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A REFINEMENT OF THE JESSEN–MERCER INEQUALITY AND A GENERALIZATION ON CONVEX HULLS IN $\mathbb{R}^k$

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Abstract. A refinement of the Jessen-Mercer inequality is obtained and shown to be an improvement of the upper bound for the Jessen's difference given in [12]. Also a generalization of the Jessen-Mercer inequality for convex functions on convex hulls in $\mathbb{R}^k$ is given and demonstrated to be an improvement of the inequalities obtained in [3]. An elegant method of producing n-exponentially convex and exponentially convex functions is applied using the Jessen-Mercer differences. Lagrange and Cauchy mean value type theorems are proved and shown to be useful in studying Stolarsky type means defined by using the Jessen-Mercer differences.

1. Introduction

Let E be a nonempty set and L be a subspace of the vector space $\mathbb{R}^E$ over $\mathbb{R}$ which contains $\mathbf{1}$, that is L having following properties

$L1$: $f,g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

$L2$: $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$.

If L additionally has property

$L3$: $(\forall f,g \in L)\ (\min\{f,g\} \in L \wedge \max\{f,g\} \in L)$,

then it is a *lattice*. Obviously, $\left(\mathbb{R}^E, \leqslant\right)$ (with standard ordering) is a lattice. It can also be easily verified that a subspace $X \subseteq \mathbb{R}^E$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of the fact that for every $x \in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|(t) = |x(t)|, \quad x^+(t) = \max\{0, x(t)\}, \quad x^-(t) = -\min\{0, x(t)\}, \qquad t \in E,$$

and $x^+ + x^- = |x|$, $x^+ - x^- = x$,

$$\min\{x,y\} = \frac{1}{2}(x + y - |x-y|), \quad \max\{x,y\} = \frac{1}{2}(x + y + |x-y|). \tag{1.1}$$

We consider *positive linear functionals* $A: L \to \mathbb{R}$, i.e., functionals having properties

$A1$: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f,g \in L$, $\alpha, \beta \in \mathbb{R}$ (linearity);

$A2$: $f \in L$, $f(t) \geqslant 0$ on $E \Rightarrow A(f) \geqslant 0$ (positivity).

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If A additionally satisfies the condition $A(\mathbf{1}) = 1$, then it is *normalized*.

Throughout the paper when using interval $[m,M]$ we assume that $-\infty < m < M < \infty$.

The following result is a variant of the well known Jessen's inequality [6] (see also [14, p. 47]) of Mercer's type [10] proved in [3]. We call it the Jessen-Mercer inequality.

THEOREM A. *Let L satisfy $L1$, $L2$ on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m,M]$, then for all $f \in L$ such that $\varphi(f)$, $\varphi(m+M-f) \in L$ (so that $m \leqslant f(t) \leqslant M$ for all $t \in E$), we have*

$$\varphi(m+M-A(f)) \leqslant \varphi(m)+\varphi(M)-A(\varphi(f)). \tag{1.2}$$

REMARK 1. In fact, to be more specific, the following series of inequalities was proved

$$\begin{aligned}\varphi(m+M-A(f)) &\leqslant A(\varphi(m+M-f)) \\ &\leqslant \frac{M-A(f)}{M-m}\varphi(M)+\frac{A(f)-m}{M-m}\varphi(m) \\ &\leqslant \varphi(m)+\varphi(M)-A(\varphi(f)).\end{aligned} \tag{1.3}$$

Furthermore, if the function φ is concave, inequalities (1.2) and (1.3) are reversed.

The reversed Jensen inequality follows easily from Jensen's inequality (see [14]).

THEOREM B. *Let $\boldsymbol{p}$ be a real n-tuple such that*

$$p_1 > 0, \quad p_i \leqslant 0 (i = 2,\dots,n), \quad P_n > 0$$

where $P_n = \sum_{i=1}^{n} p_i$. Let U be a convex set in a real vector space M, $\boldsymbol{x}_i \in U$ $(i = 1,\dots,n)$ and $\frac{1}{P_n}\sum_{i=1}^{n} p_i\boldsymbol{x}_i \in U$. If $f\colon U \to R$ is a convex function, then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i\boldsymbol{x}_i\right) \geqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i). \tag{1.4}$$

Recently, M. Klaričić Bakula *et al.* [8] obtained the following refinement of the converse Jessen's inequality [1] (see also [14, p. 98]).

THEOREM C. *Let L satisfy $L1$, $L2$, $L3$ on a nonempty set E, and let A be a positive normalized linear functional. If φ is a convex function on $[m,M]$ then for all $g \in L$ such that $\varphi(g) \in L$ we have $A(g) \in [m,M]$ and*

$$A(\varphi(g)) \leqslant \frac{M-A(g)}{M-m}\varphi(m)+\frac{A(g)-m}{M-m}\varphi(M)-A(\tilde{g})\,\delta_\varphi, \tag{1.5}$$

where

$$\tilde{g}=\frac{1}{2}-\frac{1}{M-m}\left|g-\frac{m+M}{2}\right|, \quad \delta_{\varphi}=\varphi(m)+\varphi(M)-2\varphi\left(\frac{m+M}{2}\right).$$

Using inequality (1.5) and the following Lemma we will refine the series of inequalities (1.3).

LEMMA 1. *Let ϕ be a convex function on U where U is a convex set in $\mathbb{R}^k$, $(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\in U^n$ and $\boldsymbol{p}=(p_1,\ldots,p_n)$ is nonnegative n-tuple such that $\sum_{i=1}^{n}p_i=1$. Then*

$$\begin{aligned}
&\min\{p_1,\ldots,p_n\}\left[\sum_{i=1}^{n}\phi(\boldsymbol{x}_i)-n\phi\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right)\right]\\
&\leqslant\sum_{i=1}^{n}p_i\phi(\boldsymbol{x}_i)-\phi\left(\sum_{i=1}^{n}p_i\boldsymbol{x}_i\right) &(1.6)\\
&\leqslant\max\{p_1,\ldots,p_n\}\left[\sum_{i=1}^{n}\phi(\boldsymbol{x}_i)-n\phi\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right)\right]. &(1.7)
\end{aligned}$$

Proof. This is a simple consequence of [11, p. 717, Theorem 1]. □

2. Refinement of the Jessen-Mercer inequality

Next two theorems are our main results.

THEOREM 1. *Let L satisfy $L1$, $L2$, $L3$ on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m,M]$, then for all $f\in L$ such that $\varphi(f)$, $\varphi(m+M-f)\in L$, we have*

$$\begin{aligned}
\varphi(m+M-A(f))&\leqslant A(\varphi(m+M-f))\\
&\leqslant\frac{M-A(f)}{M-m}\varphi(M)+\frac{A(f)-m}{M-m}\varphi(m)-A\left(\frac{1}{2}-\frac{1}{M-m}\left|f-\frac{m+M}{2}\right|\right)\delta_{\varphi}\\
&\leqslant\varphi(m)+\varphi(M)-A(\varphi(f))-\left[1-\frac{2}{M-m}A\left(\left|f-\frac{m+M}{2}\right|\right)\right]\delta_{\varphi}\\
&\leqslant\varphi(m)+\varphi(M)-A(\varphi(f)),
\end{aligned}$$

where

$$\delta_{\varphi}=\varphi(M)+\varphi(m)-2\varphi\left(\frac{M+m}{2}\right). \tag{2.1}$$

Proof. Using the first inequality from the series (1.3) and applying inequality (1.5) first to the function $m+M-f$, and then to the function f, we obtain

$$\begin{aligned}
&\varphi(m+M-A(f)) \\
&\leqslant A(\varphi(m+M-f)) \\
&\leqslant \frac{M-A(f)}{M-m}\varphi(M)+\frac{A(f)-m}{M-m}\varphi(m)-A\left(\frac{1}{2}-\frac{1}{M-m}\left|f-\frac{m+M}{2}\right|\right)\delta_{\varphi} \\
&=\varphi(m)+\varphi(M)-\left[\frac{M-A(f)}{M-m}\varphi(m)+\frac{A(f)-m}{M-m}\varphi(M)\right]-A\left(\frac{1}{2}-\frac{1}{M-m}\left|f-\frac{m+M}{2}\right|\right)\delta_{\varphi} \\
&\leqslant \varphi(m)+\varphi(M)-A(\varphi(f))-2A\left(\frac{1}{2}-\frac{1}{M-m}\left|f-\frac{m+M}{2}\right|\right)\delta_{\varphi} \\
&=\varphi(m)+\varphi(M)-A(\varphi(f))-\left[1-\frac{2}{M-m}A\left(\left|f-\frac{m+M}{2}\right|\right)\right]\delta_{\varphi}, \\
&\leqslant \varphi(m)+\varphi(M)-A(\varphi(f)).
\end{aligned}$$

The last inequality is a simple consequence of the easily provable facts that $\delta_{\varphi}=\varphi(M)+\varphi(m)-2\varphi\left(\frac{M+m}{2}\right)\geqslant 0$ and $1-\frac{2}{M-m}A\left(\left|f-\frac{m+M}{2}\right|\right)\geqslant 0$. □

THEOREM 2. *Let L satisfy L1, L2, L3 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m,M]$, then for all $f\in L$ such that $\varphi(f)$, $\varphi(m+M-f)\in L$, we have*

$$\begin{aligned}
&\varphi(m+M-A(f)) \\
&\leqslant \frac{M-A(f)}{M-m}\varphi(M)+\frac{A(f)-m}{M-m}\varphi(m)-\left(\frac{1}{2}-\frac{1}{M-m}\left|A(f)-\frac{m+M}{2}\right|\right)\delta_{\varphi} \\
&\leqslant \varphi(m)+\varphi(M)-A(\varphi(f))-\left[1-\frac{1}{M-m}\left(A\left(\left|f-\frac{m+M}{2}\right|\right)+\left|A(f)-\frac{m+M}{2}\right|\right)\right]\delta_{\varphi} \\
&\leqslant \varphi(m)+\varphi(M)-A(\varphi(f))-\left[1-\frac{2}{M-m}A\left(\left|f-\frac{m+M}{2}\right|\right)\right]\delta_{\varphi} \\
&\leqslant \varphi(m)+\varphi(M)-A(\varphi(f)),
\end{aligned}$$

where δ_{φ} is defined as in (2.1).

Proof. Theorem C for the function f gives us

$$A(\varphi(f))\leqslant \frac{M-A(f)}{M-m}\varphi(m)+\frac{A(f)-m}{M-m}\varphi(M)-A\left(\frac{1}{2}-\frac{1}{M-m}\left|f-\frac{m+M}{2}\right|\right)\delta_{\varphi}. \tag{2.2}$$

Let the functions $p,q\colon [m,M]\to[0,1]$ be defined by

$$p(t)=\frac{M-t}{M-m},\qquad q(t)=\frac{t-m}{M-m}.$$

For any $t \in [m,M]$ we can write

$$\varphi(t) = \varphi\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right) = \varphi(p(t)m + q(t)M).$$

By Lemma 1 for $n = 2$ it follows

$$\varphi(t) \leqslant p(t)\varphi(m) + q(t)\varphi(M) - \min\{p(t), q(t)\}\delta_{\varphi},$$

where $\delta_{\varphi} = \varphi(M) + \varphi(m) - 2\varphi\left(\frac{M+m}{2}\right)$. Using (1.1) we can write it in the form

$$\varphi(t) \leqslant \frac{M-t}{M-m}\varphi(m) + \frac{t-m}{M-m}\varphi(M) - \left(\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|\right)\delta_{\varphi}.$$

Substituting $t \leftrightarrow A(g)$, where $g \in L$ such that $A(g) \in [m,M]$, we get

$$\varphi(A(g)) \leqslant \frac{M-A(g)}{M-m}\varphi(m) + \frac{A(g)-m}{M-m}\varphi(M) - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(g) - \frac{m+M}{2}\right|\right)\delta_{\varphi}. \tag{2.3}$$

Now, applying inequality (2.3) on $g = m + M - f$ (and using linearity and normality of A), and then using inequality (2.2), we have

$$\begin{aligned}
&\varphi(m+M-A(f)) \\
&\leqslant \frac{M-A(f)}{M-m}\varphi(M) + \frac{A(f)-m}{M-m}\varphi(m) - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_{\varphi} \\
&= \varphi(m) + \varphi(M) - \left[\frac{M-A(f)}{M-m}\varphi(m) + \frac{A(f)-m}{M-m}\varphi(M)\right] \\
&\quad - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_{\varphi} \\
&\leqslant \varphi(m) + \varphi(M) - A(\varphi(f)) \\
&\quad - A\left(\frac{1}{2} - \frac{1}{M-m}\left|f - \frac{m+M}{2}\right|\right)\delta_{\varphi} - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_{\varphi} \\
&= \varphi(m) + \varphi(M) \\
&\quad - A(\varphi(f)) - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_{\varphi} \\
&\leqslant \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{2}{M-m}A\left(\left|f - \frac{m+M}{2}\right|\right)\right]\delta_{\varphi}.
\end{aligned}$$

The last inequality is obtained applying Jessen's inequality to the continuous and convex function $|x|$ so that

$$\left|A(f) - \frac{m+M}{2}\right| = \left|A\left(f - \frac{m+M}{2}\right)\right| \leqslant A\left(\left|f - \frac{m+M}{2}\right|\right). \qquad \square$$

Using Theorem 2 we can get an upper bound for the difference $A(\varphi(f)) - \varphi(A(f))$ obtained in [12].

COROLLARY 1. *Let L satisfy $L1$, $L2$, $L3$ on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m,M]$, then for all $f \in L$ such that $\varphi(f), \varphi(m+M-f) \in L$, we have*

$$A(\varphi(f)) - \varphi(A(f)) \leqslant \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\delta_\varphi,$$

where δ_φ is defined as in (2.1).

Proof. Theorem 2 gives us

$$\begin{aligned} A(\varphi(f)) \leqslant \ & \varphi(m) + \varphi(M) - \varphi(A(f) - m + M) \\ & - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_\varphi. \end{aligned} \tag{2.4}$$

Since the function φ is convex, it follows

$$\varphi(m+M-A(f)) + \varphi(A(f)) \geqslant 2\varphi\left(\frac{M+m}{2}\right). \tag{2.5}$$

Combining inequalities (2.4) and (2.5) we obtain

$$\begin{aligned} & A(\varphi(f)) - \varphi(A(f)) \\ & \leqslant \varphi(m) + \varphi(M) - [\varphi(m+M-A(f)) + \varphi(A(f))] \\ & \quad - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_\varphi \\ & \leqslant \varphi(m) + \varphi(M) - 2\varphi\left(\frac{M+m}{2}\right) \\ & \quad - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_\varphi \\ & = \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\delta_\varphi. \quad \square \end{aligned}$$

3. Generalization on convex hulls in $\mathbb{R}^k$

We present a generalization of the Jessen-Mercer inequality for convex functions on convex hulls in $\mathbb{R}^k$.

Convex hull of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ is the set

$$\left\{\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i \middle| \alpha_i \in \mathbb{R}, \alpha_i \geqslant 0, \sum_{i=1}^{n} \alpha_i = 1\right\}$$

and it is represented by $K = co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$.

Barycentric coordinates over K are continuous real functions $\lambda_1,\ldots,\lambda_n$ on K with the following properties:

$$\lambda_i(\boldsymbol{x}) \geqslant 0, \quad i=1,...,n$$
$$\sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) = 1$$
$$\boldsymbol{x} = \sum_{i=1}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i. \tag{3.1}$$

If $\boldsymbol{x}_2-\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n-\boldsymbol{x}_1$ are linearly independent vectors, then each $\boldsymbol{x}\in K$ can be written in the unique way as a convex combination of $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n$ in the form (3.1).

We also consider k-simplex $S=co(\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}\})$ in $\mathbb{R}^k$ which is a convex hull of its vertices $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}\in\mathbb{R}^k$, where vertices $\boldsymbol{v}_2-\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}-\boldsymbol{v}_1\in\mathbb{R}^k$ are lineary independent. In this case we denote k-simplex by $S=[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}]$. Barycentric coordinates $\lambda_1,\lambda_2,\ldots,\lambda_{k+1}$ over S are nonnegative linear polynomials on S and have a special form (see [2]).

With L^k we denote the linear class of functions $\boldsymbol{g}\colon E\to\mathbb{R}^k$ defined by

$$\boldsymbol{g}(t)=(g_1(t),\ldots,g_k(t)), \quad g_i\in L \quad (i=1,\ldots,k).$$

For a given linear functional A, we also consider linear operator $\widetilde{A}=(A,\ldots,A)\colon L^k\to\mathbb{R}^k$ defined by

$$\widetilde{A}(\boldsymbol{g})=(A(g_1),\ldots,A(g_k)). \tag{3.2}$$

If $A(\boldsymbol{1})=1$ is satisfied, then using (A1) we also have

$A3$: $A(f(\boldsymbol{g}))=f(\widetilde{A}(\boldsymbol{g}))$ for every linear function f on $\mathbb{R}^k$.

For $n\in\mathbb{N}$ we denote

$$\Delta_{n-1}=\left\{(\mu_1,\ldots,\mu_n)\colon \mu_i\geqslant 0, i\in\{1,\ldots,n\}, \sum_{i=1}^{n}\mu_i=1\right\}.$$

Also, if φ is a function defined on a convex subset $U\subseteq\mathbb{R}^k$ and $\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n\in U$, we denote

$$S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)=\sum_{i=1}^{n}\varphi(\boldsymbol{x}_i)-n\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right).$$

Let's notice that δ_φ from Theorem B is equal to $S_\varphi^2(m,M)$.

Obviously, if φ is convex, $S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\geqslant 0$.

The next variant of Jensen's inequality was proved by A. Matković and J. Pečarić in [9].

THEOREM 3. *Let U be a convex subset in $\mathbb{R}^k$, $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\in U$ and $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m\in co(\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\})$. If φ is a convex function on U, then the inequality*

$$\varphi\left(\frac{\sum_{i=1}^{n}p_i\boldsymbol{x}_i-\sum_{j=1}^{m}w_j\boldsymbol{y}_j}{P_n-W_m}\right)\leqslant\frac{\sum_{i=1}^{n}p_i\varphi(\boldsymbol{x}_i)-\sum_{j=1}^{m}w_j\varphi(\boldsymbol{y}_j)}{P_n-W_m} \tag{3.3}$$

holds for all positive real numbers $p_1,\ldots,p_n$ *and* $w_1,\ldots,w_m$ *satisfying the condition*

$$p_i \geqslant W_m \quad \text{for all} \quad i=1,\ldots,n,$$

where $P_n=\sum_{i=1}^{n} p_i$ *and* $W_m=\sum_{j=1}^{m} w_j$.

Next theorem generalizes and improves Theorem 3.

THEOREM 4. *Let* L *satisfy properties L1, L2, L3 on a nonempty set* E, A *be a positive linear functional on* L *and* $\widetilde{A}$ *defined as in (3.2). Let* $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n \in \mathbb{R}^k$ *and* $K=co(\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\})$. *Let* φ *be a convex function on* K *and* $\lambda_1,\ldots,\lambda_n$ *barycentric coordinates over* K. *Then for all* $\boldsymbol{g}\in L^k$ *such that* $\boldsymbol{g}(E)\subset K$ *and* $\varphi(\boldsymbol{g}),\lambda_i(\boldsymbol{g})\in L$, $i=1,\ldots,n$ *and positive real numbers* $p_1,\ldots,p_n$, *with* $P_n=\sum_{i=1}^n p_i$, *satisfying the condition*

$$p_i \geqslant A(\mathbf{1}) \text{ for all } i=1,\ldots,n, \tag{3.4}$$

we have

$$\begin{aligned}
&\varphi\left(\frac{\sum_{i=1}^n p_i\boldsymbol{x}_i-\widetilde{A}(\boldsymbol{g})}{P_n-A(\mathbf{1})}\right)\\
&\leqslant \frac{\sum_{i=1}^n p_i\varphi(\boldsymbol{x}_i)-\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\,\varphi(\boldsymbol{x}_i)-\min_i\{p_i-A(\lambda_i(\boldsymbol{g}))\}\,S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)}{P_n-A(\mathbf{1})}\\
&\leqslant \frac{\sum_{i=1}^n p_i\varphi(\boldsymbol{x}_i)-A(\varphi(\boldsymbol{g}))-S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)\left[\min_i\{p_i-A(\lambda_i(\boldsymbol{g}))\}+A\left(\min_i\{\lambda_i(\boldsymbol{g})\}\right)\right]}{P_n-A(\mathbf{1})}.
\end{aligned} \tag{3.5}$$

Proof. For each $t\in E$ we have $\boldsymbol{g}(t)\in K$. Using barycentric coordinates we have $\lambda_i(\boldsymbol{g}(t))\geqslant 0$, $i=1,\ldots,n$, $\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))=1$ and

$$\boldsymbol{g}(t)=\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Since φ is convex on K, then

$$\varphi(\boldsymbol{g}(t)) \leqslant \sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))\varphi(\boldsymbol{x}_i)-\min_i\{\lambda_i(\boldsymbol{g}(t))\}\,S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n). \tag{3.6}$$

Applying positive linear functional A on (3.6) we get

$$A(\varphi(\boldsymbol{g}))\leqslant \sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\varphi(\boldsymbol{x}_i)-A\left(\min_i\{\lambda_i(\boldsymbol{g})\}\right)S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n),$$

where

$$\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))=A\left(\sum_{i=1}^n \lambda_i(\boldsymbol{g})\right)=A(\mathbf{1})$$

and

$$A(\mathbf{1}) \geqslant A(\lambda_i(\boldsymbol{g})) \geqslant 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

Also we have

$$\widetilde{A}(\boldsymbol{g}) = \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\,\boldsymbol{x}_i.$$

Now we can write

$$\begin{aligned}\frac{\sum_{i=1}^{n} p_i\boldsymbol{x}_i - \widetilde{A}(\boldsymbol{g})}{P_n - A(\mathbf{1})} &= \frac{1}{P_n - A(\mathbf{1})}\left(\sum_{i=1}^{n} p_i\boldsymbol{x}_i - \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\,\boldsymbol{x}_i\right)\\ &= \frac{1}{P_n - A(\mathbf{1})}\sum_{i=1}^{n}(p_i - A(\lambda_i(\boldsymbol{g})))\,\boldsymbol{x}_i.\end{aligned}$$

We have

$$\frac{1}{P_n - A(\mathbf{1})}\sum_{i=1}^{n}(p_i - A(\lambda_i(\boldsymbol{g}))) = 1$$

and

$$\frac{1}{P_n - A(\mathbf{1})}(p_i - A(\lambda_i(\boldsymbol{g}))) \geqslant 0 \quad \text{for all} \quad i = 1, \ldots, n,$$

since

$$p_i \geqslant A(\mathbf{1}) \geqslant A(\lambda_i(\boldsymbol{g})) \quad \text{for all} \quad i = 1, \ldots, n.$$

Therefore, expression $\frac{\sum_{i=1}^{n} p_i\boldsymbol{x}_i - \widetilde{A}(\boldsymbol{g})}{P_n - A(\mathbf{1})}$ is a convex combination of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ and belongs to K.

Since φ is convex on K, we have

$$\begin{aligned}&\varphi\left(\frac{\sum_{i=1}^{n} p_i\boldsymbol{x}_i - \widetilde{A}(\boldsymbol{g})}{P_n - A(\mathbf{1})}\right)\\ &= \varphi\left(\frac{1}{P_n - A(\mathbf{1})}\sum_{i=1}^{n}(p_i - A(\lambda_i(\boldsymbol{g})))\,\boldsymbol{x}_i\right)\\ &\leqslant \frac{1}{P_n - A(\mathbf{1})}\sum_{i=1}^{n}(p_i - A(\lambda_i(\boldsymbol{g})))\,\varphi(\boldsymbol{x}_i) - \min_i\left\{\frac{p_i - A(\lambda_i(\boldsymbol{g}))}{P_n - A(\mathbf{1})}\right\} S_{\varphi}^{n}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\\ &= \frac{\sum_{i=1}^{n} p_i\varphi(\boldsymbol{x}_i) - \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\,\varphi(\boldsymbol{x}_i) - \min_i\{p_i - A(\lambda_i(\boldsymbol{g}))\} S_{\varphi}^{n}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)}{P_n - A(\mathbf{1})}\\ &\leqslant \frac{\sum_{i=1}^{n} p_i\varphi(\boldsymbol{x}_i) - A(\varphi(\boldsymbol{g})) - S_{\varphi}^{n}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\left[\min_i\{p_i - A(\lambda_i(\boldsymbol{g}))\} + A(\min_i\{\lambda_i(\boldsymbol{g})\})\right]}{P_n - A(\mathbf{1})}.\end{aligned}$$

□

Next Corollary shows that Theorem 4 is a generalization of Theorem 2 for convex hulls.

COROLLARY 2. *Let L satisfy properties L1, L2, L3 on a nonempty set E and A be a positive normalized linear functional on L. Let φ be a convex function on an interval $I=[m,M]\subset\mathbb{R}$. Then for all $g\in L$ such that $g(E)\subset I$ and $\varphi(g)\in L$, we have*

$$\begin{aligned}
&\varphi(m+M-A(g))\\
&\leqslant \frac{A(g)-m}{M-m}\varphi(m)+\frac{M-A(g)}{M-m}\varphi(M)-\left(\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|\right)S_{\varphi}^{2}(m,M)\\
&\leqslant \varphi(m)+\varphi(M)-A(\varphi(g))\\
&\quad-\left[1-\frac{1}{M-m}\left(\left|A(g)-\frac{m+M}{2}\right|+A\left(\left|g-\frac{m+M}{2}\right|\right)\right)\right]S_{\varphi}^{2}(m,M).
\end{aligned}\tag{3.7}$$

Proof. For each $t\in E$ we have $g(t)\in I$.

Since interval $I=[m,M]$ is 1-simplex with vertices m and M, then the barycentric coordinates have the special form:

$$\lambda_1(g(t))=\frac{M-g(t)}{M-m}\quad\text{and}\quad\lambda_2(g(t))=\frac{g(t)-m}{M-m}.$$

Applying functional A we have

$$A(\lambda_1(g))=\frac{M-A(g)}{M-m}\quad\text{and}\quad A(\lambda_2(g))=\frac{A(g)-m}{M-m}.\tag{3.8}$$

Choosing $n=2$, $p_1=p_2=1$, $x_1=m$, $x_2=M$ from (3.5) it follows

$$\begin{aligned}
&\varphi(m+M-A(g))\\
&\leqslant \varphi(m)+\varphi(M)-\left[\frac{M-A(g)}{M-m}\varphi(m)+\frac{A(g)-m}{M-m}\varphi(M)\right]\\
&\quad-\left(\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|\right)\left[\varphi(m)+\varphi(M)-2\varphi\left(\frac{m+M}{2}\right)\right]\\
&=\frac{A(g)-m}{M-m}\varphi(m)+\frac{M-A(g)}{M-m}\varphi(M)-\left(\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|\right)S_{\varphi}^{2}(m,M)\\
&\leqslant \varphi(m)+\varphi(M)-A(\varphi(g))\\
&\quad-\left[\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|+A\left(\frac{1}{2}-\frac{1}{M-m}\left|g-\frac{m+M}{2}\right|\right)\right]S_{\varphi}^{2}(m,M)\\
&=\varphi(m)+\varphi(M)-A(\varphi(g))\\
&\quad-\left[1-\frac{1}{M-m}\left(\left|A(g)-\frac{m+M}{2}\right|+A\left(\left|g-\frac{m+M}{2}\right|\right)\right)\right]S_{\varphi}^{2}(m,M).\quad\square
\end{aligned}$$

REMARK 2. The inequalities in (3.7) are also improvements of the inequalities obtained in [3].

THEOREM 5. *Let L satisfy properties L1, L2, L3 on a nonempty set E, A be a positive linear functional on L and $\widetilde{A}$ defined as in (3.2). Let $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\in\mathbb{R}^k$ and*

$K = co(\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\})$. *Let* φ *be a convex function on* K *and* $\lambda_1,\ldots,\lambda_n$ *barycentric coordinates over* K. *Then for all* $\boldsymbol{g}\in L^k$ *such that* $\boldsymbol{g}(E)\subset K$ *and* $\varphi(\boldsymbol{g}),\lambda_i(\boldsymbol{g})\in L$, $i=1,\ldots,n$ *and positive real numbers* $p_1,\ldots,p_n$ *satisfying the conditions* $P_n - A(1) > 0$, *where* $P_n=\sum_{i=1}^n p_i$, *and*

$$\frac{\sum_{i=1}^n p_i\boldsymbol{x}_i-\widetilde{A}(\boldsymbol{g})}{P_n-A(\mathbf{1})}\in K, \tag{3.9}$$

we have

$$\varphi\left(\frac{\sum_{i=1}^n p_i\boldsymbol{x}_i-\widetilde{A}(\boldsymbol{g})}{P_n-A(\mathbf{1})}\right)\geqslant\frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i\boldsymbol{x}_i\right)-A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(\boldsymbol{g})\right)}{P_n-A(\mathbf{1})}$$
$$\geqslant\frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i\boldsymbol{x}_i\right)-\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\varphi(\boldsymbol{x}_i)+\min_i\{A(\lambda_i(\boldsymbol{g}))\}S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)}{P_n-A(\mathbf{1})}. \tag{3.10}$$

Proof. For each $t\in E$ we have $\boldsymbol{g}(t)\in K$. Using barycentric coordinates we have $\lambda_i(\boldsymbol{g}(t))\geqslant 0$, $i=1,\ldots,n$, $\sum_{i=1}^n\lambda_i(\boldsymbol{g}(t))=1$ and

$$\boldsymbol{g}(t)=\sum_{i=1}^n\lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Also we have

$$\widetilde{A}(\boldsymbol{g})=\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i.$$

We can easily see that

$$\frac{1}{A(\mathbf{1})}\widetilde{A}(\boldsymbol{g})=\frac{1}{A(\mathbf{1})}\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i\in K,$$

since

$$\frac{1}{A(\mathbf{1})}\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))=1\quad\text{and}\quad\frac{1}{A(\mathbf{1})}A(\lambda_i(\boldsymbol{g}))\geqslant 0,\ \ i=1,\ldots,n.$$

Since φ is convex on K, then using Lemma 1

$$\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(\boldsymbol{g})\right)\leqslant\frac{1}{A(\mathbf{1})}\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\varphi(\boldsymbol{x}_i)-\min_i\left\{\frac{A(\lambda_i(\boldsymbol{g}))}{A(\mathbf{1})}\right\}S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n). \tag{3.11}$$

Using Theorem B and (3.11) we have

$$\varphi\left(\frac{P_n\left(\frac{1}{P_n}\sum_{i=1}^n p_i\boldsymbol{x}_i\right)-A(\mathbf{1})\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(\boldsymbol{g})\right)}{P_n-A(\mathbf{1})}\right)\geqslant\frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i\boldsymbol{x}_i\right)-A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}(\boldsymbol{g})\right)}{P_n-A(\mathbf{1})}$$
$$\geqslant\frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i\boldsymbol{x}_i\right)-A(\mathbf{1})\frac{1}{A(\mathbf{1})}\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\varphi(\boldsymbol{x}_i)+\min_i\{A(\lambda_i(\boldsymbol{g}))\}S_\varphi^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)}{P_n-A(\mathbf{1})}.$$

□

REMARK 3. If positive real numbers $p_1,\dots,p_n$ satisfy condition (3.4), then condition (3.9) is also satisfied since K is convex set. Hence (3.5) can be extended as follows

$$\frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i\boldsymbol{x}_i\right)-\sum_{i=1}^n A\left(\lambda_i(\boldsymbol{g})\right)\varphi\left(\boldsymbol{x}_i\right)+\min_i\left\{A(\lambda_i(\boldsymbol{g}))\right\}S_\varphi^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)}{P_n-A\left(\mathbf{1}\right)}$$
$$\leqslant \frac{P_n\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i\boldsymbol{x}_i\right)-A(\mathbf{1})\varphi\left(\frac{1}{A(\mathbf{1})}\widetilde{A}\left(\boldsymbol{g}\right)\right)}{P_n-A\left(\mathbf{1}\right)}$$
$$\leqslant \varphi\left(\frac{\sum_{i=1}^n p_i\boldsymbol{x}_i-\widetilde{A}\left(\boldsymbol{g}\right)}{P_n-A\left(\mathbf{1}\right)}\right)$$
$$\leqslant \frac{\sum_{i=1}^n p_i\varphi\left(\boldsymbol{x}_i\right)-\sum_{i=1}^n A\left(\lambda_i(\boldsymbol{g})\right)\varphi\left(\boldsymbol{x}_i\right)-\min_i\left\{p_i-A(\lambda_i(\boldsymbol{g}))\right\}S_\varphi^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)}{P_n-A\left(\mathbf{1}\right)}$$
$$\leqslant \frac{\sum_{i=1}^n p_i\varphi\left(\boldsymbol{x}_i\right)-A(\varphi(\boldsymbol{g}))-S_\varphi^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\left[\min_i\left\{p_i-A(\lambda_i(\boldsymbol{g}))\right\}+A\left(\min_i\left\{\lambda_i(\boldsymbol{g})\right\}\right)\right]}{P_n-A\left(\mathbf{1}\right)}.$$

COROLLARY 3. *Let L satisfy properties L1, L2, L3 on a nonempty set E and A be a positive normalized linear functional on L. Let φ be a convex function on an interval $I=[m,M]\subset\mathbb{R}$. Then for all $g\in L$ such that $g(E)\subset I$ and $\varphi(g)\in L$, we have*

$$\begin{aligned}\varphi\left(m+M-A(g)\right)&\geqslant 2\varphi\left(\frac{m+M}{2}\right)-\varphi\left(A(g)\right)\\&\geqslant 2\varphi\left(\frac{m+M}{2}\right)-\left[\frac{M-A(g)}{M-m}\varphi(m)+\frac{A(g)-m}{M-m}\varphi(M)\right]\\&\quad+\left(\frac{1}{2}-\frac{1}{M-m}\left|A(g)-\frac{m+M}{2}\right|\right)S_\varphi^2(m,M).\end{aligned}\tag{3.12}$$

Proof. Choosing $n=2,\ x_1=m,\ x_2=M,\ p_1=p_2=1$ and using (3.8), the inequalities in (3.12) easily follow from (3.10). □

Next we give generalizations of Corollary 2 and Corollary 3 for convex functions defined on k-simplices in $\mathbb{R}^k$.

Let S be a k-simplex in $\mathbb{R}^k$ with vertices $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_{k+1}\in\mathbb{R}^k$. The barycentric coordinates $\lambda_1,\dots\lambda_{k+1}$ over S are nonnegative linear polynomials which satisfy Lagrange's property

$$\lambda_i(\boldsymbol{v}_j)=\delta_{ij}=\begin{cases}1, & i=j\\0, & i\neq j.\end{cases}$$

It is known (see [2]) that for each $\boldsymbol{x}\in S$ barycentric coordinates $\lambda_1(\boldsymbol{x}),\dots,\lambda_{k+1}(\boldsymbol{x})$ have the form

$$\lambda_1(\boldsymbol{x})=\frac{\mathrm{Vol}_k\left([\boldsymbol{x},\boldsymbol{v}_2,\dots,\boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)},$$

$$\lambda_2(\boldsymbol{x}) = \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \boldsymbol{x}, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)},$$

$$\vdots$$

$$\lambda_{k+1}(\boldsymbol{x}) = \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{x}])}{\mathrm{Vol}_k(S)}, \tag{3.13}$$

where $\mathrm{Vol}_k(F)$ denotes the k-dimensional Lebesgue measure of a measurable set $F \subset \mathbb{R}^k$. Here, for example, $[\boldsymbol{v}_1, \boldsymbol{x}, \ldots, \boldsymbol{v}_{k+1}]$ denotes the subsimplex obtained by replacing $\boldsymbol{v}_2$ by $\boldsymbol{x}$, i.e., the subsimplex opposite to $\boldsymbol{v}_2$, when adding $\boldsymbol{x}$ as a new vertex.

COROLLARY 4. *Let L satisfy properties L1, L2, L3 on a nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (3.2). Let φ be a convex function on k-simplex $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, \ldots, \lambda_{k+1}$ barycentric coordinates over S. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset S$ and $\varphi(\boldsymbol{g}) \in L$ we have*

$$\begin{aligned}
&\frac{(k+1)\varphi\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\boldsymbol{v}_i\right) - \sum_{i=1}^{k+1}\lambda_i(\widetilde{A}(\boldsymbol{g}))\varphi(\boldsymbol{v}_i) + \min_i\left\{\lambda_i(\widetilde{A}(\boldsymbol{g}))\right\} S_\varphi^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1})}{k} \\
&\leqslant \frac{(k+1)\varphi\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\boldsymbol{v}_i\right) - \varphi(\widetilde{A}(\boldsymbol{g}))}{k} \\
&\leqslant \varphi\left(\frac{\sum_{i=1}^{k+1}\boldsymbol{v}_i - \widetilde{A}(\boldsymbol{g})}{k}\right) \\
&\leqslant \frac{\sum_{i=1}^{k+1}\varphi(\boldsymbol{v}_i) - \sum_{i=1}^{k+1}\lambda_i(\widetilde{A}(\boldsymbol{g}))\varphi(\boldsymbol{v}_i) - \min_i\left\{1 - \lambda_i(\widetilde{A}(\boldsymbol{g}))\right\} S_\varphi^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1})}{k} \\
&\leqslant \frac{\sum_{i=1}^{k+1}\varphi(\boldsymbol{v}_i) - A(\varphi(\boldsymbol{g})) - S_\varphi^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1})\left[\min_i\left\{1 - \lambda_i(\widetilde{A}(\boldsymbol{g}))\right\} + A(\min_i\{\lambda_i(\boldsymbol{g})\})\right]}{k}.
\end{aligned} \tag{3.14}$$

Proof. Since barycentric coordinates $\lambda_1, \ldots, \lambda_{k+1}$ over k-simplex S in $\mathbb{R}^k$ are nonnegative linear polynomials, then $A(\lambda_i(\boldsymbol{g})) = \lambda_i(\widetilde{A}(\boldsymbol{g}))$ for all $i = 1, \ldots, k+1$.

Choosing $\boldsymbol{x}_i = \boldsymbol{v}_i$ for all $i = 1, \ldots, k+1$ and $p_1 = p_2 = \ldots = p_{k+1} = 1$, the inequalities in (3.14) easily follow from (3.5) and (3.10). □

REMARK 4. As a special case of Corollary 4 for $k = 1$ and if we take p and q nonnegative real numbers such that $A(g) = \dfrac{pm + qM}{p+q}$ we get right hand side of the inequality (2.3) in [7].

REMARK 5. Using the same technique and the same special case as in Example 1 and Remark 8 in [13], from (3.14) we get the same results, that is, the k-dimensional version of the Hammer-Bullen inequality, namely

$$\frac{1}{|S|}\int_S f(t)dt - f(\boldsymbol{v}^*) \leqslant \frac{k}{k+1}\sum_{i=1}^{k+1} f(\boldsymbol{v}_i) - \frac{k}{|S|}\int_S f(t)dt,$$

and, as a special case in one dimension, an improvement of classical Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{4}S_f^2(a,b).$$

4. n-exponential convexity and exponential convexity of the Jessen-Mercer differences, applications to Stolarsky type means

Motivated by Theorems 1 and 2, we define two functionals $\Phi_i\colon L_f \to \mathbb{R}$, $i=1,2$, by

$$\begin{aligned}\Phi_1(\varphi) &= \varphi(m)+\varphi(M)-\varphi(m+M-A(f))-A(\varphi(f)) \\ &\quad - \left[1-\frac{2}{M-m}A\left(\left|f-\frac{m+M}{2}\right|\right)\right]\delta_\varphi \end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\Phi_2(\varphi) &= \varphi(m)+\varphi(M)-\varphi(m+M-A(f))-A(\varphi(f)) \\ &\quad - \left[1-\frac{1}{M-m}\left(A\left(\left|f-\frac{m+M}{2}\right|\right)+\left|\frac{m+M}{2}-A(f)\right|\right)\right]\delta_\varphi, \end{aligned} \tag{4.2}$$

where A, f and δ_φ are as in Theorem 1, $L_f = \{\varphi\colon I\to\mathbb{R}\colon \varphi(f), \varphi(m+M-f)\in L\}$, $[m,M]\subseteq I$. Obviously, Φ_1 and Φ_2 are linear.

If φ is additionally continuous and convex then Theorems 1 and 2 imply $\Phi_i(f)\geqslant 0$, $i=1,2$.

In the following with φ_0 we denote the function defined by $\varphi_0(x)=x^2$ on whatever domain we need.

Now, we give Lagrange and Cauchy mean value type theorems for the functionals Φ_i, $i=1,2$.

THEOREM 6. *Let L satisfy L1, L2 and L3 on a nonempty set E and let A be a positive normalized linear functional on L. Let $f\in L$ be such that $\varphi_0\in L_f$, $f(E)\in[m,M]$, $[m,M]\subseteq I$ and let $\varphi\in C^2(I)$ be such that $\varphi\in L_f$. If Φ_1 and Φ_2 are linear functionals defined as in (4.1) and (4.2) then there exist $\xi_i\in[m,M]$, $i=1,2$ such that*

$$\Phi_i(\varphi) = \frac{\varphi''(\xi_i)}{2}\Phi_i(\varphi_0), \quad i=1,2.$$

Proof. We give a proof for the functional Φ_1. Since $\varphi\in C^2(I)$ there exist real numbers $a=\min_{x\in[m,M]}\varphi''(x)$ and $b=\max_{x\in[m,M]}\varphi''(x)$. It is easy to show that the functions φ_1, φ_2 defined by

$$\varphi_1(x)=\frac{b}{2}x^2-\varphi(x), \quad \varphi_2(x)=f(x)-\frac{a}{2}x^2$$

are continuous and convex, therefore $\Phi_1(\varphi_1) \geqslant 0$, $\Phi_1(\varphi_2) \geqslant 0$. This implies

$$\frac{a}{2}\Phi_1(\varphi_0) \leqslant \Phi_1(\varphi) \leqslant \frac{b}{2}\Phi_1(\varphi_0).$$

If $\Phi_1(\varphi_0) = 0$, there is nothing to prove. Suppose $\Phi_1(\varphi_0) > 0$. We have

$$a \leqslant \frac{2\Phi_1(\varphi)}{\Phi_1(\varphi_0)} \leqslant b.$$

Hence, there exists $\xi_1 \in [m,M]$ such that

$$\Phi_1(\varphi) = \frac{\varphi''(\xi_1)}{2}\Phi_1(\varphi_0). \quad \square$$

THEOREM 7. *Let L satisfy L1, L2 and L3 on a non-empty set E and let A be a positive normalized linear functional on L. Let $f \in L$ be such that $\varphi_0 \in L_f$, $f(E) \in [m,M]$, $[m,M] \subseteq I$ and $\varphi_1, \varphi_2 \in C^2(I)$ such that $\varphi_1, \varphi_2 \in L_f$. If Φ_1 and Φ_2 are linear functionals defined as in (4.1) and (4.2) then there exist $\xi_i \in [m,M]$, $i = 1,2$ such that*

$$\frac{\Phi_i(\varphi_1)}{\Phi_i(\varphi_2)} = \frac{\varphi_1''(\xi_i)}{\varphi_2''(\xi_i)}, \quad i = 1,2$$

provided that the denominators are non-zero.

Proof. We give a proof for the functional Φ_1. Define $\varphi_3 \in C^2([m,M])$ by

$$\varphi_3 = c_1\varphi_1 - c_2\varphi_2, \quad \text{where } c_1 = \Phi_1(\varphi_2),\ c_2 = \Phi_1(\varphi_1).$$

Using Theorem 6 we get that there exists $\xi_1 \in [m,M]$ such that

$$\left(c_1\frac{\varphi_1''(\xi_1)}{2} - c_2\frac{\varphi_2''(\xi_1)}{2}\right)\Phi_1(\varphi_0) = 0.$$

Since $\Phi_1(\varphi_0) \neq 0$, (otherwise we have a contradiction with $\Phi_1(\varphi_2) \neq 0$, by Theorem 6), we obtain

$$\frac{\Phi_1(\varphi_1)}{\Phi_1(\varphi_2)} = \frac{\varphi_1''(\xi_1)}{\varphi_2''(\xi_1)}. \quad \square$$

Next we introduce some function properties which are going to be explored here and immediately after that we give some characterizations of these properties.

DEFINITION 1. A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i\xi_j\psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1,\dots,n$.

A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 6. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

PROPOSITION 1. *If ψ is an n-exponentially convex in the Jensen sense, then the matrix* $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ *is positive semi-definite for all $k \in \mathbb{N}$, $k \leqslant n$. Particularly,* $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$ *for all $k \in \mathbb{N}$, $k \leqslant n$.*

DEFINITION 2. A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{R}$.

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 7. It is known (and easy to show) that a function $\psi : I \to \mathbb{R}^+$ is log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using basic convexity theory, it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

We will also need the following result (see for example [14]).

PROPOSITION 2. *If Ψ is a convex function on I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$ then the following inequality is valid*

$$\frac{\Psi(x_2) - \Psi(x_1)}{x_2 - x_1} \leqslant \frac{\Psi(y_2) - \Psi(y_1)}{y_2 - y_1}. \tag{4.3}$$

If Ψ is concave on I the inequality reverses.

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

DEFINITION 3. The second order devided difference of a function $f\colon I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$[y_i; f] = f(y_i), \quad i = 0, 1, 2$$

$$[y_i, y_{i+1}; f] = \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1$$

$$[y_0, y_1, y_2; f] = \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}. \tag{4.4}$$

REMARK 8. The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $y_1 \to y_0$ in (4.4), we get

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0$$

provided that f' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1, 2$ in (4.4), we get

$$\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2}$$

provided that f'' exists.

We use an idea from [5] to give an elegant method of producing an n-exponentially convex and exponentially convex functions applying the functionals Φ_1 and Φ_2 to a given family of functions with the same property.

THEOREM 8. *Let Φ_i, $i = 1, 2$, be linear functionals defined as in (4.1) and (4.2). Let $\Upsilon = \{\varphi_s\colon s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Upsilon \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(\varphi_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(\varphi_s)$ is also continuous on J then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $s_i \in J$, $i = 1, \ldots, n$, we define the function $\chi\colon I \to \mathbb{R}$ by

$$\chi(y) = \sum_{i,j=1}^{n} \xi_i \xi_j \varphi_{\frac{s_i + s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n-exponentially convex in the Jensen sense we obtain

$$[y_0, y_1, y_2; \chi] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; \varphi_{\frac{s_i + s_j}{2}}] \geqslant 0,$$

which in turn implies that χ is a convex (and continuous) function on I, therefore $\Phi_i(\chi) \geqslant 0$, $i = 1, 2$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_i(\varphi_{\frac{s_i + s_j}{2}}) \geqslant 0.$$

We conclude that the function $s \mapsto \Phi_i(\varphi_s)$ is n-exponentially convex on J in the Jensen sense. If the function $s \mapsto \Phi_i(\varphi_s)$ is also continuous on J, then $s \mapsto \Phi_i(\varphi_s)$ is n-exponentially convex by definition. □

The following corollary is an immediate consequence of Theorem 8.

COROLLARY 5. *Let Φ_i, $i = 1,2$, be linear functionals defined as in (4.1) and (4.2). Let $\Upsilon = \{\varphi_s\colon s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Upsilon \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(\varphi_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(\varphi_s)$ is continuous on J then it is exponentially convex on J.*

COROLLARY 6. *Let Φ_i, $i = 1,2$, be linear functionals defined as in (4.1) and (4.2). Let $\Omega = \{\varphi_s\colon s \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an open interval I such that $\Omega \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following statements hold:*

(*i*) *If the function $s \mapsto \Phi_i(\varphi_s)$ is continuous on J then it is 2-exponentially convex function on J. If $s \mapsto \Phi_i(\varphi_s)$ is additionally strictly positive than it is also log-convex on J.*

(*ii*) *If the function $s \mapsto \Phi_i(\varphi_s)$ is strictly positive and differentiable on J then for every $s, q, u, v \in J$, such that $s \leqslant u$ and $q \leqslant v$, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \leqslant \mu_{u,v}(\Phi_i, \Omega), \quad i = 1,2, \tag{4.5}$$

where

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(\varphi_s)}{\Phi_i(\varphi_q)}\right)^{\frac{1}{s-q}} & , s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(\varphi_s)}{\Phi_i(\varphi_s)}\right) & , s = q. \end{cases} \tag{4.6}$$

for $\varphi_s, \varphi_q \in \Omega$ ($\mu_{s,q}(\Phi_i, \Omega), i = 1,2$ are the Stolarsky type means).

Proof. (*i*) This is an immediate consequence of Theorem 8 and Remark 7.

(*ii*) Since by (*i*) the function $s \mapsto \Phi_i(\varphi_s)$ is log-convex on J, that is, the function $s \mapsto \log \Phi_i(\varphi_s)$ is convex on J. Applying Proposition 2 we get

$$\frac{\log \Phi_i(\varphi_s) - \log \Phi_i(\varphi_q)}{s-q} \leqslant \frac{\log \Phi_i(\varphi_u) - \log \Phi_i(\varphi_v)}{u-v} \tag{4.7}$$

for $s \leqslant u, q \leqslant v, s \neq q, u \neq v$, and therefrom conclude that

$$\mu_{s,q}(\Phi_i, \Omega) \leqslant \mu_{u,v}(\Phi_i, \Omega), \quad i = 1,2.$$

Cases $s = q$ and $u = v$ follow from (4.7) as limit cases. □

REMARK 9. Note that the results from Theorem 8, Corollary 5, Corollary 6 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions φ_s such that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 8 and suitable characterization of convexity.

Now, we present several families of functions which fulfil the conditions of Theorem 8, Corollary 5 and Corollary 6 (and Remark 9). This enables us to construct a large family of functions which are exponentially convex. For a discussion related to this problem see [4].

In the rest of the section we consider only Φ_1 and Φ_2 defined as in (4.1) and (4.2) with A which is continuous and f such that compositions with any function from the chosen familly Ω_i as well as with other functions which appear as arguments of Φ_1 and Φ_2 remain in L.

EXAMPLE 1. Consider a family of functions

$$\Omega_1 = \{g_s\colon \mathbb{R} \to [0,\infty)\colon s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2 g_s}{dx^2}(x) = e^{sx} > 0$ which shows that g_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 8 we also have that $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Theorem 5 we conclude that $s \mapsto \Phi_i(g_s)$, $i = 1,2$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although mapping $s \mapsto g_s$ is not continuous for $s = 0$), so they are exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_1)$, $i = 1,2$, from (4.6) become

$$\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{2}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot g_0)}{3\Phi_i(g_0)}\right), & s = q = 0, \end{cases}$$

and using (4.5) they are monotonous functions in parameters s and q.

Using Theorem 7 it follows that for $i = 1,2$

$$M_{s,q}(\Phi_i, \Omega_1) = \log \mu_{s,q}(\Phi_i, \Omega_1)$$

satisfy $m \leqslant M_{s,q}(\Phi_i, \Omega_1) \leqslant M$, which shows that $M_{s,q}(\Phi_i, \Omega_1)$ are means (of a function g). Notice that by (4.5) they are monotonous.

EXAMPLE 2. Consider a family of functions

$$\Omega_2 = \{f_s\colon (0,\infty) \to \mathbb{R}\colon s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x\log x, & s = 1. \end{cases}$$

Here, $\frac{d^2 f_s}{dx^2}(x) = x^{s-2} = e^{(s-2)\ln x} > 0$ which shows that f_s is convex for $x > 0$ and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition. Arguing as in Example 1 we get that the mappings $s \mapsto \Phi_i(g_s)$, $i = 1,2$ are exponentially convex. Functions (4.6) in this case are equal to:

$$\mu_{s,q}(\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(f_s f_0)}{\Phi_i(f_s)}\right), & s = q \neq 0,1, \\ \exp\left(1 - \frac{\Phi_i(f_0^2)}{2\Phi_i(f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_i(f_0 f_1)}{2\Phi_i(f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_i is positive, then Theorem 7 applied for $f = f_s \in \Omega_2$ and $g = f_q \in \Omega_2$ yields that there exists $\xi \in [m,M]$ such that

$$\xi^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ is invertible for $s \neq q$, we then have

$$m \leqslant \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leqslant M, \tag{4.8}$$

which together with the fact that $\mu_{s,q}(\Phi_i,\Omega_2)$ is continuous, symmetric and monotonous (by (4.5)), shows that $\mu_{s,q}(\Phi_i,\Omega_2)$ is a mean (of a function f).

EXAMPLE 3. Consider a family of functions

$$\Omega_3 = \{h_s\colon (0,\infty) \to (0,\infty)\colon s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\ln^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

Since $s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [15]) it is exponentially convex. Obviously h_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_3)$, from (4.6) becomes

$$\mu_{s,q}(\Phi_i,\Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id\cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s\ln s}\right), & s = q \neq 1, \\ \exp\left(-\frac{2\Phi_i(id\cdot h_1)}{3\Phi_i(h_1)}\right), & s = q = 1, \end{cases}$$

and it is monotonous in parameters s and q by (4.5).

Using Theorem 7, it follows that

$$M_{s,q}(\Phi_i,\Omega_3) = -L(s,q)\log\mu_{s,q}(\Phi_i,\Omega_3),$$

satisfies $m \leqslant M_{s,q}(\Phi_i,\Omega_3) \leqslant M$, which shows that $M_{s,q}(\Phi_i,\Omega_3)$ is a mean (of a function h). $L(s,q)$ is the logarithmic mean defined by $L(s,q) = \frac{s-q}{\log s - \log q}$, $s \neq q$, $L(s,s) = s$.

EXAMPLE 4. Consider a family of functions

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}.$$

Since $s \mapsto \frac{d^2k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (see [15]) it is exponentially convex. Obviously k_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i,\Omega_4)$ from (4.6) becomes

$$\mu_{s,q}(\Phi_i,\Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id\cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s}\right), & s = q, \end{cases}$$

and it is monotonous function in parameters s and q by (4.5).

Using Theorem 7, it follows that

$$M_{s,q}(\Phi_i,\Omega_4) = -\left(\sqrt{s}+\sqrt{q}\right)\log\mu_{s,q}(\Phi_i,\Omega_4)$$

satisfies $m \leqslant M_{s,q}(\Phi_i,\Omega_4) \leqslant M$, which shows that $M_{s,q}(\Phi_i,\Omega_4)$ is a mean (of a function k).

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