Journal of
Mathematical
Inequalities
Volume 9, Number 4 (2015), 1001–1054 doi:10.7153/jmi-09-81

ASYMPTOTIC EXPANSIONS OF GAMMA AND RELATED FUNCTIONS, BINOMIAL COEFFICIENTS, INEQUALITIES AND MEANS

NEVEN ELEZOVIĆ

(*Communicated by I. Perić*)

Abstract. We give an overview of the use of asymptotic expansions of gamma and related functions — ratio of gamma functions, powers, digamma and polygamma functions. The aim is to introduce a general theory which can unify various particular formulas for factorial functions and binomial coefficients. The connection with inequalities for gamma function is established. Also, a systematic approach to asymptotic expansion of various integral means, bivariate classical and parameter means is given, with applications to comparison of means.

CONTENTS

1. Introduction 1001
2. Stirling and Laplace expansions 1003
3. Transformations of asymptotic expansions. Useful lemmas . . . 1009
4. Ratio of gamma functions. Wallis function 1011
5. Appell polynomials and asymptotic expansions 1016
6. Shifts in asymptotic expansions 1020
7. Applications: binomial coefficients 1021
8. Wallis functions and inequalities 1032
9. Integral means and the second Gautschi-Kershaw's inequality . 1034
10. Asymptotic inequalities and integral means 1037
11. Asymptotic expansions of bivariate means 1042
12. Comparison of means 1051

1. Introduction

Gamma function is uniquely determined by its property of logarithmic convexity, $\Gamma(1)=1$ and recurrence formula

$$\Gamma(x+1)=x\Gamma(x).$$

It is natural to ask what is the half-step ratio of gamma functions, especially for large values of x? One should expect

$$\varphi_1(x) := \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x}, \quad \text{and} \quad \varphi_2(x) := \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x)} \sim \sqrt{x}.$$

Mathematics subject classification (2010): 41A60, 33B15, 26D15, 26E60, 11B68.
Keywords and phrases: Asymptotic expansion, gamma function, digamma function, factorials, Stirling formula, binomial coefficients, Catalan numbers, means, integral means.

During this overview, we shall reveal the exact behaviour of these two functions.

This ratio appears in another form for the first time in the work of John Wallis. He was the first who wrote an infinite product, searching for the value of π:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots. \tag{1.1}$$

After truncation we get an approximation which can be written as

$$\frac{2^n n!}{(2n-1)!!} \approx \sqrt{\pi\Big(n+\frac{1}{2}\Big)}.$$

Since

$$\Gamma(n+1) = n!, \qquad \Gamma(n+\tfrac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi},$$

this can be written as

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \approx \sqrt{n+\frac{1}{2}}. \tag{1.2}$$

On the other hand, from probability theory it is well known that

$$\binom{2n}{n} \approx \frac{2^{2n}}{\sqrt{\pi n}}$$

and this leads to the approximation

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \approx \sqrt{n}. \tag{1.3}$$

This follows also by (keeping an odd number of terms in (1.1)). Which one of (1.2) or (1.3) is better? We shall see that the truth lies somewhere in the middle of these two approximations. Let us note the early results in this direction.

D. K. Kazarinoff (1956) [38]:

$$\sqrt{n+\frac{1}{4}} < \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < \sqrt{n+\frac{1}{2}}. \tag{1.4}$$

G. N. Watson (1959) [55]:

$$\sqrt{n+\frac{1}{4}} < \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < \sqrt{n+\frac{1}{\pi}}. \tag{1.5}$$

W. Gautschi (1959) [34] gives the first result which includes continuous parameter into this fraction. Let $0 < s < 1$:

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < (n+1)^{1-s}, \tag{1.6}$$

but the value $s = 1/2$ gives less precise bound than before.

D. Kershaw (1983) [39] improves Gautschi's bounds to the form

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s}. \tag{1.7}$$

The left bound in this inequality is the best one, it is *asymptotic* and represent the limit of the ratio when $x \to \infty$. The right bound is not the best one, since the best bound is simply the value of the ratio at the beginning of the interval. The best bounds are given by N. Elezović, C. Giordano and J. Pečarić [25]. For each $s,t > 0$ and $x > x_0 > -\min\{s,t\}$, inequality

$$\frac{s+t-1}{2} < \left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} - x < \left[\frac{\Gamma(x_0+t)}{\Gamma(x_0+s)}\right]^{\frac{1}{t-s}} - x_0 \tag{1.8}$$

holds for $|t-s| < 1$, and with opposite sign for $|t-s| > 1$.

In this paper we shall discuss in details asymptotic expansions of gamma and related functions, which leads to such estimations. The subject is old, but the technique used here is new. This enable us to generalize and improve many known particular expansions for factorial and related functions, including various form of binomial coefficients. Our intention is to replace particular expansions with those which depend on parameters. Therefore, instead numerical coefficients in an asymptotic expansion, we obtain coefficients which are polynomials in one or two variables. This enable us to give precise information of the best form of approximations. Also, using such approach various particular results can be unified.

We shall show the close connection between asymptotic expansions, inequalities and various means, especially connection with integral means. The same technique can be used in more general asymptotic expansion through polygamma functions and also in expansion of bivariate and multivariate means. Connection between special functions and means are interesting in this context. Moreover, asymptotic expansions represent an efficient tool for comparison of various means.

In this paper an overview of results obtained by the author, T. Burić and L. Vukšić in the last few years will be presented. We believe that our approach will leads to new important result in various fields where asymptotic expansions appear.

This paper is based on talks of the author on conferences MINFAA, July 2012, in Jinju, S. Korea and MIA, June 2014, in Trogir, Croatia.

2. Stirling and Laplace expansions

2.1. Stirling approximation

One of the most beautiful formulas of entire mathematics is surely the following one

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

This is the shortening of the following two expansions which are known for centuries, *Stirling's expansion*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + \dots\right), \tag{2.1}$$

and *Laplace's expansion*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \dots\right). \tag{2.2}$$

However, it seems that connection between these two expressions is not widely known, as well as various derivations of these initial expansions which can be found in literature. Let us mention the most known one.

Karatsuba-Ramanujan [39, 51],

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sqrt[6]{1 + \frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{240n^3} - \frac{11}{1920n^4}}, \tag{2.3}$$

formula of N. Batir, [3]

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sqrt[4]{1 + \frac{1}{3n} + \frac{1}{18n^2} - \frac{2}{405n^3} - \frac{31}{9720n^4}}, \tag{2.4}$$

and Wehmeier [43]

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sqrt{1 + \frac{1}{6n} + \frac{1}{72n^2} - \frac{31}{6480n^3} - \frac{139}{155520n^4}}. \tag{2.5}$$

The coefficients of these expansions are calculated on a particularly basis, without clear clue how to obtain general form. Of course, the main task of these papers was not the development of asymptotic series, but instead in proving the truncation inequalities; two consecutive terms of an asymptotic expansion usually gives lower and upper bound for the function in-between.

We shall show that complete expansions are easily computable using simple recurrence procedure.

Our first intention is to clarify these connections and to present efficient algorithm for calculation of coefficients of these and similar expansions.

2.2. Bell polynomials and Stirling formula

Bell polynomials and numbers are essential in advanced combinatorics. Besides definition, many properties can be found in [20]. We shall use here *ordinary* and not *exponential form* of these polynomials, because it is better suited to our problems. Ordinary partial Bell polynomials $(\widehat{B}_{n,k})$ are defined by double expansion

$$\exp\left(u \sum_{m\geqslant 1} x_m t^m\right) =: 1 + \sum_{n\geqslant 1} t^n \left(\sum_{k=1}^{n} \widehat{B}_{n,k}(x_1, x_2, \dots)\frac{u^k}{k!}\right). \tag{2.6}$$

It is easy to see that $\widehat{B}_{n,k}$ depends on variables $x_1, x_2, \ldots, x_{n-k+1}$. Generating function from (2.6) can be written in either form

$$\prod_{m=1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{u^j x_m^j t^{mj}}{j!}\right) \quad \text{or} \quad \sum_{k=0}^{\infty}\frac{u^k}{k!}\left(\sum_{m=1}^{\infty}x_m t^m\right)^k.$$

Hence, partial Bell polynomials have explicit form

$$\widehat{B}_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) = \sum \binom{k}{k_1,k_2,\cdots,k_r} x_1^{k_1}x_2^{k_2}\cdots x_r^{k_r} \tag{2.7}$$

and the sum is taken over all $k_1, k_2, \ldots, k_r$ which satisfy

$$\begin{gathered} k_1,k_2,\ldots,k_r \geqslant 0, \\ k_1+k_2+\ldots+k_r = k, \\ k_1+2k_2+\ldots+rk_r = n. \end{gathered}$$

The ordinary complete Bell polynomials $\widehat{Y}_n = \widehat{Y}_n(x_1,x_2,\ldots,x_n)$ are defined by

$$\exp\left(\sum_{m\geqslant 1}x_m t^m\right) =: 1+\sum_{n\geqslant 1}\widehat{Y}_n(x_1,x_2,\ldots)t^n, \tag{2.8}$$

i.e.

$$\widehat{Y}_n = \sum_{k=1}^{n}\frac{1}{k!}\widehat{B}_{n,k}, \qquad Y_0 = 1. \tag{2.9}$$

The following expansion of gamma function is known since Barnes (1899):

$$\log\Gamma(x+t) \sim (x+t-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log(2\pi) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}x^{-n}. \tag{2.10}$$

Here, $B_n(t)$ are Bernoulli polynomials defined by generating function

$$\frac{e^{xt}}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n(t)}{n!}x^n.$$

Let us denote

$$b_n(t) := \frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}. \tag{2.11}$$

Then we can apply definition of Bell polynomials to the following part of this expansion

$$F(x,t) := \exp\left(\sum_{n=1}^{\infty}b_n(t)x^{-n}\right).$$

Therefore, it follows

THEOREM 2.1. *Laplace expansion has the following explicit form*

$$\log\Gamma(x+t)\sim(x+t-\tfrac{1}{2})\log x-x+\tfrac{1}{2}\log(2\pi)+\log\Bigl(\sum_{n=1}^{\infty}\widehat{Y}_n x^{-n}\Bigr), \tag{2.12}$$

where

$$\widehat{Y}_n=\sum_{k=1}^{n}\frac{1}{k!}\widehat{B}_{n,k}(b_i(t))$$

and $(b_i(t))$ *are given in* (2.11).

Taking $t=1$ one obtain from (2.10) and (2.12) Laplace and Stirling expansion. However, the calculation of such coefficients is not easy, because of Bell polynomials' cumbersome explicit formula (2.7) and (2.9). Instead, one can use recursive procedure. We need only complete polynomials, so, the following lemma will suffice:

LEMMA 2.2. *Bell polynomials satisfy the following recursive relation,* $\widehat{Y}_0=1$ *and*

$$\widehat{Y}_n=\frac{1}{n}\sum_{k=1}^{n}kx_k\widehat{Y}_{n-k}. \tag{2.13}$$

This should be known result, but it is easier to prove it than provide the reference. The more general result is discussed in [17]. This and other necessary lemmas will be repeated in the next section.

The following result is the main formula in subsequent applications, so we will repeat original proof.

THEOREM 2.3. (Burić-Elezović (2011), [7]) *We have*

$$\log\Gamma(x+t)\sim(x+t-\tfrac{1}{2})\log x-x+\tfrac{1}{2}\log(2\pi)+\frac{1}{m}\log\left(\sum_{n=0}^{\infty}P_n(t)x^{-n}\right),$$

where $P_n(t)$ *are polynomials defined by* $P_0(t)=1$ *and*

$$P_n(t)=\frac{m}{n}\sum_{k=1}^{n}\frac{(-1)^{k+1}B_{k+1}(t)}{k+1}P_{n-k}(t),\qquad n\geqslant 1.$$

Proof. Differentiating (2.3) we get

$$\psi(x+t)\sim\log x+\frac{1}{x}(t-\tfrac{1}{2})+\frac{1}{m}\left(\sum_{n=0}^{\infty}P_nx^{-n}\right)^{-1}\left(\sum_{n=1}^{\infty}(-n)P_nx^{-n-1}\right).$$

In fact, this should be written in a way

$$m\left(\sum_{n=0}^{\infty}P_nx^{-n}\right)\left[\psi(x+t)-\log x-\frac{1}{x}(t-\tfrac{1}{2})\right]\sim\sum_{n=1}^{\infty}(-n)P_nx^{-n-1}.$$

Since (see [42, p. 33])

$$\psi(x+t) \sim \log x - \sum_{n=0}^{\infty} \frac{(-1)^{n+1} B_{n+1}(t)}{n+1} x^{-n-1}, \tag{2.14}$$

we have

$$\psi(x+t) - \log x - (t - \tfrac{1}{2})\frac{1}{x} \sim \sum_{n=1}^{\infty} \frac{(-1)^n B_{n+1}(t)}{n+1} x^{-n-1}.$$

Therefore

$$m\left(\sum_{n=0}^{\infty} P_n x^{-n}\right)\left(\sum_{k=1}^{\infty} \frac{(-1)^k B_{k+1}(t)}{k+1} x^{-k-1}\right) \sim -\sum_{n=1}^{\infty} n P_n x^{-n-1},$$

wherefrom it follows:

$$-nP_n = m\sum_{k=1}^{n} \frac{(-1)^k B_{k+1}(t)}{k+1} P_{n-k}.$$

Statement of the theorem follows. □

COROLLARY 2.4. *It holds*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[\sum_{k=0}^{\infty} P_k n^{-k}\right]^{1/m}, \tag{2.15}$$

where (P_n) *is defined by*

$$\begin{aligned} P_0 &= 1, \\ P_n &= \frac{m}{n} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{B_{2k}}{2k} P_{n-2k+1}, \qquad n \geqslant 1. \end{aligned} \tag{2.16}$$

It is interesting to see the form of the first few coefficients in this general formula:

$$\begin{aligned} n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \Bigg[& 1 + \frac{m}{2^2 \cdot 3} \cdot \frac{1}{n} + \frac{m^2}{2^5 \cdot 3^2} \cdot \frac{1}{n^2} \\ & + \left(\frac{m^3}{2^7 \cdot 3^4} - \frac{m}{2^3 \cdot 3^2 \cdot 5}\right)\frac{1}{n^3} + \left(\frac{m^4}{2^{11} \cdot 3^5} - \frac{m^2}{2^5 \cdot 3^3 \cdot 5}\right)\frac{1}{n^4} + \dots \Bigg]^{1/m}. \end{aligned} \tag{2.17}$$

From here one can see which choice of exponent m will give a good looking expansions, besides those given by Karatsuba-Ramanujan, Batir or Wehmeier, the following ones will give even better first few coefficients:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sqrt[12]{1 + \frac{1}{n} + \frac{1}{2n^2} + \frac{2}{15n^3} + \frac{1}{120n^4} + \frac{1}{840n^5} + \dots}. \tag{2.18}$$

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sqrt[24]{1+\frac{2}{n}+\frac{2}{n^2}+\frac{19}{15n^3}+\frac{8}{15n^4}+\frac{16}{105n^5}+\ldots}. \tag{2.19}$$

In the formula (2.3), (2.4), (2.17), (2.18) and (2.19), one can see that the better results are obtained for bigger values of the parameter m. Therefore, it is natural to assume that the approximation with m substituted by variable x will be better for big values of x. This is indeed the case.

THEOREM 2.5. (Burić-Elezović (2012), [9]) *The following asymptotic expression is valid*

$$\log\Gamma(x+t) \sim (x+t-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log(2\pi) + \frac{1}{12x} + \frac{1}{x}\log\left(\sum_{n=0}^{\infty} P_n(t)x^{-n}\right), \tag{2.20}$$

where $P_0(t) = 1$ *and*

$$P_n(t) = \frac{1}{n}\sum_{k=1}^{n} \frac{(-1)^k k B_{k+2}(t)}{(k+1)(k+2)} P_{n-k}(t), \qquad n \geqslant 1. \tag{2.21}$$

Proof. By comparing the following two asymptotic expansions:

$$\log\Gamma(x+t) \sim \left(x+t-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}B_{k+1}(t)}{k(k+1)} x^{-k},$$

$$\log\Gamma(x+t) \sim \left(x+t-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} + \frac{1}{x}\log\left(\sum_{n=0}^{\infty} P_n(t)x^{-n}\right),$$

we can write

$$\log\left(\sum_{n=0}^{\infty} P_n(t)x^{-n}\right) \sim \sum_{k=1}^{\infty} \frac{(-1)^k B_{k+2}(t)}{(k+1)(k+2)} x^{-k}.$$

By differentiating it follows

$$\sum_{n=1}^{\infty} nP_n(t)x^{-n-1} \sim \left(\sum_{n=0}^{\infty} P_n(t)x^{-n}\right)\left(\sum_{k=1}^{\infty} \frac{(-1)^k k B_{k+2}(t)}{(k+1)(k+2)} x^{-k-1}\right).$$

Hence,

$$nP_n(t) = \sum_{k=1}^{n} \frac{(-1)^k k B_{k+2}(t)}{(k+1)(k+2)} P_{n-k}(t)$$

and we get the statement. $\square$

COROLLARY 2.6. *The following asymptotic expansion for the factorial function is valid:*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}\left[\sum_{m=0}^{\infty} P_{2m} n^{-2m}\right]^{1/n}, \tag{2.22}$$

where (P_m) *is a sequence defined by* $P_0 = 1$ *and*

$$P_{2m} = \frac{1}{2m} \sum_{k=1}^{m} \frac{k B_{2k+2}}{(k+1)(2k+1)} P_{2m-2k}. \tag{2.23}$$

Therefore, it holds

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}\left[1 - \frac{1}{360n^2} + \frac{1447}{1814400n^4} - \frac{1170727}{1959552000n^6} + \dots\right]^{1/n}. \tag{2.24}$$

3. Transformations of asymptotic expansions. Useful lemmas

The operation of differentiation of an asymptotic series should be taken by care. Our applications are covered by next two lemmas.

LEMMA 3.1. (Erdélyi [32, p. 21]) *If the function*

$$f(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots \qquad \text{as } x \to \infty \tag{3.1}$$

is differentiable and if f' *possesses an asymptotic power series expansion, then*

$$f'(x) \sim -\frac{a_1}{x^2} - \frac{2a_2}{x^3} - \frac{3a_3}{x^4} - \dots, \qquad \text{as } x \to \infty. \tag{3.2}$$

LEMMA 3.2. (Erdélyi [32, p. 21]) *Let* R_1 *be the region* $|x| > r_1$, $\gamma_1 < \arg x < \gamma_2$, *let* $r_2 > r_1$, $\gamma_1 < \gamma_1' < \gamma_2' < \gamma_2$ *and let* R_2 *be the region* $|x| > r_2$, $\gamma_1' < \arg x < \gamma_2'$. *If* $f(x)$ *is regular in* R_1 *and*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}$$

holds uniformly in $\arg x$ *as* $x \to \infty$ *in* R_1, *then*

$$f'(x) \sim \sum_{n=1}^{\infty} (-n) a_n x^{-n-1}$$

holds uniformly in $\arg x$ *as* $x \to \infty$ *in* R_2.

The following lemma gives an algorithm for transformation of the power of an asymptotic series. This lemma has origins in Euler works on power series. See [36, Theorem 1.6c] for detailed analysis on the context of formal power series. We shall rather use it in the contents of asymptotic series, see [17] for detailed discussion. This lemma will be the crucial one in the sequel.

LEMMA 3.3. *Let* $a_0 \neq 0$,

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}.$$

Then for all real r it holds

$$[g(x)]^r \sim \sum_{n=0}^{\infty} E_n[a,r]x^{-n},$$

where

$$\begin{aligned} E_0[a,r] &= a_0^r, \\ E_n[a,r] &= \frac{1}{na_0}\sum_{k=1}^{n}[k(1+r)-n]a_kE_{n-k}[a,r]. \end{aligned} \tag{3.3}$$

In this paper we will reserve symbol $E_n[a,r]$ for coefficient of the term x^{-n} in r-th power of the asymptotic expansion of $\sum_{k=0}^{\infty} a_k x^{-k}$.

More generally, one can obtain explicit form for functional transformation of an asymptotic expansion:

LEMMA 3.4. *Let*

$$f(x) = x^r \sum_{n=0}^{\infty} b_n x^n, \qquad b_0 \neq 0,$$

and

$$A(x) = x^{-s} \sum_{n=0}^{\infty} a_n x^{-n}, \qquad a_0 \neq 0,$$

where r, s are real numbers and $s > 0$. Then $f(A(x))$ has asymptotic expansion of the form

$$f(A(x)) = x^{rs} \sum_{n=0}^{\infty} \Bigg(\sum_{k+\lfloor sj \rfloor = n} x^{-\{sj\}} E_k[a, r+j] b_j \Bigg) x^{-n}$$

Here, $\lfloor x \rfloor$ is integer part and $\{x\}$ fractional part of x.

We will use the following two special cases. The first one is already discussed in the context of Bell polynomial, see [17] for the proof and applications.

LEMMA 3.5. *Let A be a function with asymptotic expansion*

$$A(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}.$$

Then the composition $B(x) = \exp(A(x))$ has asymptotic expansion of the following form

$$B(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}$$

where $b_0 = 1$ and

$$b_n = \frac{1}{n}\sum_{k=1}^{n} k a_k b_{n-k}, \qquad n \geqslant 1. \tag{3.4}$$

The alternative procedure is also mentioned in [17]:

LEMMA 3.6. *If $A(x)$ has an asymptotic expansion*

$$A(x) \sim \sum_{n=0}^{\infty} a_n x^{-n+1}, \qquad a_0 \neq 0$$

then

$$\exp(A(x)) \sim \sum_{n=0}^{\infty} b_n x^{-n},$$

where $b_0 = 1$ and

$$b_n = \sum_{k=0}^{n} E_k[a, n-k] \frac{1}{(n-k)!}.$$

LEMMA 3.7. *If $A(x)$ has an asymptotic expansion*

$$A(x) \sim \sum_{n=0}^{\infty} a_n x^{-n+1}, \qquad a_0 \neq 0$$

then

$$\log(1 + A(x)) \sim \sum_{n=1}^{\infty} b_n x^{-n},$$

where

$$b_n = \sum_{k=0}^{n-1} E_k[a, n-k] \frac{(-1)^{n-k+1}}{n-k}.$$

4. Ratio of gamma functions. Wallis function

Asymptotic expansion of the ratio of two gamma function can be obtained using the same technique. The same is true for product/ratio of few gamma functions, since logarithm transform this to sum or difference, and all recursive formula are linear. Let us start with some notations.

4.1. Wallis fraction and Wallis power function

DEFINITION 4.1. (Wallis fraction) The fraction

$$W(x,t,s) := \frac{\Gamma(x+t)}{\Gamma(x+s)} \tag{4.1}$$

is called the Wallis' function, or the Wallis' fraction.

DEFINITION 4.2. (Wallis power function) Function

$$F(x,t,s) := \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{\frac{1}{t-s}} \tag{4.2}$$

is called the Wallis power function.

4.2. Classical expansions

In the standard reference for this subject, Tricomy-Erdélyi (1951) [52] is proved:

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(t-s+1)}(t)(t-s)_n}{n!} x^{-n}. \tag{4.3}$$

Here, $B_n^{(a)}(x)$ are generalized Bernoulli polynomials defined by:

$$\frac{x^a e^{tx}}{(e^x-1)^a} = \sum_{n=0}^{\infty} B_n^{(a)}(t) \frac{x^n}{n!}. \tag{4.4}$$

Note that $B_n^{(1)}(x) = B_n(x)$.

Our opinion is that this form of expansion is misleading. It looks like the coefficients are given by closed formulas, but there is no easy way to calculate directly generalized Bernoulli polynomials, one should apply some recurrence procedure. Also, variables s and t appear in the argument, but also as the parameter of these polynomials which can (and will) causes troubles with degree of polynomials. In [42, page 22.] the following recursive formula for generalized Bernoulli polynomials is given:

$$B_{n+1}^{(a)}(x) = B_1^{(a)}(x) B_n^{(a)}(x) - a \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{\binom{n}{2k+1} B_{2k+2}}{(2k+2)} B_{n-2k-1}^{(a)}(x). \tag{4.5}$$

We prefer the following simpler form of this recursion, see [24]:

$$B_n^{(a)}(x) = B_1^{(a)}(x) B_{n-1}^{(a)}(x) - \frac{a}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} B_{2k} B_{n-2k}^{(a)}(x). \tag{4.6}$$

4.3. Asymptotic expansion of Wallis power function

It is possible to write this expansion using only ordinary Bernoulli polynomials. J. Bukac, T. Burić and N. Elezović made the first attempt in this direction in [6]. The final result is given in the next theorem, coefficients can be calculated using simple recursion.

THEOREM 4.3. (Burić-Elezović (2011), [7]) *The Wallis power function has the following expansion:*

$$F(x,t,s) = \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}} \sim \sum_{n=0}^{\infty} P_n(t,s) x^{-n+1}.$$

$P_n(t,s)$ are polynomials of degree n defined by $P_0(t,s) = 1$ and

$$P_n(t,s) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \frac{B_{k+1}(t) - B_{k+1}(s)}{(k+1)(t-s)} P_{n-k}(t,s), \qquad n \geqslant 1.$$

$B_k(t)$ are the Bernoulli polynomials.

Proof. We can use known expansion of the logarithm of gamma function (2.10) to obtain

$$\frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)}-\log x\sim\sum_{n=1}^{\infty}a_n x^{-n},$$

Applying Lemma 3.6 to the function $g(x)=\exp(x)$, we have

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}}\sim x\sum_{n=0}^{\infty}b_n x^{-n} \tag{4.7}$$

and the existence of the expansion is ensured. Therefore, representation (4.7) is valid, with $P_0(t,s)=1$. Further, it holds

$$\frac{\partial}{\partial x}F(x,t,s)=\frac{1}{t-s}F(x,t,s)[\psi(x+t)-\psi(x+s)].$$

Since both functions $F(x,t,s)$ and $[\psi(x+t)-\psi(x+s)]$ have asymptotic expansions as $x\to\infty$, there exists asymptotic expansion for their product and, by Lemma 3.1, it can be obtained by termwise differentiation of expansion (4.7). Therefore, it holds

$$\frac{1}{t-s}F(x,t,s)[\psi(x+t)-\psi(x+s)]\sim\sum_{n=0}^{\infty}P_n(t,s)\frac{-n+1}{x^n}. \tag{4.8}$$

Using asymptotic expansion of the psi function, see (2.14), we obtain

$$\left[\sum_{j=0}^{\infty}P_j(t,s)\frac{1}{x^{j-1}}\right]\left[\sum_{k=1}^{\infty}\frac{(-1)^{k+1}[B_k(t)-B_k(s)]}{k(t-s)}\frac{1}{x^k}\right]\sim\sum_{n=0}^{\infty}P_n(t,s)\frac{-n+1}{x^n}.$$

After rearrangement of the terms of the product on the left-hand side and comparing the coefficients next to x^{-n} we get

$$\sum_{k=0}^{n}(-1)^k\frac{B_{k+1}(t)-B_{k+1}(s)}{(k+1)(t-s)}P_{n-k}(t,s)=-(n-1)P_n(t,s).$$

The member of the sum for $k=0$ has the value

$$\frac{B_1(t)-B_1(s)}{t-s}P_n(t,s)=P_n(t,s).$$

Hence,

$$nP_n(t,s)=\sum_{k=1}^{n}(-1)^{k+1}\frac{B_{k+1}(t)-B_{k+1}(s)}{(k+1)(t-s)}P_{n-k}(t,s)$$

which proves the theorem. $\square$

The first few polinomials (P_n) are:

$$\begin{aligned}
P_1 &= \frac{1}{2}(t+s-1)\\
P_2 &= \frac{1}{24}(1-t^2+2ts-s^2)\\
P_3 &= \frac{1}{48}(1-t-s-t^2+2ts-s^2+t^3-t^2s-ts^2+s^3)\\
P_4 &= \frac{1}{5760}(23-120t-120s+50t^2+140st+50s^2+120t^3-120st^2\\
&\quad -120s^2t+120s^3-73t^4+52t^3s+42t^2s^2+52ts^3-73s^4)
\end{aligned}$$

However, there exists even more simple form of this expansion which use different set of variables.

4.4. Intrinsic variables

Bernoulli polynomials has better descriptions in terms of their intrinsic variables defined by

$$\alpha = \tfrac{1}{2}(s+t-1), \qquad \beta = \tfrac{1}{4}[1-(t-s)^2].$$

See [10] for detailed analysis of this subject. In Section 5 we will explain the nature of intrinsic variables for general class of Appell polynomials.

THEOREM 4.4. (Burić-Elezović (2011), [7]) *The Wallis power function has the asymptotic expansion of the following form:*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}} \sim x + \sum_{n=0}^{\infty} Q_{n+1}(\alpha,\beta)x^{-n}, \tag{4.9}$$

where it holds $Q_0(\alpha,\beta)=1$ *and*

$$Q_n(\alpha,\beta) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\nabla_k(\alpha,\beta)Q_{n-k}(\alpha,\beta), \qquad n \geqslant 1. \tag{4.10}$$

Here, ∇_n *is a polynomial in variables* α *and* β *given by*

$$\nabla_n(\alpha,\beta) = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}B_{n-k}(\alpha)T_k(\beta). \tag{4.11}$$

and (T_k) *are defined by*

$$T_n = \beta_1^n + \beta_1^{n-1}\beta_2 + \ldots + \beta_1\beta_2^{n-1} + \beta_2^n.$$

The first few polynomials Q_n:

$$Q_1 = \alpha,$$

$$
\begin{aligned}
Q_2 &= \frac{1}{6}\beta, \\
Q_3 &= -\frac{1}{6}\alpha\beta, \\
Q_4 &= \frac{1}{6}\alpha^2\beta - \frac{1}{60}\beta - \frac{13}{360}\beta^2, \\
Q_5 &= -\frac{1}{6}\alpha^3\beta + \frac{1}{20}\alpha\beta + \frac{13}{120}\alpha\beta^2, \\
Q_6 &= \frac{1}{6}\alpha^4\beta - \frac{1}{10}\alpha^2\beta - \frac{13}{60}\alpha^2\beta^2 + \frac{1}{126}\beta + \frac{53}{2520}\beta^2 + \frac{737}{45360}\beta^3.
\end{aligned}
$$

The comparison of these two form is visible in the following example of the same polynomials, Q_6 given above and

$$
\begin{aligned}
P_6(t,s) = \frac{1}{2903040}\Big[&-18125s^6 + 6(1493t + 8316)s^5 \\
&+ 3\left(2111t^2 - 9576t - 1267\right)s^4 \\
&+ 4\left(1417t^3 - 5292t^2 + 11361t - 18900\right)s^3 \\
&+ 3\left(2111t^4 - 7056t^3 + 12558t^2 - 15120t + 10675\right)s^2 \\
&+ 6\left(1493t^5 - 4788t^4 + 7574t^3 - 7560t^2 + 917t + 4284\right)s \\
&- 18125t^6 + 49896t^5 - 3801t^4 - 75600t^3 \\
&+ 32025t^2 + 25704t - 10099\Big].
\end{aligned}
$$

4.5. Asymptotic expansion of Wallis fraction

We are able now to offer different approach to expansion of Wallis fraction.

THEOREM 4.5. (Burić-Elezović (2012), [9]) *It holds*

$$
\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s}\left(\sum_{n=0}^{\infty} P_n(t,s,r)x^{-n}\right)^{\frac{1}{r}}, \tag{4.12}
$$

where polynomials $P_n(t,s,r)$ are defined by:

$$
\begin{aligned}
&P_0(t,s,r) = 1 \\
&P_n(t,s,r) = \frac{r}{n}\sum_{k=1}^{n}(-1)^{k+1}\frac{B_{k+1}(t) - B_{k+1}(s)}{k+1}P_{n-k}(t,s,r).
\end{aligned} \tag{4.13}
$$

Proof. Differentiating the logarithm of (4.12)

$$
\log\Gamma(x+t) - \log\Gamma(x+s) \sim (t-s)\log x + \frac{1}{m}\log\left(\sum_{n=0}^{\infty} P_n(t,s)x^{-n}\right)
$$

we have

$$\psi(x+t)-\psi(x+s)\sim\frac{1}{x}(t-s)+\frac{1}{m}\left(\sum_{n=0}^{\infty}P_nx^{-n}\right)^{-1}\left(\sum_{n=1}^{\infty}(-n)P_nx^{-n-1}\right).$$

Applying (2.14) we get

$$\left(\sum_{n=0}^{\infty}P_nx^{-n}\right)\left(\sum_{k=1}^{\infty}(-1)^k\frac{B_{k+1}(t)-B_{k+1}(s)}{k+1}x^{-k-1}\right)\sim-\frac{1}{m}\sum_{n=1}^{\infty}nP_nx^{-n-1}$$

wherefrom it follows:

$$-nP_n=m\sum_{k=1}^{n}(-1)^k\frac{B_{k+1}(t)-B_{k+1}(s)}{k+1}P_{n-k}. \quad \square$$

We are able now to answer to the question from the introduction. Taking $r=1$, $t=1$, $s=1/2$ we obtain

$$\varphi_1(x)=\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\sim\sqrt{x}\left[1+\frac{1}{8x}+\frac{1}{128x^2}-\frac{5}{1024x^3}-\frac{21}{32768x^4}+\dots\right], \tag{4.14}$$

while for $r=1$, $t=\frac{1}{2}$ and $s=0$ it follows

$$\varphi_2(x)=\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x)}\sim\sqrt{x}\left[1-\frac{1}{8x}+\frac{1}{128x^2}+\frac{5}{1024x^3}-\frac{21}{32768x^4}+\dots\right]. \tag{4.15}$$

The similarity of these two expansions should be explained. There are another way to obtain coefficients of $\varphi_2(x)$. We can take $r=-1$, $t=1$, $s=\frac{1}{2}$ and use the following result:

$$P_n(t,s,-r)=(-1)^nP_n(t,s,r)$$

which immediately follows by induction from (4.13).

5. Appell polynomials and asymptotic expansions

In this subsection we shall discuss two properties of Bernoulli polynomials, the existence of intrinsic variables and appearance of Bernoulli polynomials in expansion formulas for gamma function. It will be shown that both questions are connected with Appell properties of Bernoulli polynomials.

5.1. Appell polynomials and expansions

$R_n(t)$ is Appell sequence of polynomials, if there exist analytic function $A(x)$ such that

$$A(x)e^{tx}=\sum_{n=0}^{\infty}\frac{R_n(t)}{n!}x^n. \tag{5.1}$$

Functions R_n defined by (5.1) are polynomials and

$$R_n'(t) = nR_{n-1}(t). \tag{5.2}$$

$(R_n(t))$ is determined by (5.2) and known values $a_n = R_n(0)$.

The symmetry property of the Appell polynomials is connected with the property of its generating function.

LEMMA 5.1. *It holds*

$$R_n(\lambda - t) = (-1)^n R_n(t) \tag{5.3}$$

if and only if

$$A(-x) = A(x)e^{\lambda x}. \tag{5.4}$$

Proof. From $R_n(\lambda - t) = (-1)^n R_n(t)$, one obtains

$$\begin{aligned} A(x)e^{(\lambda-t)x} &= \sum_{n=0}^{\infty} \frac{R_n(\lambda - t)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n R_n(t)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{R_n(t)}{n!} (-x)^n = A(-x)e^{-tx}, \end{aligned}$$

therefore $A(-x) = A(x)e^{\lambda x}$.

Reversly, from $A(-x) = A(x)e^{\lambda x}$ it follows

$$\begin{gathered} A(-x)e^{-tx} = A(x)e^{(\lambda-t)x} \\ \sum_{n=0}^{\infty} \frac{R_n(t)}{n!} (-1)^n x^n = \sum_{n=0}^{\infty} \frac{R_n(\lambda - t)}{n!} x^n \\ (-1)^n R_n(t) = R_n(\lambda - t). \quad \square \end{gathered}$$

For generalized Bernoulli polynomials it is easy to see that $\lambda = a$ satisfies this condition. Therefore, using Lemma 5.1 this is equivalent to

$$B_n^a(a - t) = (-1)^n B_n^a(t).$$

Let us suppose in the sequel that (R_n) is a sequence of Appell polynomials such that (5.3) is fulfilled. Then, a more natural expression of these polynomials exists.

THEOREM 5.2. (Burić-Elezović-Vukšić, [15]) *Let $R_n(t)$ be an Appell sequence and denote $v = t - \frac{\lambda}{2}$. Then there exist polynomials $C_n(v)$ and $D_n(v)$ of degree n such that it holds*

$$R_{2n}(t) = C_n(v^2), \quad R_{2n+1}(t) = vD_n(v^2). \tag{5.5}$$

These polynomials satisfy relations

$$\begin{gathered} C_n'(v) = nD_{n-1}(v), \\ D_n(v) + 2vD_n'(v) = (2n+1)C_n(v) \end{gathered} \tag{5.6}$$

and can be calculated by

$$C_n(v) = n \int_0^v D_{n-1}(v)dv + R_n\left(\frac{\lambda}{2}\right), \tag{5.7}$$

$$D_n(v) = \frac{2n+1}{2\sqrt{v}} \int_0^v \frac{C_n(v)}{\sqrt{v}} dv. \tag{5.8}$$

Proof. First, let us define polynomials $\tilde{R}_n(v) := R_n(t)$. Then, using the symmetry property, we have

$$\tilde{R}_n(-v) = \tilde{R}_n\left(\frac{\lambda}{2} - t\right) = R_n(\lambda - t) = (-1)^n R_n(t) = (-1)^n \tilde{R}_n(v).$$

Therefore, $\tilde{R}_{2n}(-v) = \tilde{R}_{2n}(v)$, i.e. $\tilde{R}_{2n}(v)$ is an even function and can be written as $C_n(v^2)$. Similary, from $\tilde{R}_{2n+1}(-v) = -\tilde{R}_{2n+1}(v)$ it follows $R_{2n+1}(t) = \tilde{R}_{2n+1}(v) = vD_n(v^2)$.

Now, let us prove (5.6). Using (5.5) and property (5.2), it holds

$$R'_{2n}(t) = C'_n(v^2) \cdot 2v = 2nR_{2n-1}(t) = 2nvD_{n-1}(v^2),$$

wherefrom it follows $C'_n(v^2) = nD_{n-1}(v^2)$.

In a same way,

$$R'_{2n+1}(t) = D_n(v^2) + 2v^2 D'_n(v^2) = (2n+1)R_{2n}(t) = (2n+1)C_n(v^2).$$

From $C'_n(v) = nD_{n-1}(v)$ it follows

$$C_n(v) = n \int_0^v D_{n-1}(v)dv + C_n(0),$$

where $C_n(0) = R_{2n}\left(\frac{\lambda}{2}\right)$. $\square$

5.2. Digamma function and Appell property

The asymptotic expansion of the digamma (psi) function is as follows.

$$\psi(x) \sim \log x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n} x^{-2n}. \tag{5.9}$$

This can be written as:

$$\psi(x) \sim \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n}{n} x^{-n}. \tag{5.10}$$

and the following generalization through Bernoulli polynomials is well known

$$\psi(x+t) \sim \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n(t)}{n} x^{-n}. \tag{5.11}$$

One may wonder about the nature of this generalization. It is connected with Appell property, which can be seen from the following result. See also description in [14].

THEOREM 5.3. (Burić-Elezović-Vukšić, [15]) *Let*

$$f(x) \sim \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{x^{n+1}} \tag{5.12}$$

be asymptotic expansion of the function $f(x)$. *Then:*

$$f(x+t) \sim \sum_{n=0}^{\infty} (-1)^n \frac{R_n(t)}{x^{n+1}}, \tag{5.13}$$

where $(R_n(t))$ *are Appell polynomials generated by* (a_n).

The proof in one direction is straightforward:

$$\begin{aligned}
f(x+t) &\sim \sum_{k=0}^{\infty} (-1)^k a_k (x+t)^{-k-1} \\
&\sim \sum_{k=0}^{\infty} (-1)^k a_k \sum_{j=0}^{\infty} (-1)^j \binom{j+k}{j} t^j x^{-k-j-1} \\
&\sim \sum_{k=0}^{\infty} (-1)^k a_k \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n}{n-k} t^{n-k} x^{-n-1} \\
&\sim \sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=0}^{n} \binom{n}{n-k} a_k t^{n-k} \right) x^{-n-1} \\
&\sim \sum_{n=0}^{\infty} (-1)^n R_n(t) x^{-n-1}.
\end{aligned}$$

See [15] for complete proof.

The natural example for this theorem is trigamma function:

$$\psi'(x) \sim \sum_{n=0}^{\infty} (-1)^n \frac{B_n}{x^{n+1}} \iff \psi'(x+t) \sim \sum_{n=0}^{\infty} (-1)^n \frac{B_n(t)}{x^{n+1}}.$$

Therefore, the easiest way to obtain generalized esymptotic expansion for digamma function (and consequently also for the gamma function) is to begin with simple expansion for trigamma function and integrate termwise obtained expansions.

6. Shifts in asymptotic expansions

In the formula of the type

$$f(x+t) \sim \sum_{n=0}^{\infty} (-1)^n \frac{R_n(t)}{x^{n+1}}, \tag{6.1}$$

one can easily manipulate with shifts in asymptotic terms, for any α we have

$$f(x+t) \sim \sum_{n=0}^{\infty} (-1)^n \frac{R_n(t-\alpha)}{(x+\alpha)^{n+1}}. \tag{6.2}$$

Therefore, properties of included Appell polynomials can be used to simplify coefficients of such expansion. The starting example is modification of Stirling formula.

COROLLARY 6.1. (Burić, Elezović [8]) *It holds*

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \left[\sum_{k=0}^{\infty} P_k\,(n+\tfrac{1}{2})^{-k}\right]^{1/m}, \tag{6.3}$$

where (P_n) *is a sequence defined by*

$$\begin{aligned} P_0 &= 1 \\ P_n &= \frac{m}{n} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(2^{-2k+1}-1)B_{2k}}{2k} P_{n-2k+1}, \qquad n \geqslant 1. \end{aligned} \tag{6.4}$$

As an example, we shall give three formulas for the choice of $m=6$, $m=12$ and $m=24$:

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \sqrt[6]{1 - \frac{1}{4(n+\frac{1}{2})} + \frac{1}{32(n+\frac{1}{2})^2} + \frac{23}{1920(n+\frac{1}{2})^3} + \dots} \tag{6.5}$$

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \sqrt[12]{1 - \frac{1}{2(n+\frac{1}{2})} + \frac{1}{8(n+\frac{1}{2})^2} + \frac{1}{120(n+\frac{1}{2})^3} + \dots} \tag{6.6}$$

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \sqrt[24]{1 - \frac{1}{n+\frac{1}{2}} + \frac{1}{2(n+\frac{1}{2})^2} - \frac{13}{120(n+\frac{1}{2})^3} + \dots} \tag{6.7}$$

The analysis of the best shifts can be made for all included formulas. There is a classical result in this direction for the ratio of two gamma functions The shifted variable is choosen as

$$w = x+t-\rho,$$

where $2\rho = t-s+1$, see [42, p. 34] and [33]

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim w^{t-s} \sum_{n=0}^{\infty} \frac{B_{2n}^{(2\rho)}(\rho)(t-s)_{2n}}{(2n)!} w^{-2n}. \tag{6.8}$$

The coefficients $B_{2k}^{(2\rho)}(\rho)$ are not appropriate here, it should be calculated not through the generalized Bernoulli polynomials. Namely $a \mapsto B_{2n}^{(a)}(x)$ and $x \mapsto B_{2n}^{(a)}(x)$ are polynomials of order $2n$, and $\rho \mapsto B_{2n}^{(2\rho)}(\rho)$ is a polynomial of order n. In [42] the following recursion is given:

$$B_{2n}^{(2\rho)}(\rho) = -2\rho \sum_{k=0}^{n-1} \frac{\binom{2n-1}{2k+1} B_{2k+2}}{2k+2} B_{2n-2k-2}^{(2\rho)}(\rho).$$

It is better to use another name and symbol for these polynomials. I used symbol $V_n(a)$ and the name *reduced Bernoulli polynomials*, see [24]. Since $V_n(a) = B_{2n}^{(2a)}(a)$, its generating function is

$$G(x) = \left(\frac{xe^{x/2}}{e^x - 1} \right)^{2a}. \tag{6.9}$$

Since G is an even function, it follows

$$G(x) = \sum_{n=0}^{\infty} \frac{V_n(a)}{(2n)!} x^{2n}. \tag{6.10}$$

We have

$$\begin{aligned} \frac{G'(x)}{G(x)} &= \frac{2a}{x} \left[1 - \frac{x}{2} - \frac{x}{e^x - 1} \right] \\ &= \frac{2a}{x} \left[1 - \frac{x}{2} - \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \right] \\ &= -2a \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n-1}. \end{aligned}$$

Using (6.10) it is easy to derive

$$V_n(a) = -\frac{a}{n} \sum_{k=0}^{n-1} \binom{2n}{2k} B_{2n-2k} V_k(a). \tag{6.11}$$

The analysis of the best shift for central binomial sequence and sequence of Catalan numbers will be given in separate section.

7. Applications: binomial coefficients

7.1. Classical approach

The classical expansion for central binomial coefficients reads as [42, p. 12]):

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}} \left[1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32768n^4} + \dots \right]. \tag{7.1}$$

In [42, p. 35] the following asymptotic expansion is connected to binomial coefficients:

$$\binom{x}{n} \sim \frac{(-1)^n n^{-(x+1)}}{\Gamma(-x)} \sum_{k=0}^{\infty} \frac{B_k^{(-x)}(x)(x+1)_k}{k!n^k}. \tag{7.2}$$

This expansion is derived from the formula for the ratio of two gamma functions

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s} \sum_{n=0}^{\infty} (-1)^n \frac{(s-t)_n}{n!} B_n^{(t-s+1)}(t) \frac{1}{x^n}. \tag{7.3}$$

Namely, for $x = -\frac{1}{2}$, the following identity holds true

$$\binom{-1/2}{n} = \frac{(-1)^n}{2^{2n}} \binom{2n}{n},$$

so expansion (7.1) follows from (7.2).

7.2. Central binomial coefficient expansion

Central binomial coefficients are closely related with the functions $\varphi_1(x)$ and $\varphi_2(x)$ from the introduction. Namely, from the duplication formula for the gamma function we have

$$\binom{2n}{n} = \frac{\Gamma(2n+1)}{\Gamma(n+1)^2} = \frac{4^n}{\sqrt{\pi}} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}. \tag{7.4}$$

However, we shall recall a more general formula, since it connect duplication formula for the gamma function and central binomial coefficients. Both of these two formulas have the same origin. Duplication formula reads as

$$\frac{\Gamma(2x)}{\Gamma(x)\Gamma(x+\frac{1}{2})} = \frac{2^{2x-1}}{\sqrt{\pi}}.$$

The quotient on the left side is similar to one used in the calculation of binomial coefficients.

THEOREM 7.1. (Burić-Elezović (2012), [9]) *It holds*

$$F(x,t,s,u) := \frac{\Gamma(2x+2t)}{\Gamma(x+s)\Gamma(x+u)} \sim \frac{2^{2x+2t-1}}{\sqrt{\pi}} x^{\gamma} \sum_{k=0}^{\infty} P_k x^{-k}, \tag{7.5}$$

where $\gamma = 2t - s - u + \frac{1}{2}$, (P_n) *is defined by* $P_0 = 1$ *and*

$$P_n = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \square_k P_{n-k}, \qquad n \geqslant 1, \tag{7.6}$$

$$\square_k = \frac{B_{k+1}(t) + B_{k+1}(t+\frac{1}{2}) - B_{k+1}(s) - B_{k+1}(u)}{k+1}. \tag{7.7}$$

Here $B_k(t)$ *are the Bernoulli polynomials.*

Proof. Differentiating expansion (7.5) with respect to variable x we get

$$F\cdot[2\psi(2x+2t)-\psi(x+s)-\psi(x+u)]$$
$$\sim \frac{2^{2x+2t-1}2\log 2}{\sqrt{\pi}}\sum_{k=0}^{\infty}P_k x^{\gamma-k}+\frac{2^{2x+2t-1}}{\sqrt{\pi}}\sum_{k=0}^{\infty}(\gamma-k)P_k x^{\gamma-k-1}.$$

Using duplication formula for the psi function

$$\psi(2x)=\tfrac{1}{2}\psi(x)+\tfrac{1}{2}\psi(x+\tfrac{1}{2})+\log 2,$$

we can write

$$\sum_{k=0}^{\infty}P_k x^{\gamma-k}\cdot[\psi(x+t)+\psi(x+t+\tfrac{1}{2})-\psi(x+s)-\psi(x+u)]\sim$$
$$2\log 2\sum_{k=0}^{\infty}P_k x^{\gamma-k}+\sum_{k=0}^{\infty}(\gamma-k)P_k x^{\gamma-k-1}.$$

Hence (using (5.11)), it follows

$$\sum_{k=0}^{\infty}P_k x^{\gamma-k}\cdot\sum_{j=0}^{\infty}(-1)^j\frac{B_{j+1}(t)+B_{j+1}(t+\frac{1}{2})-B_{j+1}(s)-B_{j+1}(u)}{j+1}x^{-j-1}$$
$$\sim\sum_{n=0}^{\infty}(\gamma-n)P_n x^{\gamma-n-1}.$$

For $k=0$ we have:

$$B_1(t)+B_1(t+\tfrac{1}{2})-B_1(s)-B_1(u)=2t-s-u+\tfrac{1}{2}=\gamma.$$

Equating the coefficients we get:

$$-nP_n=\frac{1}{n}\sum_{k=1}^{n}(-1)^k\Box_k P_{n-k},\qquad n\geqslant 1,$$

where $\Box_k$ is defined by (7.7). $\Box$

If $s=t$, $u=t+\frac{1}{2}$, then asymptotic expansion (7.5) collapses to the duplication formula for the gamma function since in this case we have $\gamma=0$, $\Box_k=0$, $P_n=0$ for all $n\geqslant 0$.

COROLLARY 7.2. *The central binomial coefficient has the following asymptotic expansion:*

$$\binom{2n}{n}\sim\frac{2^{2n}}{\sqrt{n\pi}}\sum_{k=0}^{\infty}P_k x^{-k} \tag{7.8}$$

where (P_n) *is defined by* $P_0=1$ *and*

$$P_n=\frac{1}{n}\sum_{k=1}^{\lfloor(n+1)/2\rfloor}\frac{[2^{-2k}-1]B_{2k}}{k}P_{n-2k+1},\qquad n\geqslant 1, \tag{7.9}$$

where B_{2k} *are the Bernoulli numbers.*

The expansion (7.8) *is exactly* (7.1).

Proof. We can write

$$\binom{2n}{n} = \frac{\Gamma(2n+1)}{\Gamma(n+1)^2}.$$

Hence, $t = \frac{1}{2}$, $s = u = 1$ wherefrom $\gamma = -\frac{1}{2}$. The statement follows easily since $B_k(1) = (-1)^k B_k$ and $B_k(\frac{1}{2}) = -(1-2^{-k+1})B_k$ and $B_{2k+1} = 0$ for $k \geqslant 1$. □

7.3. Asymptotic expansion of general binomial coefficients

The next important class of binomial coefficients is given in the following theorem.

THEOREM 7.3. (Burić-Elezović (2013), [11]) *The following asymptotic expansion holds true:*

$$\binom{px}{rx} \sim \sqrt{\frac{p}{2\pi rsx}} \left(\frac{p^p}{r^r s^s}\right)^x \sum_{n=0}^{\infty} P_n x^{-n}, \tag{7.10}$$

where polynomials (P_n) *satisfy* $P_0 = 1$ *and*

$$P_n = \frac{1}{n} \sum_{k=1}^{n} \frac{(-1)^k}{k+1} (r^{-k} + s^{-k} - p^{-k}) B_{k+1} P_{n-k}. \tag{7.11}$$

Examples:

$$\binom{3n}{n} \sim \frac{3^{3n+\frac{1}{2}}}{2^{2n+1}\sqrt{\pi n}} \left(1 - \frac{7}{72n} + \frac{49}{10368n^2} + \frac{6425}{2239488n^3} - \frac{187103}{644972544n^4} + \ldots\right)$$

$$\binom{4n}{n} \sim \frac{4^{4n+\frac{1}{4}}}{3^{3n+\frac{1}{2}}\sqrt{\pi n}} \left(1 - \frac{13}{144n} + \frac{169}{41472n^2} + \frac{48635}{17915904n^3} - \frac{2614703}{10319560704n^4} + \ldots\right)$$

$$\binom{5n}{n} \sim \frac{5^{5n+\frac{1}{2}}}{4^{4n+\frac{3}{4}}\sqrt{\pi n}} \left(1 - \frac{7}{80n} + \frac{49}{12800n^2} + \frac{37847}{46080000n^3} - \frac{1167761}{14745600000n^4} + \ldots\right)$$

$$\binom{5n}{2n} \sim \frac{5^{5n+\frac{1}{2}}}{2^{2n+1}3^{3n+\frac{1}{2}}\sqrt{\pi n}} \left(1 - \frac{19}{360n} + \frac{361}{259200n^2} + \frac{165337}{1399680000n^3} + \ldots\right)$$

7.4. The best shift for central binomial coefficients

On very interesting page [43] one can find this expansion:

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{N\pi/2}} \left(2 - \frac{2}{N^2} + \frac{21}{N^4} - \frac{671}{N^6} + \frac{45080\frac{3}{4}}{N^8}\right), \tag{7.12}$$

where $N = 8n+2$.

The analysis of the best shift is summarized in the following result.

THEOREM 7.4. (Elezović (2014), [21]) *The following asymptotic expansion is valid:*

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi(n+\alpha)}} \left(\sum_{m=0}^{\infty} P_m(\alpha)(n+\alpha)^{-m} \right)^{1/r}, \tag{7.13}$$

where $P_0 = 1$ *and*

1. *for* $\alpha = \frac{1}{4}$

$$P_m = \frac{r}{m} \sum_{k=1}^{\lfloor m/2 \rfloor} 2^{-2k-1} E_k P_{m-2k}; \tag{7.14}$$

2. *for* $\alpha = \frac{1}{2}$

$$P_m = \frac{r}{m} \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} \frac{(1-2^{-2k})B_{2k}}{k} P_{m-2k+1}; \tag{7.15}$$

3. *for* $\alpha = \frac{3}{4}$

$$P_m = \frac{r}{m} \sum_{k=1}^{\lfloor m/2 \rfloor} 2^{-2k-1}(2-E_k) P_{m-2k}; \tag{7.16}$$

Here, E_k *are Euler's numbers.*

The best choice is $\alpha = \frac{1}{4}$. Using $r = 1$ we get

$$\begin{aligned} \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi(n+\frac{1}{4})}} \Bigg[1 &- \frac{1}{64\left(n+\frac{1}{4}\right)^2} + \frac{21}{8192\left(n+\frac{1}{4}\right)^4} - \frac{671}{524288\left(n+\frac{1}{4}\right)^6} \\ &+ \frac{180323}{134217728\left(n+\frac{1}{4}\right)^8} - \frac{20898423}{8589934592\left(n+\frac{1}{4}\right)^{10}} + \cdots \Bigg], \end{aligned} \tag{7.17}$$

$$\begin{aligned} \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi(n+\frac{1}{2})}} \Bigg[1 &+ \frac{1}{8\left(n+\frac{1}{2}\right)} + \frac{1}{128\left(n+\frac{1}{2}\right)^2} - \frac{5}{1024\left(n+\frac{1}{2}\right)^3} \\ &- \frac{21}{32768\left(n+\frac{1}{2}\right)^4} + \frac{399}{262144\left(n+\frac{1}{2}\right)^5} + \frac{869}{4194304\left(n+\frac{1}{2}\right)^6} + \cdots \Bigg], \end{aligned} \tag{7.18}$$

$$\begin{aligned} \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi(n+\frac{3}{4})}} \Bigg[1 &+ \frac{1}{4\left(n+\frac{3}{4}\right)} + \frac{5}{64\left(n+\frac{3}{4}\right)^2} + \frac{5}{256\left(n+\frac{3}{4}\right)^3} \\ &+ \frac{21}{8192\left(n+\frac{3}{4}\right)^4} + \frac{21}{32768\left(n+\frac{3}{4}\right)^5} + \frac{715}{524288\left(n+\frac{3}{4}\right)^6} + \cdots \Bigg]. \end{aligned} \tag{7.19}$$

7.5. An interesting symmetry

This is connected with properties of functions φ_1 and φ_2.

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}\Big[1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32768n^4} - \frac{399}{262144n^5} + \frac{869}{4194304n^6} + \cdots\Big], \tag{7.20}$$

$$1\Big/\binom{2n}{n} \sim \frac{\sqrt{\pi n}}{4^n}\Big[1 + \frac{1}{8n} + \frac{1}{128n^2} - \frac{5}{1024n^3} - \frac{21}{32768n^4} + \frac{399}{262144n^5} + \frac{869}{4194304n^6} + \cdots\Big]. \tag{7.21}$$

7.6. Multinomial central coefficients

Using formula

$$\binom{kn}{n,n,\dots,n} = \frac{(kn)!}{(n!)^k} = \frac{\Gamma(kn+1)}{\Gamma^k(n+1)}$$

and lemmas from Section 2, the algorithm for computation of coefficients can be easily prepared.

THEOREM 7.5. (Burić-Elezović-Šimić (2013), [13]) *Central multinomial coefficient has following asymptotic expansion*

$$\binom{kn}{n,n,\dots,n} \sim \left(\sqrt{2\pi n}\right)^{1-k} k^{kn+\frac{1}{2}} \left(\sum_{j=0}^{\infty} c_j n^{-j}\right)^{1/m}, \tag{7.22}$$

where

$$a_0 = 1, \quad a_j = \frac{m}{j} \sum_{i=1}^{\lfloor (j+1)/2 \rfloor} \frac{B_{2i}}{2i} a_{j-2i+1}, \quad j \geqslant 1, \tag{7.23}$$

$$b_0 = 1, \quad b_j = \frac{1}{j} \sum_{i=1}^{j} [(k+1)i - j] a_i b_{j-i}, \quad j \geqslant 1, \tag{7.24}$$

$$c_0 = 1, \quad c_j = a_j k^{-j} - \sum_{i=1}^{j} b_i c_{j-i}, \quad j \geqslant 1, \tag{7.25}$$

and B_k are Bernoulli numbers.

Let us give here the first terms of the first few expansions, in the case $m = 1$ we obtain:

$$\binom{3n}{n,n,n} \sim \frac{3^{3n+1/2}}{2\pi n} \cdot \left(1 - \frac{2}{9n} + \frac{2}{81n^2} + \frac{14}{2187n^3} - \frac{34}{19683n^4} + \dots\right), \tag{7.26}$$

$$\binom{4n}{n,n,n,n} \sim \frac{4^{4n}}{\pi n\sqrt{2\pi n}} \cdot \left(1 - \frac{5}{16n} + \frac{25}{512n^2} + \frac{49}{8192n^3} - \frac{1605}{524288n^4} + \dots\right). \tag{7.27}$$

Let us note that $m = 1/k$ is good choice for exponent, which can be seen from the following expansions, see [13]:

$$\binom{3n}{n,n,n} \sim \frac{3^{3n+1/2}}{2\pi n} \cdot \left(1 - \frac{2}{3n} + \frac{2}{9n^2} - \frac{2}{81n^3} - \frac{2}{243n^4} + \dots\right)^{1/3}, \tag{7.28}$$

$$\binom{4n}{n,n,n,n} \sim \frac{4^{4n}}{\pi n\sqrt{2\pi n}} \cdot \left(1 - \frac{5}{4n} + \frac{25}{32n^2} - \frac{9}{32n^3} - \frac{95}{2048n^4} + \dots\right)^{1/4}. \tag{7.29}$$

7.7. Coefficients close to central

Let us denote

$$S_r^k = 1^k + 2^k + \ldots + r^k.$$

THEOREM 7.6. (Burić-Elezović (2013), [11]) *The following asymptotic expansion holds true:*

$$\binom{2x}{x-r} \sim \frac{4^x}{\sqrt{\pi x}} \sum_{n=0}^{\infty} P_n x^{-n}, \tag{7.30}$$

where polynomials (P_n) satisfy $P_0 = 1$ and

$$\begin{aligned} P_n &= \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} r^{2k} P_{n-2k} \\ &+ \frac{1}{n} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \left[\frac{2^{-2k}-1}{k} B_{2k} - 2S_{r-1}^{2k-1} - r^{2k-1} \right] P_{n-2k+1}. \end{aligned} \tag{7.31}$$

Examples:

$$\binom{2n}{n-1} \sim \frac{4^n}{n\sqrt{\pi}} \left(1 - \frac{9}{8n} + \frac{145}{128n^2} - \frac{1155}{1024x^3} + \frac{36939}{32768n^4} + \dots\right),$$

$$\binom{2n}{n-2} \sim \frac{4^n}{n\sqrt{\pi}} \left(1 - \frac{33}{8n} + \frac{1345}{128n^2} - \frac{23835}{1024x^3} + \frac{1599339}{32768n^4} + \dots\right).$$

7.8. Comparison with central coefficient

The ratio

$$F(x,r) = \binom{2x}{x-r} \Big/ \binom{2x}{x} \tag{7.32}$$

is connected with ratio of gamma functions:

$$F(x,r) = \frac{\Gamma(x+1)^2}{\Gamma(x+1-r)\Gamma(x+1+r)} \tag{7.33}$$

and can be expanded into asymptotic series using the same principle.

THEOREM 7.7. (Burić-Elezović (2013), [11]) *The function* (7.32) *has the asymptotic expansion:*

$$F(x,r)=\sum_{m=0}^{\infty}Q_m x^{-m}, \tag{7.34}$$

where polynomials (Q_k) *are defined by* $Q_0=1$ *and*

$$Q_m=\frac{1}{m}\sum_{k=1}^{\lfloor (m+1)/2\rfloor} r^{2k-1}Q_{m-2k+1}-\frac{1}{m}\sum_{k=1}^{\lfloor m/2\rfloor}(2S_{r-1}^k+r^k)Q_{m-k}. \tag{7.35}$$

$$\binom{2n}{n-2}=\binom{2n}{n}\left[1-\frac{4}{n}+\frac{10}{n^2}-\frac{22}{n^3}+\frac{46}{n^4}-\frac{94}{n^5}+\ldots\right],$$
$$\binom{2n}{n-3}=\binom{2n}{n}\left[1-\frac{9}{n}+\frac{45}{n^2}-\frac{177}{n^3}+\frac{621}{n^4}-\frac{2045}{n^5}+\ldots\right].$$

The shift analysis can be made in this context. If we write

$$F(x,r)=\sum_{m=0}^{\infty}Q_m(x+\alpha)^m, \tag{7.36}$$

then

$$Q_m=\frac{1}{m}\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k+1}[2B_{k+1}(1-\alpha)-B_{k+1}(1-r-\alpha)-B_{k+1}(1+r-\alpha)]Q_{m-k}. \tag{7.37}$$

From the value of the first few coefficients the best shift α can be deduced:

$$\begin{aligned} Q_1&=-r^2,\\ Q_2&=\frac{r^2}{2}+\frac{r^4}{2}-\alpha r^2,\\ Q_3&=-\frac{r^2}{6}-\frac{2r^4}{3}-\frac{r^6}{6}+(r^2+r^4)\alpha-r^2\alpha^2. \end{aligned} \tag{7.38}$$

The coefficient Q_2 will be equal to zero if

$$\alpha=\frac{r^2}{2}+\frac{1}{2}. \tag{7.39}$$

Hence

$$\binom{2x}{x-r}\approx\binom{2x}{x}\left[1-\frac{2r^2}{2x+r^2+1}+\ldots\right]. \tag{7.40}$$

The right hand side gives a good lower estimate for the binomial coefficient from the left:

$$\binom{2n}{n-r}>\left[1-\frac{2r^2}{2n+r^2+1}\right]\binom{2n}{n}. \tag{7.41}$$

This inequality can be proved using the following theorem:

THEOREM 7.8. (Burić-Elezović (2013), [11]) *Let function f be defined as*

$$f(x)=\frac{\Gamma(x+1)^2}{\Gamma(x+1-r)\Gamma(x+1+r)}\cdot\frac{2x+r^2+1}{2x-r^2+1}. \tag{7.42}$$

Then $\log f(x)$ is completely monotonic on $(r-1,+\infty)$.

Proof. Using the following integral representations [2]:

$$\log\Gamma(x)=\int_0^\infty\left[(x-1)e^{-t}-\frac{1-e^{(1-x)t}}{e^t-1}\right]\frac{dt}{t},$$

$$\log x=\int_0^\infty[e^{-t}-e^{-xt}]\frac{dt}{t}$$

it is easy to obtain

$$\log f(x)=\int_0^\infty 2h(t)\frac{e^{-xt}}{t(e^t-1)}\,dt.$$

Here function $h(t)$ is defined by

$$h(t)=1-\cosh(rt)-\cosh\left(\tfrac{1-r^2}{2}t\right)+\cosh\left(\tfrac{1+r^2}{2}t\right)=\sum_{k=1}^\infty\frac{c_{2k}}{(2k)!}t^{2k}$$

where

$$c_{2k}=\left(\frac{1+r^2}{2}\right)^{2k}-\left(\frac{1-r^2}{2}\right)^{2k}-r^{2k}.$$

It is obvious that it holds $c_2=0$ and $c_{2k}>0$ for $k>1$. Hence, $h(t)>0$ for all $t>0$ and the claim follows. □

As a consequence, we have $f(x)>1$ for all $x>r-1$. Therefore, (7.41) follows.

7.9. Asymptotic expansion of Catalan numbers

Catalan numbers are given by the following explicit formula:

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)}. \tag{7.43}$$

Hence, Catalan numbers can be expressed as a ratio of two gamma functions

$$C_n=\frac{4^n}{\sqrt{\pi}}\cdot\frac{\Gamma(n+\frac12)}{\Gamma(n+2)}. \tag{7.44}$$

Putting $x=n+\alpha$, $t=\frac12-\alpha$, $s=2-\alpha$, from (4.5) we get

$$C_n\sim\frac{4^n}{\sqrt{\pi}}x^{-3/2}\left(\sum_{m=0}^\infty P_m x^{-m}\right)^{1/r}, \tag{7.45}$$

with $P_0 = 1$ and

$$P_m = \frac{r}{m} \sum_{k=1}^{m} c_k(\alpha) P_{m-k}, \tag{7.46}$$

where we denote

$$c_k(\alpha) = \frac{B_{n+1}(\alpha + \frac{1}{2}) - B_{k+1}(\alpha - 1)}{k+1}. \tag{7.47}$$

As before, 0 and $\frac{1}{2}$ are the natural choice for α. Two other good values follow from $\alpha + \frac{1}{2} = 1 - (\alpha - 1)$ and $\alpha + \frac{1}{2} = -(\alpha - 1)$, wherefrom one gets $\alpha = \frac{3}{4}$ and $\alpha = \frac{1}{4}$, respectively.

THEOREM 7.9. (Elezović (2014), [21]) *The following asymptotic expansion holds:*

$$C_n \sim \frac{4^n}{\sqrt{\pi}} (n+\alpha)^{-3/2} \left(\sum_{m=0}^{\infty} P_m(\alpha)(n+\alpha)^{-m} \right)^{1/r}, \tag{7.48}$$

where $P_0 = 1$ *and*

1. *for* $\alpha = 0$

$$P_m = \frac{r}{m} \sum_{k=1}^{m} \left[\frac{(2^{-k} - 2)B_{k+1}}{k+1} + (-1)^k \right] P_{m-k}; \tag{7.49}$$

2. *for* $\alpha = \frac{1}{2}$

$$P_m = \frac{r}{m} \sum_{k=1}^{m} \left[\frac{(2 - 2^{-k})B_{k+1}}{k+1} + \frac{(-1)^{k+1}}{2^k} \right] P_{m-k}; \tag{7.50}$$

3. *for* $\alpha = \frac{3}{4}$

$$P_m = \frac{r}{m} \sum_{k=1}^{\lfloor m/2 \rfloor} 2 \cdot 4^{-2k-1} (4 - E_{2k}) P_{m-2k}; \tag{7.51}$$

4. *for* $\alpha = \frac{1}{4}$

$$P_m = \frac{r}{m} \sum_{k=1}^{m} [2^{-2k-1} E_k + (-\tfrac{3}{4})^k] P_{m-k}. \tag{7.52}$$

We give here expansions for $r = 1$ and in two most important cases, $\alpha = 0$ and $\alpha = \frac{3}{4}$.

$$C_n \sim \frac{4^n}{\sqrt{\pi n^3}} \left[1 - \frac{9}{8n} + \frac{145}{128n^2} - \frac{1155}{1024n^3} + \frac{36939}{32768n^4} - \frac{295911}{262144n^5} + -\frac{4735445}{4194304n^6} + \cdots \right], \tag{7.53}$$

$$C_n \sim \frac{4^n}{\sqrt{\pi(n+\frac{3}{4})^3}}\Bigg[1+\frac{5}{64\left(n+\frac{3}{4}\right)^2}+\frac{21}{8192\left(n+\frac{3}{4}\right)^4}+\frac{715}{524288\left(n+\frac{3}{4}\right)^6} - \frac{162877}{134217728\left(n+\frac{3}{4}\right)^8}+\frac{19840275}{8589934592\left(n+\frac{3}{4}\right)^{10}}+\cdots\Bigg]. \qquad (7.54)$$

As one can see, the expansion in terms of $n+\frac{3}{4}$ has additional property that it contains only odd terms. Having in mind that Catalan number differs from central binomial coefficient by factor of $(n+1)$, and binomial coefficient has similar property for expansions through $n+\frac{1}{4}$, this property of Catalan numbers seems very strange. But, it can be clearly seen from previous theorem. See [21] for detailed discussion.

7.10. Sum of binomial coefficients and Catalan numbers

If asymptotic expansion of some series $a(n)$ is known, then in some situations it is possible to determine asymptotic expansion of its finite sum.

Suppose that the following expansion is known

$$a(n) \sim b^n \sum_{k=0}^{\infty} P_k(\alpha)(n+\alpha)^{-k-r}, \qquad (7.55)$$

where $r>0$ is a real number, $b>1$ and P_k are polynomials, $P_0=1$.

The next theorem is a slight extension of [21, Theorem 6.1]

THEOREM 7.10. *If $A(n)$ are given by (7.55) then*

$$\sum_{k=0}^{n} a(k) \sim \frac{b^{n+1}}{b-1}\sum_{k=0}^{\infty} S_k(\alpha)(n+\alpha)^{-k-r}, \qquad (7.56)$$

where the coefficients of this expansion satisfy $S_0(\alpha)=1$ and

$$S_k(\alpha) = P_k(\alpha) + \frac{1}{b-1}\sum_{j=0}^{k-1}(-1)^{k-j}\binom{-j-r}{k-j}S_j(\alpha). \qquad (7.57)$$

Proof. Denote $\Sigma(n)=\sum_{k=0}^{n} a(k)$. Suppose that

$$\Sigma(n) \sim C\cdot b^n n^{-r} + O(n^{-r-1}).$$

Then

$$\begin{aligned}
\Sigma(n) &\sim a(n)+\Sigma(n-1)\\
&\sim b^n n^{-r} + C\cdot b^{n-1}(n-1)^{-r} + O(n^{-r-1})\\
&\sim b^n n^{-r} + C\cdot b^{n-1} n^{-r} + O(n^{-r-1})\\
&\sim C\cdot b^n n^{-r} + O(n^{-r-1})
\end{aligned}$$

and from here it follows that $C = b/(b-1)$. The fact that $\Sigma(n)$ indeed has the asymptotic behavior of this type may be proved in the same way as it is done for the case $r = 1/2$ in [44].

Hence, we obtain that $\Sigma(n)$ has the asymptotic expansion of the following form:

$$\Sigma(n) = \frac{b^{n+1}}{(b-1)} \sum_{k=0}^{\infty} S_k(\alpha)(n+\alpha)^{-k-r}. \tag{7.58}$$

Then, using the asymptotic expansion (7.55), we get

$$\begin{aligned}
&\frac{b^{n+1}}{(b-1)} \sum_{k=0}^{\infty} S_k(\alpha)(n+\alpha)^{-k-r} \\
&\quad = b^n \sum_{k=0}^{\infty} P_k(\alpha)(n+\alpha)^{-k-r} + \frac{b^n}{(b-1)} \sum_{k=0}^{\infty} S_k(\alpha)(n+\alpha-1)^{-k-r} \\
&\quad = b^n \sum_{k=0}^{\infty} P_k(\alpha)(n+\alpha)^{-k-r} \\
&\qquad + \frac{b^n}{(b-1)} \sum_{k=0}^{\infty} S_k(\alpha)(n+\alpha)^{-k-r} \sum_{j=0}^{\infty} (-1)^j \binom{-k-r}{j} (n+\alpha)^{-j}.
\end{aligned}$$

Hence

$$bS_k(\alpha) = (b-1)P_k(\alpha) + \sum_{j=0}^{k} (-1)^{k-j} \binom{-j-r}{k-j} S_j(\alpha).$$

Extracting from the right side the member S_k, we get (7.57). $\square$

Taking $b = 4$, $\alpha = 0$ and $r = 1/2$ or $r = 3/2$, it is easy to obtain the following asymptotics for the sum of binomial coefficients and sum of Catalan numbers:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{2k}{k} &\sim \frac{4^{n+1}}{3\sqrt{\pi n}} \Big(1 + \frac{1}{24n} + \frac{59}{384n^2} + \frac{2425}{9216n^3} + \frac{576793}{884736n^4} \\
&\qquad + \frac{5000317}{2359296n^5} + \frac{953111599}{113246208n^6} + \dots\Big), \\
\sum_{k=0}^{n} C_n &\sim \frac{4^{n+1}}{3n\sqrt{\pi n}} \Big(1 - \frac{5}{8n} + \frac{475}{384n^2} + \frac{1225}{9216n^3} + \frac{395857}{98304n^4} \\
&\qquad + \frac{27786605}{2359296n^5} + \frac{6798801295}{113246208n^6}\Big).
\end{aligned}$$

8. Wallis functions and inequalities

The main property of Wallis function is given in the next theorem.

THEOREM 8.1. (Elezović-Giordano-Pečarić (2000) [25]) *Let $s,t > 0$, $r = \min(s,t)$. Then*

$$z(x) := \left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} - x \tag{8.1}$$

is convex and decreasing function on $(-r,+\infty)$ *for* $|t-s|<1$, *and concave and increasing on the same interval for* $|t-s|>1$.

Let α and β be lower and upper bounds for this function. Because of the monotonicity we immediate obtain Gautschi-Kershaw first inequality in its most natural form:

$$x+\alpha<\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}}<x+\beta. \tag{8.2}$$

THEOREM 8.2. (Elezović-Giordano-Pečarić (2000), [25]) *For each* $x>0$ *and* $s,t>0$, *inequality*

$$x+\frac{s+t-1}{2}<\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}}<x+\left[\frac{\Gamma(t)}{\Gamma(s)}\right]^{\frac{1}{t-s}} \tag{8.3}$$

holds for $|t-s|<1$, *and with opposite sign for* $|t-s|>1$. *The bounds are the best possible.*

It should be noticed that condition $x>0$ is posed because of the right side of this inequality. The left side holds for $x>-r$, as before.

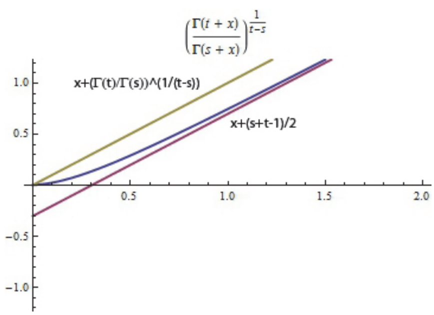

Figure 1. Upper and lower bound in the first Gautschi's inequality.

Gautschi-Kershaw inequality is just the very beginning of an asymptotic expansion. Taking into account the second term of this expansion, one can write inequality which is waiting for the proof:

CONJECTURE 8.3.

$$\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}}<x+\frac{(1-(t-s)^2)}{24x}. \tag{8.4}$$

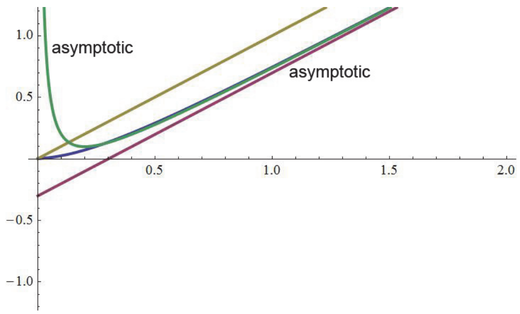

Figure 2. The second asymptotic term added.

F. Qi [50] give an extensive study of the various inequalities connected to the ratio of gamma functions.

9. Integral means and the second Gautschi-Kershaw's inequality

9.1. Second Gautschi inequality

In 1983 D. Kershaw [39] splitted original Gautschi's inequality (1.6) into two of different type, the first one is given in (1.7) and the second one reads as

$$\exp[(1-s)\psi(x+\sqrt{s})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]. \tag{9.1}$$

In 2005 D. Kershaw [40] proved the following inequality:

$$\psi(x+\sqrt{st}) < \frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)} < \psi\left(x+\frac{s+t}{2}\right) \tag{9.2}$$

for $x \geqslant 0$ and $0 < s \leqslant t$, and in 2008. F. Qi raises the left bound from geometrical to logarithmic mean

$$L(s,t) = \frac{t-s}{\log t - \log s},$$

but this was already proved in [25]. Moreover, in [25] the best bound was given:

THEOREM 9.1. (Elezović-Pečarić (2000) [27]) *For each $x \geqslant 0$, $s,t > 0$ it holds*

$$\psi\Big(x+I_\psi(s,t)\Big) < \frac{1}{t-s}\int_s^t \psi(x+u)\,\mathrm{d}u < \psi(x+\frac{s+t}{2}). \tag{9.3}$$

Here

$$I_\psi(s,t) = \psi^{-1}\left(\frac{1}{t-s}\int_s^t \psi(u)du\right)$$

is integral ψ-mean of s and t, ψ^{-1} denotes the inverse function of ψ. Namely, in [27] it is proved that

$$\sqrt{st} \leqslant L(s,t) \leqslant I_\psi(s,t).$$

Since

$$I_\psi(x+s,x+t)-x\to\frac{s+t}{2}\quad\text{as}\quad x\to\infty,$$

this implies that

$$\frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)}\sim\psi\left(x+\frac{s+t}{2}\right)\quad\text{as}\quad x\to\infty. \tag{9.4}$$

Therefore, the bounds given in (9.3) are the best possible. The left bound is obtained for $x=0$, and the right one is asymptotical.

THEOREM 9.2. (Elezović-Giordano-Pečarić (2000), [25]) *Let* $s,t>0$, $r=\min(s,t)$. *Function*

$$v(x)=\psi\Big(x+\frac{s+t}{2}\Big)-\frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)}$$

is completely monotonic on $(-r,+\infty)$.

It is known that if v is completely monotonic then $-v'$ is completely monotonic, hence $\exp(v)$ is also completely monotonic. Bustoz and Ismail [16] proved that $-v'$ is completely monotonic, for $t=1$ and $0<s<1$.

Proof of Theorem 9.2. We shall use the following representation of gamma and digamma functions:

$$\log\Gamma(x)=\int_0^\infty\left[(x-1)e^{-u}-\frac{e^{-u}-e^{-xu}}{1-e^{-u}}\right]\frac{du}{u},$$

$$\psi(x)=\int_0^\infty\left[\frac{e^{-u}}{u}-\frac{e^{-xu}}{1-e^{-u}}\right]du.$$

Hence, for any value of φ we have

$$\begin{aligned}\psi(x+\varphi)&-\frac{1}{t-s}[\log\Gamma(x+t)-\log\Gamma(x+s)]\\&=\int_0^\infty\frac{e^{-xu}}{1-e^{-u}}\left[-e^{-\varphi u}-\frac{e^{-tu}-e^{-su}}{u(t-s)}\right]\\&=\int_0^\infty\frac{e^{-xu}}{1-e^{-u}}g(u)du,\end{aligned}$$

where

$$g(u)=-e^{-\varphi u}+\frac{1}{t-s}\int_s^t e^{-\tau u}d\tau.$$

By Hermite-Hadamard inequality for convex function $\tau\mapsto e^{-\tau u}$, we have $g(u)>0$ for $\varphi\geqslant A(s,t)$. Therefore, for this value of φ

$$(-1)^k v^{(k)}(x)=\int_0^\infty u^k\frac{e^{-xu}}{1-e^{-u}}g(u)du>0$$

and the theorem is proved. $\square$

9.2. Integral mean formulation

Let f be monotone on $[s,t]$. Integral mean of f is defined by

$$I_f = I_f(s,t) = f^{-1}\left[\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u\right]. \tag{9.5}$$

Equivalent form of the second Gautschi-Kershow's inequality is: find the best constants α, β such that

$$\psi(x+\alpha) < \frac{1}{t-s}\int_s^t \psi(x+u)\,\mathrm{d}u < \psi(x+\beta). \tag{9.6}$$

Equivalent notation to inequality (9.3) is:

$$x + I_\psi(s,t) < I_\psi(x+s,x+t) < x + A(s,t). \tag{9.7}$$

The natural question connected with this inequality is to find asymptotic expansion of the function G defined by

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}} = \exp\psi(G(x)). \tag{9.8}$$

Equivalent form is to find asymptotic expansion of the integral mean of digamma function:

$$\begin{aligned} I_\psi(x+s,x+t) &= \psi^{-1}\left(\frac{1}{t-s}\int_s^t \psi(x+u)\,\mathrm{d}u\right) = G(x) \\ &= \sum_{k=0}^{\infty} c_k(t,s)x^{-k+1}. \end{aligned} \tag{9.9}$$

This problem is solved in the following theorem.

THEOREM 9.3. (Chen-Elezović-Vukšić (2013), [17]) *Let (a_k) and (b_k) are related to (c_k) through:*

$$a_n = \frac{c_n}{c_0} - \frac{1}{nc_0}\sum_{k=1}^{n-1} k a_k c_{n-k}, \qquad n \geqslant 1. \tag{9.10}$$

$$b_n(r) = \frac{1}{nc_0}\sum_{k=1}^{n}[k(1+r)-n]c_k b_{n-k}(r). \tag{9.11}$$

Then integral mean of digamma function has asymptotic expansion

$$\begin{aligned} c_n(t,s) = \frac{1}{n}\sum_{k=1}^{n-1} k a_k c_{n-k}(t,s) + \sum_{k=1}^{n}\frac{B_k(1)}{k}b_{n-k}(k) \\ + \frac{B_{n+1}(1-t)-B_{n+1}(1-s)}{n(n+1)(t-s)}. \end{aligned} \tag{9.12}$$

The first few coefficients are:

$$\begin{aligned}
c_0 &= 1,\\
c_1 &= \frac{1}{2}(t+s),\\
c_2 &= -\frac{1}{24}(t-s)^2,\\
c_3 &= \frac{1}{48}(t-s)^2(s+t-1),\\
c_4 &= -\frac{1}{5760}(t-s)^2\Big[73(t^2+s^2)+94ts-120(t+s)+20\Big],\\
c_5 &= \frac{1}{3840}(t-s)^2\Big[33(t^3+s^3)+47ts(t+s)\\
&\qquad -73(t^2+s^2)-94ts+20(t+s)+20)\Big],
\end{aligned}$$

This implies, among others, the hypothesis about improvement of the second Gautschi's inequality.

CONJECTURE 9.4. The following inequality is satisfied for all $x > x_0$:

$$\exp\Big(\psi(x+\tfrac{1}{2}(t+s)-\tfrac{1}{24}(t-s)^2x^{-1})\Big) < \left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}.$$

Also, this was a clear signal that asymptotic expansion can be efficient tool in investigation of various means.

The coefficients of the asymptotic expansions of the mean $M(x+t,x+s)$ usually has simpler form (especially in the case of symmetric mean, this one is not symmetric) if one uses another set of variables, $\alpha=(s+t)/2$, $\beta=(s-t)/2$. We shall use these substitutions in the sequell.

10. Asymptotic inequalities and integral means

10.1. Integral means

Let $I\subset\mathbb{R}$ be an interval and let f be a strictly monotone continuous function on I and $s,t\in I$, $s<t$. Then there exists the unique $\vartheta\in[s,t]$ for which

$$\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u = f(\vartheta).$$

ϑ is called *integral f-mean* of s and t, and is denoted by

$$I_f(s,t) = f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u\right), \tag{10.1}$$

see [4, 5] for details.

Many classical means can be interpreted as integral means, for suitably chosen function f. For example,

$$
\begin{aligned}
&f(x)=x, && I_x(s,t)=\frac{s+t}{2}=A(s,t),\\
&f(x)=\log x, && I_{\log x}(s,t)=\frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}}=I(s,t),\\
&f(x)=\frac{1}{x}, && I_{1/x}(s,t)=\frac{s-t}{\log s-\log t}=L(s,t),\\
&f(x)=\frac{1}{x^2}, && I_{1/x^2}(s,t)=\sqrt{st}=G(s,t),\\
&f(x)=x^r,\quad r\neq 0,-1 && I_{x^r}(s,t)=\left(\frac{t^{r+1}-s^{r+1}}{(r+1)(t-s)}\right)^{\frac{1}{r}}=L_r(s,t),
\end{aligned}
$$

where A, I, L, G, and L_r are arithmetic, identric, logarithmic, geometric and generalized logarithmic mean.

Let us denote:

$$A(x)\sim x^w\sum_{n=0}^{\infty}a_nx^{-n},\quad B(x)\sim x^u\sum_{n=0}^{\infty}b_nx^{-n},\quad C(x)\sim x^v\sum_{n=0}^{\infty}c_nx^{-n}. \tag{10.2}$$

10.2. Finding asymptotic expansion

N. Elezović and L. Vukšić [26] solved general problem of finding coefficients of an asymptotic expansion $A(x)$ such that the formal equation

$$B(A(x))=C(x) \tag{10.3}$$

is satisfied. The algorithm depends on the sign of exponent w. In applications to mean, this exponent is always equal to 1 and this case is covered in the following result:

THEOREM 10.1. (Elezović-Vukšić (2013), [26]) *Suppose $u\neq 0$. Then there exists asymptotic expansion $A(x)$ of with $w>0$, if and only if:*

1. *u and v have the same sign and $w=v/u$ is a rational number,*
2. *If nw is not a positive integer then $b_n=0$,*
3. *b_0 and c_0 are of the same sign.*

Then:

$$
\begin{aligned}
a_n=-\frac{a_0}{b_0uE_0[a,u]}\Bigg[&\sum_{j=1}^{\lfloor n/w\rfloor}b_jE_{n-wj}[a,u-j]\\
&+\frac{b_0}{na_0}\sum_{k=1}^{n-1}[k(1+u)-n]a_kE_{n-k}[a,u]-c_n\Bigg].
\end{aligned} \tag{10.4}
$$

THEOREM 10.2. (Elezović-Vukšić (2013), [26]) *Let* $B(x)$ *and* $C(x)$ *be asymptotic series given by* (10.2)*. Let us suppose that the following conditions are fulfilled:*

1. *representation of* B *is finite, ie.*

$$B(x) \sim x^u \sum_{n=0}^{M} b_n x^{-n}, \tag{10.5}$$

where M *is such that* $b_M \neq 0$.

2.

$$w = \frac{v}{u-M} < 0. \tag{10.6}$$

Then there exists series $A(x)$ *of the form* (10.2) *such that* (10.3) *is satisfied. Coefficients of the series* $A(x)$ *can be calculated using recursive formula:*

$$\begin{aligned} a_0 &= \left(\frac{c_0}{b_M}\right)^{M-u}, \\ a_n &= \frac{a_0^{1-u+M}}{(u-M)b_M}\left(c_n - \frac{b_M}{na_0}\sum_{k=1}^{n-1}[k(1+u-M)-n]a_k E_{n-k}[a,u-M]\right. \\ &\qquad \left. - \sum_{j=1}^{\min\{M,-n/w\}} b_{M-j}E_{n+wj}[a,u-M+j]\right). \end{aligned} \tag{10.7}$$

The following example ilustrates both situations.

EXAMPLE 10.3. Let us find asymptotic series which satisfies the equation $A^2(x)+1/A(x) = x^2$. Here $B(x) = x^2 + 1/x$, $C(x) = x^2$. Hence, for the first solution we have $u = 2$, $v = 2$, $w = 1$ and (10.4) will give

$$A(x) \sim x\left(1 - \frac{1}{2x^3} - \frac{3}{8x^6} - \frac{1}{2x^9} - \frac{105}{128x^{12}} - \frac{3}{2x^{15}} - \frac{3003}{1024x^{18}} - \frac{6}{x^{21}} + \dots\right).$$

The expansion of the function B is finite, and this enables another solution for which $M = 3$, hence $w = -2$ and (10.7) gives

$$A(x) \sim \frac{1}{x^2}\left(1 + \frac{1}{x^6} + \frac{3}{x^{12}} + \frac{12}{x^{18}} + \frac{55}{x^{24}} + \frac{273}{x^{30}} + \frac{1428}{x^{36}} - \frac{7752}{x^{42}} + \dots\right).$$

10.3. Asymptotic expansion of integral mean

Theorem 10.1 can be applied to the problem of finding asymptotic expansion

$$A(x) \sim x \sum_{m=0}^{\infty} a_n x^{-n} \tag{10.8}$$

of an integral mean of function f, if the asymptotic expansion of a function f is known:

$$f(x) \sim x^u \sum_{n=0}^{\infty} b_n x^{-n}.$$

The right side is given by

$$C(x) = \frac{1}{t-s} \int_s^t f(x+u)\,\mathrm{d}u \sim x^v \sum_{n=0}^{\infty} c_n x^{-n}.$$

Coefficients (c_n) can be calculated in a straightforward way:

$$c_n = \sum_{k=0}^{n} \frac{b_k}{n+1-k} \binom{u-k}{n-k} \frac{t^{n+1-k} - s^{n+1-k}}{t-s}. \tag{10.9}$$

The algorithm for calculation of integral mean reads as:

THEOREM 10.4. *Let function f have the following asymptotic expansion*

$$f(x) \sim x^u \sum_{n=0}^{\infty} b_n x^{-n}. \tag{10.10}$$

Then the integral mean of function f has the form

$$I_f(x+s, x+t) \sim x \sum_{n=0}^{\infty} a_n x^{-n} \tag{10.11}$$

and coefficients a_n satisfy the following recursive relation:

$$\begin{aligned} a_0 &= 1, \\ a_n &= -\frac{1}{b_0 u} \Bigg(\sum_{j=1}^{n} b_j E_{n-j}[a, u-j] \\ &\qquad + \frac{b_0}{n} \sum_{k=1}^{n-1} (k(1+u)-n) a_k E_{n-k}[a,u] - c_n \Bigg), \end{aligned} \tag{10.12}$$

where coefficients (c_n) are defined by (10.9).

EXAMPLE 10.5. Asymptotic expansion of generalized logarithmic mean. Suppose $f(x) = x^r$, $r \neq 0$, $r \neq -1$. Then the integral mean has the form (10.11) where $a_0 = 1$ and

$$a_n = -\frac{1}{r} \left(\frac{1}{n} \sum_{k=1}^{n-1} (k(1+r)-n) a_k E_{n-k}[a,r] - c_n \right)$$

where

$$c_n = \frac{1}{n+1} \binom{r}{n} \frac{t^{n+1} - s^{n+1}}{t-s}.$$

The first few coefficients are

$$\begin{aligned}
a_0 &= 1,\\
a_1 &= \alpha,\\
a_2 &= \tfrac{1}{6}(r-1)\beta^2,\\
a_3 &= -\tfrac{1}{6}(r-1)\alpha\beta^2,\\
a_4 &= \tfrac{1}{360}(r-1)\beta^2[(-2r^2-5r+13)\beta^2+60\alpha^2],\\
a_5 &= -\tfrac{1}{120}(r-1)\alpha\beta^2[(-2r^2-5r+13)\beta^2+20\alpha^2].
\end{aligned} \tag{10.13}$$

Here we use $\alpha = (s+t)/2$ and $\beta = (s-t)/2$.

10.4. Logarithmic case

This theorem does not cover initial example of integral mean of psi function, because of the logarithm term in asymptotic expansion. Introduction of logarithm term into function f

$$f(x) \sim b \cdot \log x + x^{-1} \sum_{n=0}^{\infty} b_n x^{-n} \tag{10.14}$$

leads to different algorithm. We shall give only the final form, interested readers can find details in [28].

THEOREM 10.6. *Let*

$$f(x) \sim b \log x + x^{-1} \sum_{n=0}^{\infty} b_n x^{-n}. \tag{10.15}$$

Then the coefficients from the asymptotic expansion

$$I_f(x+s, x+t) \sim x \sum_{n=0}^{\infty} a_n x^{-n} \tag{10.16}$$

can be calculated as follows

$$\begin{aligned}
a_0 &= 1,\\
a_n &= -\frac{d_n}{n} + \frac{c_{n-1}}{b} + \frac{1}{n}\sum_{k=1}^{n-1} kL_k a_{n-k} - \frac{1}{b}\sum_{k=0}^{n-1} b_k E_{n-1-k}[a, -k-1],
\end{aligned} \tag{10.17}$$

where

$$L_n = a_n - \frac{1}{n}\sum_{k=1}^{n-1} kL_k a_{n-k}. \tag{10.18}$$

Using this teorem one can deduce coefficients of integral means of psi function which are given before. Another example is asymptotic expansion of identric mean.

EXAMPLE 10.7. Identric $I(s,t)$ mean is integral mean of the function $f(x) = \log x$, therefore all coefficients b_k are equal to zero. Applying the algorithm stated in Theorem 10.6, we obtain the following coefficients:

$$I(x+s,x+t) \sim x\sum_{n=0}^{\infty} a_k x^{-k},$$

where

$$\begin{aligned} a_0 &= 1, \\ a_1 &= \alpha, \\ a_2 &= -\tfrac{1}{6}\beta^2, \\ a_3 &= \tfrac{1}{6}\alpha\beta^2, \\ a_4 &= -\tfrac{1}{360}\beta^2(60\alpha^2+13\beta^2), \\ a_5 &= \tfrac{1}{120}\alpha\beta^2(20\alpha^2+13\beta^2), \\ a_6 &= -\tfrac{1}{45360}\beta^2(7560\alpha^4+9828\alpha^2\beta^2+737\beta^4). \end{aligned} \tag{10.19}$$

Identric mean is the special case of generalized logarithmic mean, obtained by taking a limit $r \to 0$. So, it is not a surprise that these coefficients coincide to ones calculated in (10.13), if we choose there $r = 0$.

11. Asymptotic expansion of bivariate means

11.1. Bivariate means

Asymptotic expansion is very efficient tool in analysis of bivariate and multivariate means, since it describes behaviour of means when the data are translated by some large quantity x. Therefore, we will analyse the asymptotic behaviour of the function $F(x+s,x+t)$, where F is bivariate mean and x tends to ∞. By a bivariate mean we understand a function $M\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies

$$\min(s,t) \leqslant M(s,t) \leqslant \max(s,t).$$

It follows that $M(s,s) = s$ for all $s > 0$. Means considered here will be homogeneous and symmetric.

Condition of homogeneity is not necessary, but it will simplify comparison of means, and all classical bivariate and parameter means has this property. Some other means are not homogeneous, like quasi-arithmetic mean or integral mean of nonhomogeneous function.

If the mean is homogeneous, then we have

$$F(x+s,x+t) = xF(1+\frac{s}{x}, 1+\frac{t}{x}).$$

So, the asymptotic expansion is essentially equivalent to the power series expansion of the function $F(1+s,1+t)$ for small values of s and t. It will be shown that it is sufficient to consider symmetric case $F(1-t,1+t)$. H. W. Gould and M. E. Mays [35] consider expansion of the function $F(1,1+t)$, but this is non-symmetric case and results are not so clear as in our approach.

Let F be homogeneous bivariate mean. We expect to obtain asymptotic expansions of this mean in the form

$$F(x+s,x+t)=x+\frac{s+t}{2}+\sum_{n=2}^{\infty}c_n(t,s)x^{-n+1}. \tag{11.1}$$

Here, c_n are homogeneous polynomials of two variables of degree n, which depend, of course, also on the parameters of the involved mean.

Such expansions have many important properties. First, they are introduction for the similar analysis for general n-variable means. Further, they reveal many important properties of the means under consideration, for example, the comparison of various means. In [41], [45]–[48] similar problems were studied.

The notation will be much simpler with s and t being replaced by variables α and β, $t=\alpha+\beta$ and $s=\alpha-\beta$. Then

$$\alpha=\frac{t+s}{2},\qquad \beta=\frac{t-s}{2}.$$

We shall use also

$$\gamma=st=\alpha^2-\beta^2,\qquad \delta=\frac{s^2+t^2}{2}=\alpha^2+\beta^2.$$

In all examples, the asymptotic expansions will be stated in terms of α and β.

Finally, let us denote

$$S_n=t^n-s^n,\qquad T_n=\tfrac{1}{2}(s^n+t^n).$$

These sequences can be calculated by recursive relations

$$S_n=2\alpha S_{n-1}-\gamma S_{n-2},\qquad n\geqslant 2,$$

where $S_0=0$ and $S_1=2\beta$, and

$$T_n=2\alpha T_{n-1}-\gamma T_{n-2},\qquad n\geqslant 2,$$

where $T_0=1$ and $T_1=\alpha$.

11.2. Asymptotic expansions of bivariate classical means

N. Elezović and L. Vukšić [29] considered bivariate classical means, contraharmonic mean (N), quadratic mean (Q), centroidal mean (C), arithmetic mean (A), identric mean (I), Heronian mean (He), logarithmic mean (L), geometric mean (G), cologarithmic mean (cL), coidentric mean (cI), harmonic mean (H). They are listed in the falling order, since all these means are comparable:

$$N\geqslant Q\geqslant C\geqslant A\geqslant I\geqslant He\geqslant L\geqslant G\geqslant cL\geqslant cI\geqslant H.$$

If $M(s,t)$ is any of the means above, our intention is to find asymptotic expansion for the function $x \mapsto M(x+s,x+t)$ as $x \to \infty$.

For all x it holds

$$A(x+s,x+t) = x + A(s,t),$$

and this is (degenerated) asymptotic expansion of arithmetic mean. The asymptotic expansion of all other means are given, in explicit or recursive form. It is shown that it is sufficient to consider the case $\alpha = 0$, since general case can be deduced from this one. Denote: The main expansion has the form

$$M(x+s,x+t) \sim x + \alpha + \sum_{n=0}^{\infty} c_{n+2}(\alpha,\beta)x^{-n-1} \tag{11.2}$$

and this relation is valid for all values of s and t, not only for the positive ones. This can be written as

$$M(x+\alpha-\beta,x+\alpha+\beta) \sim x + \alpha + \sum_{n=0}^{\infty} c_{n+2}(\alpha,\beta)x^{-n-1}.$$

Now, choose in (11.2) $s=-\beta$, $t=\beta$. Then corresponding α is equal to zero, so

$$M(x-\beta,x+\beta) \sim x + \sum_{n=0}^{\infty} c_{n+2}(0,\beta)x^{-n-1},$$

and this holds for any sufficiently large x, hence, also for $x+\alpha$:

$$M(x+\alpha-\beta,x+\alpha+\beta) \sim x + \alpha + \sum_{n=0}^{\infty} c_{n+2}(0,\beta)(x+\alpha)^{-n-1}.$$

THEOREM 11.1. *The coefficients (c_n) of the asymptotic expansion of the mean M satisfy*

$$c_{n+2}(\alpha,\beta) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\alpha^{n-k}c_{k+2}(0,\beta), \qquad n \geqslant 0. \tag{11.3}$$

Proof. From the analysis above, one can write

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{n+2}(\alpha,\beta)x^{-n-1} &= \sum_{n=0}^{\infty} c_{n+2}(0,\beta)(x+\alpha)^{-n-1} \\
&= \sum_{n=0}^{\infty} c_{n+2}(0,\beta)\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}\alpha^k x^{-n-1-k} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\alpha^{n-k}c_{k+2}(0,\beta)\right)x^{-n-1},
\end{aligned}$$

which had to be proved. $\square$

However, we shall here repeat formulas for general expansions.

THEOREM 11.2. (Elezović-Vukšić (2014), [29]) *Classical bivariate means have following asymptotic expansions:*

$$M(x+s,x+t) = x \cdot \sum_{n=0}^{\infty} c_n x^{-n} \tag{11.4}$$

where $c_0 = 1$, $c_1 = \alpha$. *The other coefficients are given by*

1. *Geometric mean:*

$$c_n = \left(\frac{3}{n} - 2\right)\alpha c_{n-1} + \left(\frac{3}{n} - 1\right)(\alpha^2 - \beta^2)c_{n-2}. \tag{11.5}$$

2. *Quadratic mean:*

$$c_n = \left(\frac{3}{n} - 2\right)\alpha c_{n-1} + \left(\frac{3}{n} - 1\right)(\alpha^2 + \beta^2)c_{n-2}. \tag{11.6}$$

3. *Harmonic mean:*

$$c_n = (-1)^n \beta^2 \alpha^{n-1}. \tag{11.7}$$

4. *Contraharmonic mean:*

$$c_n = (-1)^{n-1} \beta^2 \alpha^{n-1}. \tag{11.8}$$

5. *Centroidal mean:*

$$c_n = \frac{(-1)^{n-1}}{3} \beta^2 \alpha^{n-1}. \tag{11.9}$$

6. *Logarithmic mean:*

$$c_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{S_{k+1}}{2(k+1)\beta} c_{n-k}. \tag{11.10}$$

7. *Cologarithmic mean:*

$$c_n = \frac{(-1)^n}{\beta(n+2)} \left[\frac{\alpha S_{n+1}}{n+1} - \frac{(\alpha^2 - \beta^2)S_n}{n}\right] x^{-n}. \tag{11.11}$$

8. *Identric mean:*

$$c_n = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} \frac{S_{k+1}}{2(k+1)\beta} c_{n-k}. \tag{11.12}$$

9. *Coidentric mean:*

$$c_n = \frac{1}{n} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2(k+1)\beta} [2k\alpha S_k - (2k+1)(\alpha^2 - \beta^2)S_{k-1}] c_{n-k}. \tag{11.13}$$

In the following table, the coefficients of some of these means are listed, with the choice $\alpha = 0$.

	x	β^2/x	β^4/x^3	β^6/x^5	β^8/x^7
N	1	1	0	0	0
Q	1	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{16}$	$-\frac{5}{128}$
A	1	0	0	0	0
I	1	$-\frac{1}{6}$	$-\frac{13}{360}$	$-\frac{737}{45360}$	$-\frac{50801}{5443200}$
L	1	$-\frac{1}{3}$	$-\frac{4}{45}$	$-\frac{44}{945}$	$-\frac{428}{14175}$
G	1	$-\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{5}{128}$
H	1	-1	0	0	0

For example, one has

$$L(x+s,x+t) \sim x - \frac{\beta^2}{3x} - \frac{4\beta^4}{45x^3} - \frac{44\beta^6}{945c^5} - \frac{428b^8}{14175x^7} + \dots.$$

As a first byproduct of these results, the following inequalities are proved in [29].

THEOREM 11.3. *Let $\alpha > 0$. Then the following inequalities are valid for all $x > -s$:*

$$\begin{aligned} G(x+s,x+t) &> x + \alpha - \frac{\beta^2}{2x}, \\ H(x+s,x+t) &> x + \alpha - \frac{\beta^2}{x}, \\ L(x+s,x+t) &> x + \alpha - \frac{\beta^2}{3x}, \\ I(x+s,x+t) &> x + \alpha - \frac{\beta^2}{6x}. \end{aligned} \tag{11.14}$$

For $\alpha \leqslant 0$ these inequalities are valid with opposite sign.

Note that the first neglected term in asymptotic expansion of the functions from the left is negative.

11.3. Asymptotic expansions of bivariate parameter means

Some classes of parameter means are important in applications, we shall refer here to generalized logarithmic mean, power means, Stolarsky and Gini means. Here is a short overview of results given by N. Elezović and L. Vukšić [30]. We shall restrict ourselves to the definition, algorithm for calculations of coefficients and the table with first few coefficients.

1. Generalized logarithmic mean. Let r be a real number. The generalized logarithmic mean is defined for all $s,t>0$ by

$$L_r(s,t)=\begin{cases}\left(\dfrac{t^{r+1}-s^{r+1}}{(r+1)(t-s)}\right)^{1/r}, & r\neq -1,0,\\ \dfrac{t-s}{\log t-\log s}, & r=-1,\\ \dfrac{1}{e}\left(\dfrac{t^t}{s^s}\right)^{1/(t-s)}, & r=0.\end{cases} \tag{11.15}$$

THEOREM 11.4. *Generalized logarithmic mean can be expanded into asymptotic series*

$$L_r(x+s,x+t)=x\sum_{n=0}^{\infty}c_nx^{-n},$$

where sequence (c_n) is defined by $c_0=1$ and

$$c_n=\frac{1}{2n}\sum_{k=1}^{n}\left(\frac{k}{r}-\frac{n}{r+1}\right)\binom{r+1}{k+1}\frac{S_{k+1}}{\beta}c_{n-k}. \tag{11.16}$$

The first few coefficients in this asymptotic expansion are:

$$\begin{aligned}
c_0&=1, & c_3&=-\tfrac{1}{6}(r-1)\alpha\beta^2,\\
c_1&=\alpha, & c_4&=-\tfrac{1}{360}(r-1)\beta^2[(2r^2+5r-13)\beta^2-60\alpha^2],\\
c_2&=\tfrac{1}{6}(r-1)\beta^2, & c_5&=\tfrac{1}{120}(r-1)\alpha\beta^2[(2r^2+5r-13)\beta^2-20\alpha^2].
\end{aligned}$$

Parameter means are defined by various formulas according to the full or special meaning of some of theirs parameters. But, there exists only one set of coefficients, for general value of parameters. Expansions of the critical cases which correspond to particular choice of parameters follow from these general values. For example, taking $r=-1$ and $r=0$ we obtain the special cases: the asymptotic expansion of logarithmic and identric mean:

$$\begin{aligned}
L(x+s,x+t)&=x+\alpha-\frac{\beta^2}{3x}+\frac{\alpha\beta^2}{3x^2}-\frac{\beta^2(15\alpha^2+4\beta^2)}{45x^3}+\frac{\alpha\beta^2(5\alpha^2+4\beta^2)}{15x^4},\\
I(x+s,x+t)&=x+\alpha-\frac{\beta^2}{6x}+\frac{\alpha\beta^2}{6x^2}-\frac{\beta^2(60\alpha^2+13\beta^2)}{360x^3}+\frac{\alpha\beta^2(20\alpha^2+13\beta^2)}{120x^4}.
\end{aligned}$$

2. Power mean. The r-th power mean is defined for all $s,t>0$ by

$$M_r(s,t)=\begin{cases}\left(\dfrac{t^r+s^r}{2}\right)^{1/r}, & r\neq 0,\\ \sqrt{st}, & r=0.\end{cases} \tag{11.17}$$

The important particular cases of this mean are arithmetic mean $A=M_1$, quadratic mean $Q=M_2$ and harmonic mean $H=M_{-1}$. Geometric mean $G=M_0$ is obtained as limit of means M_r for $r\to 0$.

THEOREM 11.5. *The power mean has the asymptotic expansion of the form*

$$M_r(x+s,x+t)=x\sum_{k=0}^{\infty}c_kx^{-k},$$

where $c_0=1$ *and*

$$c_n=\frac{1}{n}\sum_{k=1}^{n}\left[k\left(1+\frac{1}{r}\right)-n\right]\binom{r}{k}T_kc_{n-k}. \tag{11.18}$$

The first few coefficients are

$$\begin{aligned}
c_0&=1,\\
c_1&=\alpha,\\
c_2&=\tfrac{1}{2}(r-1)\beta^2,\\
c_3&=-\tfrac{1}{2}(r-1)\alpha\beta^2,\\
c_4&=\tfrac{1}{24}(r-1)\beta^2(12\alpha^2+(3+r-2r^2)\beta^2),\\
c_5&=-\tfrac{1}{8}(r-1)\alpha\beta^2(4\alpha^2+(3+r-2r^2)\beta^2).
\end{aligned}$$

3. Stolarsky mean or the extended mean of order p,r is defined for all $s,t>0$ by

$$E_{p,r}(s,t)=\left[\frac{r(t^p-s^p)}{p(t^r-s^r)}\right]^{1/(p-r)}, \qquad p\neq r,\ p,r\neq 0. \tag{11.19}$$

It is symmetric both on t and s as well on p and r. Therefore, we may suppose that $s\leqslant t$ and $r\leqslant p$. The excluded cases are obtained by limit procedure:

$$\begin{aligned}
E_{r,r}(s,t)&=\frac{1}{e^{1/r}}\left(\frac{t^{t^r}}{s^{s^r}}\right)^{1/(t^r-s^r)}, && r=p\neq 0,\\
E_{0,r}(s,t)&=\left[\frac{t^r-s^r}{r(\log t-\log s)}\right]^{1/r}, && r\neq 0,\\
E_{0,0}(s,t)&=\sqrt{st}.
\end{aligned}$$

Let us denote

$$a_n(q) = \binom{q}{n+1} \frac{t^{n+1} - s^{n+1}}{q(t-s)}. \tag{11.20}$$

Then the asymptotic expansion of Stolarsky mean can be obtained by the following algorithm.

THEOREM 11.6. *Let $r \neq p$, $r, p \neq 0$. The Stolarsky mean has the asymptotic expansion of the form*

$$E_{p,r}(x+s, x+t) = x \sum_{n=0}^{\infty} c_n x^{-n},$$

where (c_n) is obtained by following algorithm, $c_0 = 1$ and

$$b_0 = 1,$$

$$b_n = a_n(p) - \sum_{k=1}^{n} a_k(r) b_{n-k}, \qquad n \geqslant 1 \tag{11.21}$$

$$c_n = \frac{1}{n} \sum_{k=1}^{n} \left[k\left(1 + \frac{1}{p-r}\right) - n \right] b_k c_{n-k}, \qquad n \geqslant 1. \tag{11.22}$$

The first few coefficients are:

$$\begin{aligned}
c_0 &= 1, \qquad c_1 = \alpha, \\
c_2 &= \tfrac{1}{6}(-3+p+r)\beta^2, \\
c_3 &= -\tfrac{1}{6}(-3+p+r)\alpha\beta^2, \\
c_4 &= \tfrac{1}{360}\beta^2 \Big[60(p+r-3)\alpha^2 + (-2(p+r)(p^2+r^2) \\
&\qquad + 5(p+r)^2 + 10(p+r) - 45)\beta^2 \Big], \\
c_5 &= -\tfrac{1}{120}\alpha\beta^2 \Big[20(p+r-3)\alpha^2 + (-2(p+r)(p^2+r^2) \\
&\qquad + 5(p+r)^2 + 10(p+r) - 45)\beta^2 \Big].
\end{aligned}$$

4. Gini means are defined for all $s, t > 0$ by

$$G_{p,r}(s,t) = \begin{cases} \left(\dfrac{t^p + s^p}{t^r + s^r} \right)^{\frac{1}{p-r}}, & p \neq r, \\ \exp\left(\dfrac{s^p \log s + t^p \log t}{s^p + t^p} \right), & p = r \neq 0, \\ \sqrt{st}, & p = r = 0. \end{cases} \tag{11.23}$$

Some of the special cases of the Gini means are power mean $G_{0,r} = M_r$ and Lehmer mean $G_{r+1,r}$.

Let us denote

$$a_k(q) = \binom{q}{k} \frac{t^k + s^k}{2}.$$

THEOREM 11.7. *Let $r \neq p$ and $r, p \neq 0$. The Gini mean has asymptotic expansion*

$$G_{p,r}(x+t, x+s) = x \sum_{n=0}^{\infty} c_n x^{-n},$$

where coefficients c_n are obtained by the following algorithm:

$$\begin{aligned} c_0 &= 1; \\ c_n &= \frac{1}{n} \sum_{k=1}^{n} \left[k \left(1 + \frac{1}{p-r} \right) - n \right] b_k c_{n-k}, \end{aligned} \tag{11.24}$$

and

$$b_n = a_n(p) - \sum_{k=1}^{n} a_k(r) b_{n-k}. \tag{11.25}$$

The first few coefficients are

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \alpha, \\ c_2 &= \tfrac{1}{2}(p+r-1)\beta^2, \\ c_3 &= -\tfrac{1}{2}(p+r-1)\alpha\beta^2, \\ c_4 &= \tfrac{1}{24}\beta^2[12(p+r-1)\alpha^2 \\ &\qquad + (-3-2p^3+p^2(3-2r)+2r+3r^2-2r^3+p(2+6r-2r^2))\beta^2], \\ c_5 &= \tfrac{1}{8}\alpha\beta^2[-4(-1+p+r)\alpha^2 \\ &\qquad + (3+2p^3-2r-3r^2+2r^3+p^2(-3+2r)+2p(-1-3r+r^2))\beta^2], \\ &\vdots \end{aligned}$$

COROLLARY 11.8. (Lehmer mean) *The asymptotic expansion of Lehmer mean*

$$G_{r+1,r}(s,t) = \frac{t^{r+1} + s^{r+1}}{t^r + s^r} \tag{11.26}$$

reads as folows:

$$G_{r+1,r}(x+s, x+t) = x \sum_{n=0}^{\infty} a_n x^{-n},$$

where

$$\begin{aligned} c_0 &= 1, \\ c_n &= \frac{1}{n} \sum_{k=1}^{n} (2k-n) b_k c_{n-k}, \end{aligned} \tag{11.27}$$

$$b_n = a_n(r+1) - \sum_{k=1}^{n} a_k(r) b_{n-k}. \tag{11.28}$$

The first few coefficients are

$$
\begin{aligned}
c_0 &= 1,\\
c_1 &= \alpha,\\
c_2 &= r\beta^2,\\
c_3 &= -r\alpha\beta^2,\\
c_4 &= \tfrac{1}{3}r\beta^2[3\alpha^2-(r^2-1)\beta^2],\\
c_5 &= r\alpha\beta^2[-\alpha^2+(r^2-1)\beta^2],\\
c_6 &= \frac{1}{15}r\beta^2[15\alpha^4-30(r^2-1)\alpha^2\beta^2+(2r^4-5r^2+3)\beta^4],\\
&\vdots
\end{aligned}
$$

12. Comparison of means

Knowing asymptotic expansion of means is a powerful tool for comparison of its values. Let us repeat some definitions from [22, 29, 30].

DEFINITION 12.1. Let M be bivariate function, and

$$M(x+s,x+t)=c_k(s,t)x^{-k+1}+O(x^{-k}). \tag{12.1}$$

If $c_k(s,t)>0$ for all s and t, then we say that M is *asymptotically* greater than zero, and write $M\succ 0$.

If $M(x+s,x+t)\geqslant 0$ for all values $x,s,t>0$, then $M\succ 0$. Namely, for x large enough, the sign of $M(x+s,x+t)$ is the same as the sign of the first term in its asymptotic expansion.

Therefore, one may consider asymptotic inequalities as a necessary condition for the inequality between comparable means.

For example, in [26] the following expansions are derived

$$
\begin{aligned}
Q(x+s,x+t)&=x+\alpha+\frac{\beta^2}{2x}-\frac{\alpha\beta^2}{2x^2}+\frac{\beta^2(4\alpha^2-\beta^2)}{8x^3}+\frac{\alpha\beta^2(4\alpha^2+3\beta^2)}{8x^4}+\dots,\\
G(x+s,x+t)&=x+\alpha-\frac{\beta^2}{2x}+\frac{\alpha\beta^2}{2x^2}-\frac{\beta^2(4\alpha^2+\beta^2)}{8x^3}+\frac{\alpha\beta^2(4\alpha^2+3\beta^2)}{8x^4}+\dots.
\end{aligned}
$$

Since it holds $A(x+s,x+t)=x+\alpha$, adding the previous two expansions one obtain

$$2A-G-Q\sim\frac{\beta^4}{4x^3}.$$

Hence,

$$2A-G-Q\succ 0.$$

This is a strong suggestion that the inequality

$$2A - G - Q \geqslant 0$$

may be valid. This is of course the true inequality.

The similar analysis can be made for various combination of means. Such analysis was done in a series of papers [22, 30, 53], unifying various particular results in this field. See, for example, paper [31] in this issue of the journal.

Calculation of asymptotic expansion is possible for means defined by limit procedures, arithmetic-geometric mean is such example. By knowing coefficients of these means enables us to determine its approximate value for large arguments, and also to compare it with other means. See, for example, paper [12] in this issue of the journal.

Acknowledgement. This work has been fully supported by Croatian Science Foundation under the project 5435.

REFERENCES

[1] J. ABAD, J. SESMA, *Two new asymptotic expansions of the ratio of two gamma functions*, J. Comput. Appl. Math. **173** (2005), 359–363.

[2] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.

[3] N. BATIR, *Very accurate approximations for the factorial function*, J. Math. Inequal., **4**, 3 (2010), 335–344.

[4] P. S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, 2003.

[5] P. S. BULLEN, D. S. MITRINOVIĆ, P. M. VASIĆ, *Means and theirs inequalities*, D. Reidel, Dordrecht, 1988.

[6] J. BUKAC, T. BURIĆ AND N. ELEZOVIĆ, *Stirling's formula revisited via some new and classical inequalities*, Math. Inequal. Appl. **14** (2011), 235–245.

[7] T. BURIĆ, N. ELEZOVIĆ, *Bernoulli polynomials and asymptotic expansions of the quotient of gamma functions*, J. Comput. Appl. Math., **235**, 11 (2011), 3315–3331.

[8] T. BURIĆ, N. ELEZOVIĆ, *New asymptotic expansions of the gamma function and improvements of Stirling's type formulas*, J. Comput. Anal. Appl., **13**, 4 (2011), 785–795.

[9] T. BURIĆ, N. ELEZOVIĆ, *New asymptotic expansions of the quotient of gamma functions*, Integral Transforms Spec. Funct., **23** (2012), 355–368.

[10] T. BURIĆ, N. ELEZOVIĆ, *Approximations of the Euler-Mascheroni constant and harmonic numbers*, Appl. Math. Comput., **222** (2013), 604–611.

[11] T. BURIĆ, N. ELEZOVIĆ, *Asymptotic expansions of the binomial coefficients*, J. Appl. Math. Comput., **46** (2014), 135–145.

[12] T. BURIĆ, N. ELEZOVIĆ, *Asymptotic expansion of the arithmetic-geometric mean and related inequalities*, J. Math. Inequal. 9, 4 (2015), 1181–1190.

[13] T. BURIĆ, N. ELEZOVIĆ, R. ŠIMIĆ, *Asymptotic expansions of the multiple quotients of gamma functions with applications*, Math. Inequal. Appl., **16**, 4 (2013), 1159–1170.

[14] T. BURIĆ, N. ELEZOVIĆ, L. VUKŠIĆ, *Asymptotic expansions of the gamma function and Wallis function through polygamma functions*, Integral Transorms Spec. Funct., **25** (2014), 163–172.

[15] T. BURIĆ, N. ELEZOVIĆ, L. VUKŠIĆ, *Appell polynomials and asymptotic expansions*, Mediterranian J. Math., to appear, DOI 10.1007/s00009-015-0529-z

[16] J. BUSTOZ, M. E. M. ISMAIL, *On Gamma Function Inequalities* Math. Comp. **47** (1986), 659–667.

[17] C.-P. CHEN, N. ELEZOVIĆ, L. VUKŠIĆ, *Asymptotic formulae associated with the Wallis power function and digamma function*, J. Class. Anal., **2**, 2 (2013), 151–166.

[18] C.-P. CHEN, N. ELEZOVIĆ, L. VUKŠIĆ, *Asymptotic expansions of integral mean of polygamma functions*, Math. Inequal. Appl., **18**, 1 (2015), 255–266.

[19] C. P. CHEN, F. QI, *The best bounds in Wallis' inequality*, Proc. Amer. Math. Soc., **133**, 2 (2005), 397–401.

[20] L. COMTET, *Advanced combinatorics*, Dordrecht, D. Reidel Publishing Company, 1974.

[21] N. ELEZOVIĆ, *Asymptotic expansions of central binomial coefficients and Catalan numbers*, J. Integer Sequences., **17**, 2 (2014), 1–14.

[22] N. ELEZOVIĆ, *Asymptotic inequalities and comparison of classical means*, J. Math. Inequal., **9**, 1 (2015), 177–196.

[23] N. ELEZOVIĆ, *Estimations of psi function and harmonic numbers*, Appl. Math. Comput. **258** (2015), 192–205.

[24] N. ELEZOVIĆ, *Generalized Bernoulli polynomials and numbers, revisited*, Mediterranean J. Math., to appear, DOI 10.1007/s00009-014-0498-7

[25] N. ELEZOVIĆ, C. GIORDANO AND J. PEČARIĆ, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252.

[26] N. ELEZOVIĆ, L. LIN, L. VUKŠIĆ, *Inequalities and asymptotic expansions of the Wallis sequence and the sum of the Wallis ratio*, J. Math. Inequal., **7**, 4 (2013), 679–695.

[27] N. ELEZOVIĆ, J. PEČARIĆ, *Differential and integral f-means and applications to digamma function*, Math. Inequal. Appl. **3** (2000), 189–196.

[28] N. ELEZOVIĆ, L. VUKŠIĆ, *Asymptotic expansions of integral means and applications to the ratio of gamma functions*, Appl. Math. Comput., **235** (2014), 187–200.

[29] N. ELEZOVIĆ, L. VUKŠIĆ, *Asymptotic expansions of bivariate classical means and related inequalities* J. Math. Inequal., **8**, 4 (2014), 707–724.

[30] N. ELEZOVIĆ, L. VUKŠIĆ *Asymptotic expansions and comparison of bivariate parameter means*, Math. Inequal. Appl., **17**, 4 (2014) 1225–1244.

[31] N. ELEZOVIĆ, L. VUKŠIĆ, Neuman-Sándor mean, asymptotic expansions and related inequalities, J. Math. Inequal, **9**, 4 (2015), 1337–1348.

[32] A. ERDÉLYI, *Asymptotic expansions*, Dover Publications, New York, 1956.

[33] JERRY L. FIELDS, *A note on the asymptotic expansion of a ratio of gamma functions*, Proc. Edinburgh Math. Soc., **15** (1966), 43–45.

[34] W. GAUTSCHI, *Inequalities for gamma and incomplete gamma function*, J. Math. Phys. **39** (1959), 77–81.

[35] H. W. GOULD, M. E. MAYS, *Series expansions of means*, J. Math. Anal. Appl. **101** (1984), 611–621.

[36] P. HENRICI *Applied and computational complex analysis*, Vol. 1, John Willey & Sons, New York, 1974.

[37] E. A. KARATSUBA, *On the asymptotic representation of the Euler gamma function by Ramanujan*, J. Comput. Appl. Math. **135** (2001), 225–240.

[38] D. K. KAZARINOFF, *On Wallis' formula*, Edinburgh Math. Notes, **40** (1956), 19–21.

[39] D. KERSHAW, *Some extentions of W. Gautschi's inequalities for the gamma function* Math. of Comp. **41** (1983), 607–611.

[40] D. KERSHAW, *Upper and lower bounds for a ratio involving the gamma function*, Anal. Appl. (Singap.) **3** (2005), 293–295.

[41] E. B. LEACH, M. C. SHOLANDER, *Extended mean values II*, J. Math. Anal. Appl. **92** (1983) 207–223.

[42] Y. L. LUKE, *The Special Functions and Their Approximations*, Vol. I, Academic Press, New York, 1969.

[43] LUSCHNY, web page at `http://www.luschny.de/math/factorial/`.

[44] S. MATTAREI, *Asymptotics of partial sums of central binomial coefficients and Catalan numbers*, preprint, `http://arXiv.math.CO/0906.4290`.

[45] E. NEUMAN, ZS. PÁLES, *On comparison of Stolarsky and Gini means*, J. Math. Anal. Appl. **278** (2003) 274–284.

[46] E. NEUMAN, J. SANDOR *Inequalities involving Stolarsky and Gini means*, Mathematica Pannonica 14/1 (2003) 29–44.

[47] ZS. PÁLES, *Inequalities for differences of powers*, J. Math. Anal. Appl. **131** (1988) 271–281.

[48] ZS. PÁLES, *Inequalities for sums of powers*, J. Math. Anal. Appl. **131** (1988) 265–270.

[49] F. QI *A new lower bound in the second Kershaw's double inequality*, J. Comput. Appl. Math. **214** (2008), 610–616.

[50] F. QI. *Bounds for the ratio of two gamma functions*, J. Inequal. Appl., Vol. 2010, Article ID 493058, (201), 84 p.

[51] S. RAMANUJAN, *The lost notebook and other unpublished papers*, Intr. by G. E. Andrews, Narosa Publ. H.-Springer, New Delhi-Berlin, 1988.

[52] F. G. TRICOMI AND A. ERDÉLYI, *The asymptotic expansion of a ratio of Gamma functions*, Pacific. J. Math., **1** (1951), 133–142.

[53] L. VUKŠIĆ, *Seiffert means, asymptotic expansions and related inequalities*, Rad HAZU, to appear.

[54] J. WALLIS, *Arithmetica Infinitorum*, Oxford, England, 1656; Facsimile of relevant pages available in: J. A. Stedall, Catching Proteus: The collaborations of Wallis and Brouncker. I. Squaring the circle, Notes and Records Roy. Soc. London **54** (3) (2000), 293–316.

[55] G. N. WATSON, *A note on gamma function*, Edinburgh Math. Notes, **42** (1959), 7–9.

(Received November 17, 2014)

Neven Elezović
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3, 10000 Zagreb, Croatia
e-mail: neven.elez@fer.hr