisfy (2.5), so μ is a Steffensen-Popoviciu measure.

Suppose that $w_1 > 0$, $w_i < 0$, $W_n = 1$ and $\sum_{i=1}^{n} w_i x_i \in [a,b]$. In this case $w_1 > 1$ and $V_1 = 1 - w_1 < 0$ so Jensen-Steffensen conditions are not satisfied. We have

$$\begin{aligned}
&\sum_{i=0}^{m} v_i (y_m - y_i) \geqslant 0 \\
&\quad \Leftrightarrow x_m - a - w_1(x_m - x_1) - \cdots - w_{m-1}(x_m - x_{m-1}) \geqslant 0 \\
&\quad \Leftrightarrow w_1 x_1 + \cdots + w_{m-1} x_{m-1} + (1 - w_1 - \cdots - w_{m-1}) x_m \geqslant a,
\end{aligned}$$

which follows from

$$\begin{aligned}
&w_1 x_1 + \cdots + w_{m-1} x_{m-1} + (1 - w_1 - \cdots - w_{m-1}) x_m \\
&\quad = w_1 x_1 + \cdots + w_{m-1} x_{m-1} + (w_m + \cdots + w_n) x_m \\
&\quad \geqslant w_1 x_1 + \cdots + w_{m-1} x_{m-1} + w_m x_m + \cdots + w_n x_n \geqslant a.
\end{aligned}$$

Analogously

$$\begin{aligned}
&\sum_{i=m}^{n+1} v_i (y_i - y_m) \geqslant 0 \\
&\quad \Leftrightarrow (1 - w_{m+1} - \cdots - w_n) x_m + w_{m+1} x_{m+1} + \cdots w_n x_n \leqslant b,
\end{aligned} \tag{2.6}$$

which follows from

$$\begin{aligned}
&(1 - w_{m+1} - \cdots - w_n) x_m + w_{m+1} x_{m+1} + \cdots w_n x_n \\
&\quad \leqslant (1 - w_{m+1} - \cdots - w_n) x_m + w_{m+1} x_m + \cdots w_n x_m = x_m \leqslant b.
\end{aligned}$$

This holds for $m \geqslant 1$ (since $w_1 > 0$). For $m = 0$ the second line of (2.6) reduces to

$$w_1 x_1 + \cdots + w_n x_n \leqslant b$$

which holds by an assumption.

Integral case is based on the Jensen-Boas variant of the the Jensen inequality. Since there is no continuous analogue of the Reverse Jensen's inequality, we can consider just Jensen-Steffensen conditions.

THEOREM 2.8. *Let* $g : [\alpha,\beta] \to (a,b)$ *be a continuous and monotonic function and let* $\lambda : [\alpha,\beta] \to \mathbb{R}$ *be either continuous or of a bounded variation satisfying*

$$\lambda(\alpha) \leqslant \lambda(t) \leqslant \lambda(\beta),\ \lambda(\beta) - \lambda(\alpha) > 0. \tag{2.7}$$

If $\phi : (a,b) \to \mathbb{R}$ *is a convex function, then*

$$\phi\left(\frac{1}{\lambda[a,b]}\int_\alpha^\beta g(t)d\lambda(t)\right) \leqslant \frac{1}{\lambda[a,b]}\int_\alpha^\beta \phi(g(t))\,d\lambda(t). \tag{2.8}$$

Boas conditions (2.7) are obviously equivalent to Jensen-Steffensen conditions for integrals in the sense that

$$0 \leqslant \int_\alpha^x d\lambda(t) \leqslant \int_\alpha^\beta d\lambda(t),\ \int_\alpha^\beta d\lambda(t) > 0, \tag{2.9}$$

and these conditions are exactly the conditions for a measure λ to be a Steffensen-Popoviciu measure (see also [19]).

Usually this is expressed in terms of absolutely continuous measures $d\lambda(t) = p(t)dt$, where p is an integrable function (not necessarily non-negative).

In [2] it was proved that if $\lambda : [\alpha,\beta] \to \mathbb{R}$ is of a bounded variation satisfying (2.7) (or equivalently (2.9)), then the measure $\mu = \delta_\alpha + \delta_\beta + \lambda_c$, where $\lambda_c(x) = \frac{\lambda(\beta)-\lambda(x)}{\lambda(\beta)-\lambda(\alpha)}$ (and δ_a is the Dirac measure concentrated at a), also satisfies conditions (2.7). Inequality (2.8) for the measure μ gives:

$$\begin{aligned}
&\phi\left(g(a)+g(b)-\frac{1}{\lambda[a,b]}\int_\alpha^\beta g(t)d\lambda(t)\right)\\
&\quad\leqslant \phi(g(a))+\phi(g(b))-\frac{1}{\lambda[a,b]}\int_\alpha^\beta \phi(g(t))\,d\lambda(t).
\end{aligned}$$

In this way, using Theorem 2.5, it was proved that the measure

$$\mu = \delta_\alpha + \delta_\beta + \lambda_c$$

is a Steffensen-Popoviciu measure.

3. Boundary domination in the Jensen-Mercer inequality via majorization

In the preceding section boundary weights domination, in some sense crucial for the Jensen-Mercer inequality, was expressed using Jensen-Popoviciu measures. Another way in describing boundary domination is in using the Majorization Theorem. Compare with (1.3). We give two typical results in this area with short proofs to keep the paper reasonably self-contained.

The following theorem is proved in [21]

THEOREM 3.1. *Let $f: I \to \mathbb{R}$ ($I \subseteq \mathbb{R}$ an interval) be a continuous convex function, $\mathbf{a} = (a_1, \ldots, a_m) \in I^m$, and $\mathbf{X} = (x_{ij})$ be a $n \times m$ matrix such that $x_{ij} \in I$ for all i, j.*
If $\mathbf{a}$ majorizes each row of $\mathbf{X}$, then

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij}\right) \leqslant \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i f(x_{ij}), \tag{3.1}$$

where $\sum_{i=1}^{n} w_i = 1$, $w_i \geqslant 0$ for all i.

Proof. Since $x_i \prec a$, $i = 1, \ldots, n$, it follows by the Majorization Theorem that

$$\sum_{j=1}^{m} f(x_{ij}) \leqslant \sum_{j=1}^{m} f(a_j) \Leftrightarrow f(x_{im}) \leqslant \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1} f(x_{ij}).$$

We have:

$$\begin{aligned}
f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij}\right) &= f\left(\sum_{i=1}^{n} w_i \left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right)\right) \\
&\leqslant \sum_{i=1}^{n} w_i f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right) \leqslant \sum_{i=1}^{n} w_i \left(\sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1} f(x_{ij})\right) \\
&= \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i f(x_{ij}).
\end{aligned}$$

A continuous version was given in [8].

THEOREM 3.2. (simplified version) *Let $a = a_0 < a_1 < b_1 < b_2 = b$, $I = (a_1, b_1)$, $I^c = [a,b] \setminus I = [a, a_1] \cup [b_1, b]$. Let $\lambda : [a,b] \to \mathbb{R}$ be a function of bounded variation such that $\lambda(a) \leqslant \lambda(t) \leqslant \lambda(a_1)$ on (a, a_1), $\lambda(b_1) \leqslant \lambda(t) \leqslant \lambda(b)$ on (b_1, b) and $L = \int_{I^c} d\lambda(t) > 0$ and $g : [a,b] \to J$ be a decreasing continuous function. Let (X, Σ, μ) be a (positive) measure space with $\mu(X) > 0$, $f : X \times [a,b] \to J$ be a measurable function such that $t \mapsto f(s,t)$ is decreasing and continuous for each $s \in X$ and $\int_a^x f(s,t) d\lambda(t) \leqslant \int_a^x g(t) d\lambda(t)$ for every $x \in [a,b]$, with equality for $x = b$. If $\phi : J \to \mathbb{R}$ is a continuous convex function, then*

$$\begin{aligned}
&\phi\left[\frac{1}{L}\left(\int_a^b g(t) d\lambda(t) - \frac{1}{\mu(X)} \int_I \int_X f(s,t) d\mu(s) d\lambda(t)\right)\right] \\
&\quad \leqslant \frac{1}{L}\left(\int_a^b \phi(g(t)) d\lambda(t) - \frac{1}{\mu(X)} \int_I \int_X \phi(f(s,t)) d\mu(s) d\lambda(t)\right).
\end{aligned}$$

Proof. Obviously

$$\begin{aligned}
&\int_a^b g(t)d\lambda(t) - \frac{1}{\mu(X)}\int_{a_1}^{b_1}\int_X f(s,t)d\mu(s)d\lambda(t) \\
&= \frac{1}{\mu(X)}\int_X \left(\int_a^b g(t)d\lambda(t) - \int_{a_1}^{b_1} f(s,t)d\lambda(t)\right)d\mu(s) \\
&= \frac{1}{\mu(X)}\int_X\int_{I^c} f(s,t)d\lambda(t)d\mu(s).
\end{aligned}$$

Jensen's inequality gives:

$$\begin{aligned}
&\phi\left[\frac{1}{L}\left(\int_a^b g(t)d\lambda(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)d\lambda(t)\right)\right] \\
&\leqslant \frac{1}{\mu(X)}\int_X \phi\left(\frac{1}{L}\int_{I^c} f(s,t)d\lambda(t)\right)d\mu(s).
\end{aligned}$$

Finally

$$\begin{aligned}
&\phi\left(\frac{1}{L}\int_{I^c} f(s,t)d\lambda(t)\right) = \phi\left(\frac{1}{L}\left(\int_a^{a_1} f(s,t)d\lambda(t) + \int_{b_1}^b f(s,t)d\lambda(t)\right)\right) \\
&\leqslant \frac{1}{L}\left(\int_a^{a_1} d\lambda(t)\phi\left(\frac{1}{\int_a^{a_1} d\lambda(t)}\int_a^{a_1} f(s,t)d\lambda(t)\right)\right. \\
&\left.+\int_{b_1}^b d\lambda(t)\phi\left(\frac{1}{\int_{b_1}^b d\lambda(t)}\int_{b_1}^b f(s,t)d\lambda(t)\right)\right) \\
&\leqslant \frac{1}{L}\left(\int_a^{a_1}\phi\left(f(s,t)\right)d\lambda(t) + \int_{b_1}^b \phi\left(f(s,t)\right)d\lambda(t)\right) \\
&= \frac{1}{L}\left(\int_a^b \phi\left(f(s,t)\right)d\lambda(t) - \int_{a_1}^{b_1}\phi\left(f(s,t)\right)d\lambda(t)\right) \\
&\leqslant \frac{1}{L}\left(\int_a^b \phi\left(g(t)\right)d\lambda(t) - \int_{a_1}^{b_1}\phi\left(f(s,t)\right)d\lambda(t)\right).
\end{aligned}$$

4. Multidimensional Jensen-Mercer inequality

The following theorem was proved in [10]. Its setting is still one-dimensional but the proof is a model for similar proofs for polytopes with barycentric coordinates.

THEOREM 4.1. *Let $f: I \to \mathbb{R}$ (I an interval in $\mathbb{R}$) be a function and let $(M_1,\ldots,M_m)$, $m \geqslant 1$, be an $m-$tuple of fixed means of $n \geqslant 2$ variables on I. Inequality*

$$f\left(\frac{\sum_{i=1}^n p_i x_i - \sum_{j=1}^m w_j M_j(\mathbf{x})}{P_n - W_m}\right) \leqslant \frac{\sum_{i=1}^n p_i f(x_i) - \sum_{j=1}^m w_j f(M_j(\mathbf{x}))}{P_n - W_m}$$

holds for all $\mathbf{x} \in I^n$, where $\mathbf{p} \in \mathbb{R}_+^{\mathbf{n}}$, $\mathbf{w} \in \mathbb{R}_+^m$ are such that $p_i \geqslant \sum_{j=1}^m w_j$, $i = 1,\ldots n$, if and only if f is convex on I.

Proof. The crucial step in the proof is the existence of "barycentric" coordinates $\lambda_i^j \in [0,1]$ such that $M_j(\mathbf{x}) = \sum_{i=1}^n \lambda_i^j x_i$, $j = 1, \ldots, m$, $\sum_{i=1}^n \lambda_i^j = 1$.
Mercer type inequality is not mentioned in this paper. It is easy to see that for $n = 2$, $p_1 = p_2 = 1$, $W_m = 1$, $x_1 = a$, $x_2 = b$, $M_j(a,b) = x_j \in [a,b]$, $j = 1, \ldots, m$, Theorem 4.1 implies Theorem 1.1.

A multidimensional version was given in [7] as follows.

THEOREM 4.2. *Let $A : L \to \mathbb{R}$ be a positive linear functional (L a vector space of real functions on some set E which contains constant functions). Let $x_1, \ldots, x_n \in \mathbb{R}^k$ and $K = \operatorname{conv}(\{x_1, \ldots, x_n\})$. Let $f : K \to \mathbb{R}$ be a convex function and $\lambda_1, \ldots, \lambda_n$ be barycentric coordinates on K. If $g \in L^k$ is such that $g(E) \subseteq K$ and $p_1, \ldots, p_n$ with $P_n = \sum_{i=1}^n p_i$ satisfying $p_i \geqslant A(1)$, $i = 1, \ldots, n$, then*

$$
\begin{aligned}
& f\left(\frac{\sum_{i=1}^n p_i x_i - A[g]}{P_n - A(1)}\right) \\
& \leqslant \frac{\sum_{i=1}^n p_i f(x_i) - \sum_{i=1}^n A(\lambda_i(g)) f(x_i)}{P_n - A(1)} \leqslant \frac{\sum_{i=1}^n p_i f(x_i) - A(f(g))}{P_n - A(1)},
\end{aligned}
$$

where $A[g] = (A(g_1), \ldots, A(g_k))$.

An equivalent version was proved in [5]. We give a short proof of this theorem as a model for proving similar results in $\mathbb{R}^k$. The basic tool in such a type of proofs is an existence of barycentric coordinates.

THEOREM 4.3. *Let Ω be a polytope in $\mathbb{R}^k$, $\{v_0, v_1, \ldots, v_n\}$ its vertices and $f : \Omega \to \mathbb{R}$ be a convex function. Suppose that $A : L \to \mathbb{R}$ is a positive normalized functional and $g = (g_1, \ldots, g_k) \in L^k$ such that $\operatorname{Im}(g) \subseteq \Omega$. If $0 < \beta \leqslant \alpha$, then*

$$
\begin{aligned}
& f\left(\frac{\alpha \sum_{i=0}^n v_i - \beta A[g]}{(n+1)\alpha - \beta}\right) \\
& \leqslant \frac{\alpha}{(n+1)\alpha - \beta} \sum_{i=0}^n f(v_i) - \frac{\beta}{(n+1)\alpha - \beta} A[f(g)],
\end{aligned}
$$

where $A[g] = (A(g_1), \ldots, A(g_k))$. More generally,

$$
f\left(\sum_{i=0}^n \alpha_i v_i - \beta A[g]\right) \leqslant \sum_{i=0}^n \alpha_i f(v_i) - \beta A[f(g)],
$$

provided $\sum_{i=0}^n \alpha_i = 1 + \beta$, $\alpha_i \geqslant \beta \geqslant 0$, $i = 0, \ldots, n$.

Proof. Using the Jensen and the Jessen inequality it follows:

$$\begin{aligned}
f\left(\sum_{i=0}^{n}\alpha_i v_i-\beta A[g]\right)&=f\left(\sum_{i=0}^{n}\alpha_i v_i-\beta\sum_{i=0}^{n}A\left(\lambda_i(g)\right)v_i\right)\\
&=f\left(\sum_{i=0}^{n}\left(\alpha_i-\beta A\left(\lambda_i(g)\right)\right)v_i\right)\leqslant\sum_{i=0}^{n}\left(\alpha_i-\beta A\left(\lambda_i(g)\right)\right)f\left(v_i\right)\\
&=\sum_{i=0}^{n}\alpha_i f\left(v_i\right)-\beta\sum_{i=0}^{n}A\left(\lambda_i(g)\right)f\left(v_i\right)\leqslant\sum_{i=0}^{n}\alpha_i f\left(v_i\right)-\beta A\left(f(g)\right).
\end{aligned}$$

Boundary domination in these type of results is expressed through the simple domination of weights $\alpha_i\geqslant\beta$, $i=0,\ldots,n$. Notice that from $\sum_{i=0}^{n}\alpha_i=\beta+1$ it follows that $\beta\leqslant\frac{1}{n}$.

The following corollary, using Theorem 2.5, gives a construction of Steffensen-Popoviciu measures on simplices.

COROLLARY 4.4. *Let $(X,\mathscr{A},\mu)$ be a probability measure space and let $g:X\to\Omega$ be a measurable function. Let $0<\beta\leqslant\alpha_i$, $i=0,1,\ldots,n$ and $f:\Omega\to\mathbb{R}$ be a continuous convex function. Then*

$$\begin{aligned}
&f\left(\frac{\sum_{i=0}^{n}\alpha_i v_i-\beta\int_X g d\mu}{\sum_{i=0}^{n}\alpha_i-\beta}\right)\\
&\quad\leqslant\frac{1}{\sum_{i=0}^{n}\alpha_i-\beta}\sum_{i=0}^{n}\alpha_i f\left(v_i\right)-\frac{\beta}{\sum_{i=0}^{n}\alpha_i-\beta}\int_X f(g)d\mu.
\end{aligned}$$

COROLLARY 4.5. *Let $f:[a,b]\to\mathbb{R}$ be a convex function, $x_i\in[a,b]$, $w_i\geqslant 0$, $i=1,\ldots,n$, such that $\sum_{i=1}^{n}w_i=1$. If $0<\beta\leqslant\alpha_i$, $i=1,2$, then*

$$\begin{aligned}
&f\left(\frac{\alpha_1 a+\alpha_2 b-\beta\sum_{i=1}^{n}w_i x_i}{\alpha_1+\alpha_2-\beta}\right)\\
&\quad\leqslant\frac{\alpha_1 f(a)+\alpha_2 f(b)}{\alpha_1+\alpha_2-\beta}-\frac{\beta}{\alpha_1+\alpha_2-\beta}\sum_{i=1}^{n}w_i f\left(x_i\right).
\end{aligned}\tag{4.1}$$

The following lemma is given in [17]. Usually it is used as a tool in improving Jensen's type inequalities with weights.

LEMMA 4.6. *Let $\phi:U\to\mathbb{R}$ (U convex set in a vector space) be a convex function, $x_1,\ldots,x_n\in U$, $p_1,\ldots,p_n$ be non-negative with $P_n>0$. Then*

$$\begin{aligned}
&\min\left\{p_1,\ldots,p_n\right\}\left[\sum_{i=1}^{n}\phi\left(x_i\right)-n\phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right]\\
&\quad\leqslant\sum_{i=1}^{n}p_i\phi\left(x_i\right)-P_n\phi\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right).
\end{aligned}$$

Set

$$S_{\phi}\left(x_1,\ldots,x_n\right)=\sum_{i=1}^{n}\phi\left(x_i\right)-n\phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right). \tag{4.2}$$

Typical application of Lemma 4.6 can be seen it the following theorem proved in [15].

THEOREM 4.7. *Let L be a vector space of real functions on some non-empty set E containing constants, A be a positive linear functional on L. Let $x_1,\ldots,x_n\in\mathbb{R}^k$ and $K=\operatorname{co}\{x_1,\ldots,x_n\}$. Let ϕ be a convex function on K and $\lambda_1,\ldots,\lambda_n$ be barycentric coordinates over K. Then for all $g\in L^k$ such that $g(E)\subseteq K$ and $\phi(g)$, $\lambda_i(g)\in L$, $i=1,\ldots,n$ and positive real numbers $p_1,\ldots,p_n$ satisfying $p_i\geqslant A(1)$, $i=1,\ldots,n$,*

$$\begin{aligned}
&\phi\left(\frac{\sum_{i=1}^{n}p_ix_i-A[g]}{P_n-A(1)}\right)\\
&\leqslant\frac{1}{P_n-A(1)}\left[\sum_{i=1}^{n}p_i\phi\left(x_i\right)-\sum_{i=1}^{n}A\left(\lambda_i(g)\right)\phi\left(x_i\right)\right.\\
&\left.-\min_i\left\{p_i-A\left(\lambda_i(g)\right)\right\}S_{\phi}\left(x_1,\ldots,x_n\right)\right]\\
&\leqslant\frac{1}{P_n-A(1)}\left[\sum_{i=1}^{n}p_i\phi\left(x_i\right)-A\left(\phi(\mathbf{g})\right)\right.\\
&\left.-\left[\min_i\left\{p_i-A\left(\lambda_i(g)\right)\right\}+A\left(\min_i\left\{\lambda_i(g)\right\}\right)\right]S_{\phi}\left(x_1,\ldots,x_n\right)\right],
\end{aligned} \tag{4.3}$$

where $A[g]=\left(A\left(g_1\right),\ldots,A\left(g_k\right)\right)$.

5. General functional Jensen-Mercer inequality with applications to simplices

All results in the previous section can be deduced from the following theorem. This theorem and similar are simple consequences of the classical Jensen inequality (in its various forms) and its improvements and the boundary domination in this case is expressed through the crude domination of weights. The problem is to find more subtle conditions to express boundary domination in the multidimensional case.

THEOREM 5.1. *Let L_1,L_2 be vector spaces of real functions which contain constant functions over sets E_1,E_2 respectively. Let A,B be positive linear functionals on L_1,L_2 respectively. Let $x_1,\ldots,x_n\in\mathbb{R}^k$ and $K=\operatorname{co}\left(\{x_1,\ldots,x_n\}\right)$. Let ϕ be a convex function on K and $\lambda_1,\ldots,\lambda_n$ barycentric coordinates on K (generated by $x_1,\ldots,x_n$). If $g\in L_1^k$, $h\in L_2^k$ such that $g\left(E_1\right)\subseteq K$, $\phi(g)$, $\lambda_i(g)\in L_1$, $h\left(E_2\right)\subseteq K$,*

$\phi(h),\ \lambda_i(h) \in L_2,\ i = 1,\dots,n$, then

$$\begin{aligned}
&\phi\left(\frac{B[h]-A[g]}{B(1)-A(1)}\right) \\
&\leqslant \frac{1}{B(1)-A(1)}\left[\sum_{i=1}^{n} B\left(\lambda_i(h)\right)\phi\left(x_i\right) - \sum_{i=1}^{n} A\left(\lambda_i(g)\right)\phi\left(x_i\right)\right. \\
&\left. -\min_i\left\{B\left(\lambda_i(h) - A\left(\lambda_i(g)\right)\right\} S_\phi\left(x_1,\dots,x_n\right)\right] \\
&\leqslant \frac{1}{B(1)-A(1)}\left[\sum_{i=1}^{n} B\left(\lambda_i(h)\right)\phi\left(x_i\right) - A\left(\phi(g)\right)\right. \\
&\left. -\left[\min_i\left\{B\left(\lambda_i(\mathrm{h})\right) - A\left(\lambda_i(g)\right)\right\} + A\left(\min_i\left\{\lambda_i(g)\right\}\right)\right] S_\phi\left(x_1,\dots,x_n\right)\right],
\end{aligned} \tag{5.1}$$

providing $B(1) > A(1)$ and $B\left(\lambda_i(h)\right) \geqslant A\left(\lambda_i(g)\right),\ i = 1,\dots,n$.

Proof. Since $A[g] = \sum_{i=1}^{n} A\left(\lambda_i(g)\right)x_i$ and $B[h] = \sum_{i=1}^{n} B\left(\lambda_i(h)\right)x_i$, we have using Lemma 4.6 twice

$$\begin{aligned}
&\phi\left(\frac{B[h]-A[g]}{B(1)-A(1)}\right) \leqslant \frac{1}{B(1)-A(1)}\sum_{i=1}^{n}\left[B\left(\lambda_i(h)\right) - A\left(\lambda_i(g)\right)\right]\phi\left(x_i\right) \\
&-\min_i\left\{\frac{B\left(\lambda_i(h)\right) - A\left(\lambda_i(g)\right)}{B(1)-A(1)}\right\} S_\phi\left(x_1,\dots,x_n\right) \\
&\leqslant \frac{1}{B(1)-A(1)}\left[\sum_{i=1}^{n} B\left(\lambda_i(h)\right)\phi\left(x_i\right) - A\left(\phi(g)\right)\right. \\
&\left. -\left(\min_i\left\{B\left(\lambda_i(h)\right) - A\left(\lambda_i(g)\right)\right\} + A\left(\min_i\left\{\lambda_i(g)\right\}\right)\right) S_\phi\left(x_1,\dots,x_n\right)\right].
\end{aligned}$$

Many applications of the previous theorem can be given. We give a case of simplices. Let K in Theorem 5.1 be a simplex with vertices $x_1,\dots,x_{k+1}$, $E_2 = [v_1,\dots,v_{k+1}]$ be a simplex with vertices $v_1,\dots,v_{k+1}$ and $E_1 = [w_1,\dots,w_{k+1}]$ be a simplex with vertices $w_1,\dots,w_{k+1}$ such that $E_1 \subset E_2 \subseteq K \subset \mathbb{R}^k$. Let $g : E_1 \to K$ and $h : E_2 \to K$ be inclusion functions. For arbitrary $g : E_1 \to \mathbb{R}$ set $A(g) = \int_{E_1} g(x)dx$ and for arbitrary $h : E_2 \to \mathbb{R}$ set $B(h) = \int_{E_2} h(x)dx$. Obviously $A(1) = \mathrm{Vol}\left(E_1\right)$ and $B(1) = \mathrm{Vol}\left(E_2\right)$. Also,

$$A[g] = \int_{E_1} xdx = \mathrm{Vol}\left(E_1\right)E_1^*,\ B[h] = \int_{E_2} xdx = \mathrm{Vol}\left(E_2\right)E_2^*,$$

where E_1^* and E_2^* are barycentres of E_1 and E_2 respectively.

It is easy to see that

$$B\left(\lambda_i(h)\right) = \int_{E_2} \lambda_i(x)dx = \mathrm{Vol}\left(E_2\right)\lambda_i\left(E_2^*\right),$$

and

$$A\left(\lambda_i(g)\right) = \int_{E_1} \lambda_i(x)dx = \operatorname{Vol}\left(E_1\right)\lambda_i\left(E_1^*\right),\ i = 1,\ldots,k+1.$$

It is obvious that $B(1) - A(1) = \operatorname{Vol}\left(E_2\right) - \operatorname{Vol}\left(E_1\right) > 0$. The second condition $B\left(\lambda_i(h)\right) \geqslant A\left(\lambda_i(g)\right),\ i = 1,\ldots,k+1$, is obviously equivalent to

$$\operatorname{Vol}\left(E_2\right)\lambda_i\left(E_2^*\right) \geqslant \operatorname{Vol}\left(E_1\right)\lambda_i\left(E_1^*\right),\ i = 1,\ldots,k+1,$$

which by $E_1^* = \frac{1}{k+1}\sum_{i=1}^{k+1} w_i,\ E_2^* = \frac{1}{k+1}\sum_{i=1}^{k+1} v_i$ is equivalent to

$$\begin{aligned}\operatorname{Vol}\left(E_2\right)&\sum_{j=1}^{k+1}\operatorname{Vol}\left(\left[x_1,\ldots,x_{i-1},v_j,x_{i+1},\ldots,x_{k+1}\right]\right)\\ &\geqslant \operatorname{Vol}\left(E_1\right)\sum_{j=1}^{k+1}\operatorname{Vol}\left(\left[x_1,\ldots,x_{i-1},w_j,x_{i+1},\ldots,x_{k+1}\right]\right)\end{aligned}$$

for $i = 1,\ldots,k+1$. If $E_2 = K$, then obviously

$$\operatorname{Vol}^2\left(E_2\right) \geqslant \operatorname{Vol}\left(E_1\right)\sum_{j=1}^{k+1}\operatorname{Vol}\left(\left[v_1,\ldots,v_{i-1},w_j,v_{i+1},\ldots,v_{k+1}\right]\right)$$

for $i = 1,\ldots,k+1$. We extract this interesting inequality (although a simple consequence of $\int_{E_1}\lambda_i(x)dx \leqslant \int_{E_2}\lambda_i(x)dx,\ E_1 \subseteq E_2,\ i = 1,\ldots,k+1$).

THEOREM 5.2. *Let $E_2 = [v_1,\ldots,v_{k+1}],\ E_1 = [w_1,\ldots,w_{k+1}]$ be simplices in $\mathbb{R}^k$ such that $E_1 \subseteq E_2$. Then*

$$\operatorname{Vol}^2\left(E_2\right) \geqslant \operatorname{Vol}\left(E_1\right)\sum_{j=1}^{k+1}\operatorname{Vol}\left[v_1,\ldots,v_{i-1},w_j,v_{i+1},\ldots,v_{k+1}\right], \tag{5.2}$$

for every $i = 1,\ldots,k+1$ (with agreement $v_0 = v_{k+2} = \emptyset$). Inequality (5.2) is sharp and equality is obtained for $E_1 = E_2$.

In the rest of the text we denote $\operatorname{Vol}(E)$ by $|E|$ for a measurable $E \subset \mathbb{R}^k$.

It is easy to see that inequality (5.1) in the case of simplices gives

$$\begin{aligned}&\phi\left(\frac{|E_2|\,E_2^* - |E_1|\,E_1^*}{|E_2| - |E_1|}\right) = \phi\left(\left(E_2\setminus E_1\right)^*\right)\\ &\leqslant \frac{1}{|E_2| - |E_1|}\left[\sum_{i=1}^{k+1}\int_{E_2}\lambda_i(x)dx\phi\left(x_i\right) - \int_{E_1}\phi(x)dx\right.\\ &\left.-\left(\min_i\left\{\int_{E_2}\lambda_i(x)dx - \int_{E_1}\lambda_i(x)dx\right\} + \int_{E_1}\min_i\lambda_i(x)dx\right)(k+1)\left(\overline{\phi}_V - \phi\left(K^*\right)\right)\right],\end{aligned}$$

where $\overline{\phi}_V = \frac{1}{k+1}\sum_{i=1}^{k+1}\phi(x_i)$, which in the case $E_1 \subset E_2 = K$ reduces to inequality

$$\begin{aligned}\phi\left((E_2\setminus E_1)^*\right) \leqslant \frac{1}{|E_2|-|E_1|}\Big[&|E_2|\overline{\phi}_V - |E_1|\overline{\phi}_{E_1}\\ &-\left(\frac{|E_2|}{k+1} - |E_1|\max_i \lambda_i(E_1^*) + |E_1|\overline{\min_i\lambda_{iE_1}}\right)(k+1)\left(\overline{\phi}_V - \phi(E_2^*)\right)\Big]\end{aligned} \tag{5.3}$$

Note that the first part of the right-hand side of inequality (5.3)

$$\phi\left(\frac{|E_2|E_2^* - |E_1|E_1^*}{|E_2|-|E_1|}\right) \leqslant \frac{1}{|E_2|-|E_1|}\left(|E_2|\bar{\phi}_V - |E_1|\bar{\phi}_{E_1}\right) \tag{5.4}$$

is obtained without applying Lemma 4.6. It is easy to see (using discrete Jensen's inequality) that

$$\frac{|E_2|\phi(E_2^*) - |E_1|\phi(E_1^*)}{|E_2|-|E_1|} \leqslant \phi\left(\frac{|E_2|E_2^* - |E_1|E_1^*}{|E_2|-|E_1|}\right).$$

Decomposing $E_2 \setminus E_1 = \cup_{i=1}^{l} S_i$ where S_i are convex sets with mutually disjoint interiors, we have

$$\begin{aligned}\phi\left((E_2\setminus E_1)^*\right) &= \phi\left(\frac{1}{|E_2|-|E_1|}\int_{E_2\setminus E_1} x dx\right)\\ &= \phi\left(\frac{1}{\sum_{i=1}^{l}|S_i|}\sum_{i=1}^{l}\int_{S_i} x dx\right) \leqslant \frac{1}{\sum_{i=1}^{l}|S_i|}\sum_{i=1}^{l}|S_i|\,\phi\left(\frac{1}{|S_i|}\int_{S_i} x dx\right)\\ &\leqslant \frac{1}{\sum_{i=1}^{l}|S_i|}\sum_{i=1}^{l}\int_{S_i}\phi(x)dx = \frac{1}{|E_2|-|E_1|}\int_{E_2\setminus E_1}\phi(x)dx.\end{aligned} \tag{5.5}$$

It is easy to see (using the right-hand side of Hermite-Hadamard inequality for simplices) that estimation given in (5.5) is better than estimation obtained in (5.4).

In this way the following sequence of inequalities holds for simplices $E_1 \subset E_2$ and convex $\phi : E_2 \to \mathbb{R}$:

$$\begin{aligned}\frac{|E_2|\phi(E_2^*) - |E_1|\phi(E_1^*)}{|E_2|-|E_1|} &\leqslant \phi\left(\frac{|E_2|E_2^* - |E_1|E_1^*}{|E_2|-|E_1|}\right) = \phi\left((E_2\setminus E_1)^*\right)\\ &\leqslant \frac{1}{|E_2|-|E_1|}\int_{E_2\setminus E_1}\phi(x)dx \leqslant \frac{1}{|E_2|-|E_1|}\left(|E_2|\overline{\phi}_V - |E_1|\overline{\phi}_{E_1}\right),\end{aligned}$$

where $\overline{\phi}_V = \frac{1}{k+1}\sum_{i=1}^{k+1}\phi(v_i)$ (v_i vertices of E_2) and $\overline{\phi}_{E_1} = \frac{1}{|E_1|}\int_{E_1}\phi(x)dx$.

Rearranging (5.3) it follows

$$\begin{aligned}\overline{\phi}_{E_1} \leqslant &\left(\frac{|E_2|}{|E_1|} - (k+1)\left(\max_i\lambda_i(E_1^*) - \overline{\min_i\lambda_{iE_1}}\right)\right)\phi(E_2^*)\\ &-\left(\frac{|E_2|}{|E_1|} - 1\right)\phi\left((E_2\setminus E_1)^*\right) + (k+1)\left(\max_i\lambda_i(E_1^*) - \overline{\min_i\lambda_{iE_1}}\right).\end{aligned} \tag{5.6}$$

In the case $E_1^* = E_2^*$, which easily implies $(E_2 \setminus E_1)^* = E_2^*$ and $\lambda_i(E_1^*) = \frac{1}{k+1}$, $i = 1, \ldots, k+1$, estimation (5.6) implies

$$\overline{\phi}_{E_1} \leqslant (k+1)\overline{\min_i \lambda_{iE_1}}\,\phi(E_2^*) + \left(1 - (k+1)\overline{\min_i \lambda_{iE_1}}\right)\overline{\phi}_V. \tag{5.7}$$

Notice that inequality (5.7) holds also for $E_1 = E_2$. In this case, using $\overline{\min_i \lambda_{iE_2}} = \frac{1}{(k+1)^2}$ (see [23]), it follows

$$\overline{\phi}_{E_2} \leqslant \frac{1}{k+1}\phi(E_2^*) + \frac{k}{k+1}\overline{\phi}_V,$$

which is Bullen-Hammer inequality proven by many authors (see for example [26], [18],[23]). In this sense inequalities (5.6), (5.7) are generalizations of the Bullen-Hammer inequality.

In some of the above estimations we presented several comparisons between different mean values of convex functions over simplices. Related to this and also giving another way of expressing boundary domination of a convex function (concentration convexity at endpoints in the terminology of C. Niculescu in [20]) the following conjecture it seems to hold.

CONJECTURE 5.3. *Suppose that $E_1 \subseteq E_2 \subset \mathbb{R}^k$ are simplices with common barycentre and let $\phi : E_2 \to \mathbb{R}$ be a convex function. Then*

$$\frac{1}{|E_1|}\int_{E_1} \phi(x)dx \leqslant \frac{1}{|E_2|}\int_{E_2} \phi(x)dx. \tag{5.8}$$

In the one-dimensional case this conjecture is certainly true (see [20] for signed measure and [22]). For convex polytopes the conjecture doesn't hold (see also [22]). Notice that, since

$$\frac{1}{|E_1|}\int_{E_1} \phi(x)dx = \frac{1}{|E_2|}\left[\int_{E_1} \phi(x)dx + \frac{|E_2|-|E_1|}{|E_1|}\int_{E_1} \phi(x)dx\right],$$

inequality

$$\frac{1}{|E_1|}\int_{E_1} \phi(x)dx \leqslant \frac{1}{|E_2|-|E_1|}\int_{E_2 \setminus E_1} \phi(x)dx$$

implies (5.8). In an analogous way inequality (5.8) implies

$$\frac{1}{|E_2|}\int_{E_2} \phi(x)dx \leqslant \frac{1}{|E_2|-|E_1|}\int_{E_2 \setminus E_1} \phi(x)dx.$$

Some consequences:

1. (without refinement)

$$\phi\left(\frac{|E_2|E_2^* - |E_1|E_1^*}{|E_2|-|E_1|}\right) \leqslant \frac{1}{|E_2|-|E_1|}\left(|E_2|\,\bar{\phi}_V - |E_1|\,\bar{\phi}_{E_1}\right)$$

2. $E_1^* = E_2^* \Rightarrow$

$$\overline{\phi}_{E_1} \leqslant \left(1-(k+1)\overline{\min_i\{\lambda_i\}}_{E_1}\right)\overline{\phi}_V + (k+1)\overline{\min_i\{\lambda_i\}}_{E_1}\phi\left(E_2^*\right).$$

Conjecture: $\overline{\phi}_{E_1} \leqslant \overline{\phi}_{E_2}$?

3. $E_1 = E_2 \Rightarrow \overline{\phi}_{E_2} - \phi\left(E_2^*\right) \leqslant k\left(\bar{\phi}_V - \overline{\phi}_{E_2}\right).$

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