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INFINITE REFINEMENTS OF THE DISCRETE JENSEN'S INEQUALITY DEFINED BY RECURSION

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Abstract. In this paper we give very general refinements of the discrete Jensen's inequality for convex and mid-convex functions defined by recursion. Conditions are given for strict inequality which is rare in this topic. In some cases explicit formulas are obtained. The results contain and generalize earlier statements. As an application we define some new quasi-arithmetic means and study their (strict) monotonicity.

1. Introduction

The main purpose of this paper is to give a new refinement of the famous discrete Jensen's inequality which says that

THEOREM A. (see [1]) *Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n \in \mathbb{N}_+$ is fixed. Let $p_1,\ldots,p_n$ be nonnegative numbers with $\sum_{j=1}^{n} p_j = 1$.*

(a) If $f : C \to \mathbb{R}$ is either a convex or a mid-convex function and in the latter case the numbers p_j $(1 \leqslant j \leqslant n)$ are rational, then

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leqslant \sum_{j=1}^{n} p_j f(x_j). \tag{1}$$

(b) If $n \geqslant 2$, there are at least two different elements between the vectors $x_1,\ldots,x_n$, the weights $p_1,\ldots,p_n$ are all positive, and f is strictly convex, then strict inequality holds in (1).

The function $f : C \to \mathbb{R}$ is called convex if

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha) f(y), \quad x,y \in C, \quad 0 \leqslant \alpha \leqslant 1, \tag{2}$$

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and mid-convex or Jensen-convex if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}f(x)+\frac{1}{2}f(y), \quad x,y \in C.$$

f is strictly convex if the inequality in (2) becomes strict inequality for $x \neq y$ and $0 < \alpha < 1$.

The set of nonnegative integers and positive integers will be denoted by $\mathbb{N}$ and $\mathbb{N}_+$, respectively.

We say that $p_1, \dots, p_n$ generate a discrete distribution if they are nonnegative numbers with $\sum_{j=1}^{n} p_j = 1$.

Many papers dealing with various refinements of the above inequality have appeared in the literature: see, for example, the recent book [5] and the references therein. We recall two results concerning special refinements proved recently in [4] and [3], respectively.

The first assertion gives the weighted version of Theorem 1 in [6].

THEOREM B. (see [4, Theorem 1]) *Let C be a convex subset of a real vector space X, and $\{x_1, \dots, x_n\}$ be a finite subset of C, where $n \in \mathbb{N}_+$ is fixed. Let $p_1, \dots, p_n$ generate a discrete distribution. Assume $f : C \to \mathbb{R}$ is either a convex or a mid-convex function and in the latter case the numbers p_j $(1 \leqslant j \leqslant n)$ are rational. Define*

$$\begin{aligned} G_k &= G_{k,n}(x_1, \dots, x_n; p_1, \dots, p_n) \\ &:= \frac{1}{\binom{n+k-1}{k-1}} \sum_{\substack{i_1+\dots+i_n=n+k-1 \\ i_j \in \mathbb{N}_+; \quad 1 \leqslant j \leqslant n}} \left(\sum_{j=1}^{n} i_j p_j\right) f\left(\frac{\sum_{j=1}^{n} i_j p_j x_j}{\sum_{j=1}^{n} i_j p_j}\right), \quad k \in \mathbb{N}_+. \end{aligned} \tag{3}$$

Then

$$f\left(\sum_{j=1}^{n} p_j x_j\right) = G_1 \leqslant \dots \leqslant G_k \leqslant G_{k+1} \leqslant \dots \leqslant \sum_{j=1}^{n} p_j f(x_j).$$

The second result is a parameter dependent refinement of the discrete Jensen's inequality. We state a variant of Theorem 1 (a) in [3], suitable for our purposes.

THEOREM C. *Let C be a convex subset of a real vector space X, and $\{x_1, \dots, x_n\}$ be a finite subset of C, where $n \in \mathbb{N}_+$ is fixed. Let $p_1, \dots, p_n$ generate a discrete distribution, and let $\lambda_i > 1$ $(1 \leqslant i \leqslant n)$. Suppose $f : C \to \mathbb{R}$ is either a convex or a mid-convex function and in the latter case the numbers p_i and λ_i $(1 \leqslant i \leqslant n)$ are rational.*

Introduce

$$d(\lambda) = d(\lambda_1, \dots, \lambda_n) := \sum_{j=1}^{n} \frac{1}{\lambda_j - 1}, \tag{4}$$

and for $k \in \mathbb{N}_+$ *define*

$$\begin{aligned} D_k(\lambda) &= D_{k,n}(x_1,\ldots,x_n;p_1,\ldots,p_n;\lambda_1,\ldots,\lambda_n) \\ &:= \frac{1}{(d(\lambda)+1)^{k-1}} \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} \frac{(k-1)!}{(i_1-1)!\ldots(i_n-1)!} \\ &\quad \times \prod_{j=1}^{n} \frac{1}{(\lambda_j-1)^{i_j-1}} \left(\sum_{j=1}^{n}\lambda_j^{i_j-1}p_j\right) f\left(\frac{\sum_{j=1}^{n}\lambda_j^{i_j-1}p_jx_j}{\sum_{j=1}^{n}\lambda_j^{i_j-1}p_j}\right). \end{aligned} \tag{5}$$

Then

$$f\left(\sum_{j=1}^{n} p_jx_j\right) = D_1(\lambda) \leqslant \ldots \leqslant D_k(\lambda) \leqslant D_{k+1}(\lambda) \leqslant \ldots \leqslant \sum_{j=1}^{n} p_jf(x_j). \tag{6}$$

It can be seen that there are essential similarities between Theorem B and Theorem C:

(i) Both of them are infinite refinements of the discrete Jensen's inequality;

(ii) G_k and $D_k(\lambda)$ are sums over the same set

$$S_k := \left\{ (i_1,\ldots,i_n) \in \mathbb{N}_+^n \mid \sum_{j=1}^{n} i_j = n+k-1 \right\}, \quad k \in \mathbb{N}_+; \tag{7}$$

(iii) G_k and $D_k(\lambda)$ are special cases of the general expression

$$\begin{aligned} T_k &= T_{k,n}(x_1,\ldots,x_n;p_1,\ldots,p_n;g_1,\ldots,g_n) \\ &:= \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_k(i_1,\ldots,i_n) \left(\sum_{j=1}^{n} g_j(i_j)p_j\right) f\left(\frac{\sum_{j=1}^{n} g_j(i_j)p_jx_j}{\sum_{j=1}^{n} g_j(i_j)p_j}\right), \end{aligned} \tag{8}$$

where

$$(g_j(m))_{m\in\mathbb{N}_+}, \quad 1 \leqslant j \leqslant n$$

are real sequences and

$$u_k(i_1,\ldots,i_n), \ (i_1,\ldots,i_n) \in S_k, \quad k \in \mathbb{N}_+$$

are real numbers.

Considering $G_{k,n}$ and $D_{k,n}(\lambda)$, we have the sequences

$$g_j(m) := m, \quad m \in \mathbb{N}_+, \quad 1 \leqslant j \leqslant n \tag{9}$$

and

$$g_j(m) := \lambda_j^{m-1}, \quad m \in \mathbb{N}_+, \quad 1 \leqslant j \leqslant n, \tag{10}$$

respectively. Common properties of these two sequences are the next: they are strictly increasing positive sequences for which

$$g_1(1) = \ldots = g_n(1). \tag{11}$$

In this paper, among others, we shall show that for arbitrarily choosen increasing positive sequences $(g_j(m))_{m\in\mathbb{N}_+}$ $(1 \leqslant j \leqslant n)$ satisfying (11) there exist finite positive sequences

$$(u_k(i_1,\ldots,i_n))_{(i_1,\ldots,i_n)\in S_k}, \quad k \in \mathbb{N}_+ \tag{12}$$

such that $(T_k)_{k\in\mathbb{N}_+}$ defined in (8) gives a refinement of the discrete Jensen's inequality. Regarding $G_{k,n}$ and $D_{k,n}(\lambda)$, the finite sequences above are

$$u_k(i_1,\ldots,i_n) := \frac{1}{\binom{n+k-1}{k-1}}, \quad (i_1,\ldots,i_n) \in S_k, \quad k \in \mathbb{N}_+ \tag{13}$$

and

$$\begin{aligned} u_k(i_1,\ldots,i_n) := &\frac{1}{(d(\lambda)+1)^{k-1}} \cdot \frac{(k-1)!}{(i_1-1)!\ldots(i_n-1)!} \\ &\times \prod_{j=1}^{n} \frac{1}{(\lambda_j-1)^{i_j-1}}, \quad (i_1,\ldots,i_n) \in S_k, \quad k \in \mathbb{N}_+, \end{aligned} \tag{14}$$

respectively, that is they are defined explicitly. This is hardly to be expected in the general case. Really, we shall be able to give the finite sequences (12) by recursive definition.

Our first main result is the following:

THEOREM 1. *Let $n \in \mathbb{N}_+$ be fixed, and let*

$$(g_j(m))_{m\in\mathbb{N}_+}, \quad 1 \leqslant j \leqslant n \tag{15}$$

be strictly increasing sequences such that

$$\alpha := g_1(1) = \ldots = g_n(1) > 0. \tag{16}$$

Define the finite sequences

$$(u_k(i_1,\ldots,i_n))_{(i_1,\ldots,i_n)\in S_k}, \quad k \in \mathbb{N}_+ \tag{17}$$

recursively by

$$u_1(1,\ldots,1) := \frac{1}{\alpha}, \tag{18}$$

and for every $(i_1,\ldots,i_n) \in S_{k+1}$ (see (7))

$$\begin{aligned} u_{k+1}(i_1,\ldots,i_n) := &\sum_{\{l\in\{1,\ldots,n\} \mid i_l\neq 1\}} \frac{1}{1+\frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)}+\sum\limits_{\substack{j=1\\ j\neq l}}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}} \\ &\times \frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)} u_k(i_1,\ldots,i_{l-1},i_l-1,i_{l+1},\ldots,i_n). \end{aligned} \tag{19}$$

Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C. Let $p_1,\ldots,p_n$ generate a discrete distribution. If $f:C\to\mathbb{R}$ is either a convex or a mid-convex function and in the latter case the numbers p_i $(1\leqslant i\leqslant n)$ and the sequences (15) are rational, then

$$f\left(\sum_{j=1}^{n}p_jx_j\right)=T_1\leqslant\ldots\leqslant T_k\leqslant T_{k+1}\leqslant\ldots\leqslant\sum_{j=1}^{n}p_jf(x_j), \tag{20}$$

where T_k is introduced in (8).

Some conditions under which all inequalities in (20) become strict are given in the next statement. The most important situations are covered. Similar results are quite rare in this topic.

THEOREM 2. *Suppose the conditions of Theorem 1 are satisfied, and suppose that $f:C\to\mathbb{R}$ is strictly convex. If $n\geqslant 2$, there are at least two different elements between the vectors $x_1,\ldots,x_n$, and $p_1,\ldots,p_n$ are all positive, then*

$$f\left(\sum_{j=1}^{n}p_jx_j\right)=T_1<\ldots<T_k<T_{k+1}<\ldots<\sum_{j=1}^{n}p_jf(x_j).$$

Now we consider a special case of Theorem 1, in which the sequences (17) can be given explicitly.

In the further part of the paper we use the following notational convention: the empty product is equal to 1.

THEOREM 3. *Let $n\in\mathbb{N}_+$ be fixed, and let*

$$(g_j(m))_{m\in\mathbb{N}_+},\quad 1\leqslant j\leqslant n$$

be strictly increasing sequences such that

$$\alpha:=g_1(1)=\ldots=g_n(1)>0,$$

and for every $k\in\mathbb{N}_+$ the numbers

$$\sum_{j=1}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)},\quad (i_1,\ldots,i_n)\in S_k$$

depend only on k, that is

$$c(k):=\sum_{j=1}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)},\quad (i_1,\ldots,i_n)\in S_k. \tag{21}$$

If $p_1,\ldots,p_n$ generate a discrete distribution, then for every $k\in\mathbb{N}_+$

$$\begin{aligned}u_k(i_1,\ldots,i_n)&=\frac{1}{\alpha}\prod_{j=1}^{k-1}\frac{1}{1+c(j)}\cdot\prod_{j=1}^{n}\left(\prod_{m=1}^{i_j-1}\frac{g_j(m)}{g_j(m+1)-g_j(m)}\right)\\&\quad\times\frac{(k-1)!}{(i_1-1)!\ldots(i_n-1)!},\quad (i_1,\ldots,i_n)\in S_k,\end{aligned} \tag{22}$$

where $u_k(i_1,\ldots,i_n)$ *is defined by (18) and (19).*

2. Discussion and applications

Theorem 1 contains Theorem B and Theorem C, as the following example shows. Moreover, we obtain a common generalization of these results.

EXAMPLE 1. Suppose $n\in\mathbb{N}_+$ is fixed, and suppose $p_1,\ldots,p_n$ generate a discrete distribution. Let $\alpha>0$, $a\geqslant 0$ and $b_j\in\mathbb{R}$ $(1\leqslant j\leqslant n)$ such that the numbers $a+b_j$ are all positive. Define the sequences $(g_j(m))_{m\in\mathbb{N}_+}$ by

$$g_j(m):=\alpha\prod_{i=1}^{m-1}\left(1+\frac{1}{ai+b_j}\right),\quad m\in\mathbb{N}_+,\quad 1\leqslant j\leqslant n.$$

Then these sequences are strictly increasing,

$$\alpha=g_1(1)=\ldots=g_n(1)>0,$$

and for every $k\in\mathbb{N}_+$

$$c(k):=\sum_{j=1}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}=a(n+k-1)+\sum_{j=1}^{n}b_j,\quad (i_1,\ldots,i_n)\in S_k.$$

Thus they satisfy all the hypotheses of Theorem 3, according to which for every $k\in\mathbb{N}_+$

$$u_k(i_1,\ldots,i_n)=\frac{1}{\alpha}\prod_{j=1}^{k-1}\frac{1}{1+a(n+j-1)+\sum\limits_{l=1}^{n}b_l}\cdot\prod_{j=1}^{n}\left(\prod_{m=1}^{i_j-1}(am+b_j)\right)$$
$$\times\frac{(k-1)!}{(i_1-1)!\ldots(i_n-1)!},\quad (i_1,\ldots,i_n)\in S_k. \tag{23}$$

By choosing $\alpha=a=1$ and $b_j=0$ $(1\leqslant j\leqslant n)$, we have the sequences (9), and (23) gives (13).

By taking $\alpha=1$, $a=0$ and $b_j=\frac{1}{\lambda_j-1}$ $(1\leqslant j\leqslant n)$, where $\lambda_j>1$ $(1\leqslant j\leqslant n)$, we have the sequences (10), and (23) gives (14).

The considerations in the previous example and Theorem 1 lead to a common generalization of Theorem B and Theorem C:

COROLLARY 1. *Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C, where $n\in\mathbb{N}_+$ is fixed. Let $p_1,\ldots,p_n$ generate a discrete distribution. Let $\alpha>0$, $a\geqslant 0$ and $b_j\in\mathbb{R}$ $(1\leqslant j\leqslant n)$ such that the numbers $a+b_j$ are* *all positive. If $T_k \ (k \in \mathbb{N}_+)$ is defined by*

$$T_k := \left(\prod_{j=1}^{k-1} \frac{1}{1+a(n+j-1)+\sum_{l=1}^{n} b_l} \right) \cdot \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j \in \mathbb{N}_+; \quad 1 \leqslant j \leqslant n}} \left(\prod_{j=1}^{n} \left(\prod_{l=1}^{i_j-1} (al+b_j) \right) \right.$$

$$\left. \times \frac{(k-1)!}{(i_1-1)!\ldots(i_n-1)!} \left(\sum_{j=1}^{n} \prod_{l=1}^{i_j-1} \left(1+\frac{1}{al+b_j}\right) p_j \right) \cdot f\left(\frac{\sum_{j=1}^{n} \prod_{l=1}^{i_j-1} \left(1+\frac{1}{al+b_j}\right) p_j x_j}{\sum_{j=1}^{n} \prod_{l=1}^{i_j-1} \left(1+\frac{1}{al+b_j}\right) p_j} \right) \right),$$

and $f : C \to \mathbb{R}$ is either a convex or a mid-convex function and in the latter case the numbers $p_i \ (1 \leqslant i \leqslant n)$, α, a and $b_j \in \mathbb{R} \ (1 \leqslant j \leqslant n)$ are all rational, then

$$f\left(\sum_{j=1}^{n} p_j x_j\right) = T_1 \leqslant \ldots \leqslant T_k \leqslant T_{k+1} \leqslant \ldots \leqslant \sum_{j=1}^{n} p_j f(x_j). \tag{24}$$

By applying Theorem 2, conditions for strict inequality in (24) can be obtained. According to (20), the sequence $(T_k)_{k\in\mathbb{N}_+}$ is increasing and bounded, hence it tends to a finite limit, which is known for the special sequences $(G_k)_{k\in\mathbb{N}_+}$ and $(D_k(\lambda))_{k\in\mathbb{N}_+}$. In [3] probability theoretical method is applied to determine $\lim_{k\to\infty} D_k(\lambda)$, if X is a normed space and f is a continuous convex function. In [4] a totally different argument gives $\lim_{k\to\infty} G_k$, if f is a convex function.

PROBLEM 1. Find the limit of the sequence $(T_k)_{k\in\mathbb{N}_+}$.

As an application we introduce some new quasi-arithmetic means and study their monotonicity.

DEFINITION 1. Let $n \geqslant 2$ be a fixed integer, and let

$$(g_j(m))_{m\in\mathbb{N}_+}, \quad 1 \leqslant j \leqslant n$$

be strictly increasing sequences such that

$$\alpha := g_1(1) = \ldots = g_n(1) > 0.$$

Let $I \subset \mathbb{R}$ be an interval, $x_j \in I \ (1 \leqslant j \leqslant n)$, let $p_1, \ldots, p_n$ generate a discrete distribution, and let φ, $\psi : I \to \mathbb{R}$ be continuous and strictly monotone functions. We define the quasi-arithmetic means with respect to the sequence $(T_k)_{k\in\mathbb{N}_+}$ by

$$M^g_{\psi,\varphi}(k) := \psi^{-1}\left(\sum_{\substack{i_1+\ldots+i_n=n+k-1\\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^n g_j(i_j)\,p_j\right)\right.$$
$$\left.\times(\psi\circ\varphi^{-1})\left(\frac{\sum_{j=1}^n g_j(i_j)\,p_j\varphi(x_j)}{\sum_{j=1}^n g_j(i_j)\,p_j}\right)\right),\quad k\in\mathbb{N}_+. \tag{25}$$

REMARK 1. $M^g_{\psi,\varphi}(k)$ really defines a mean, since as we shall see in (38)

$$\sum_{j=1}^n\left(\sum_{\substack{i_1+\ldots+i_n=n+k-1\\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_k(i_1,\ldots,i_n)\,g_j(i_j)\,p_j\right)=1,\quad k\in\mathbb{N}_+,$$

where the members are positive.

Another well known mean is also needed.

DEFINITION 2. Let $n\geqslant 2$ be a fixed integer, let $I\subset\mathbb{R}$ be an interval, $x_j\in I$ $(1\leqslant j\leqslant n)$, and let $p_1,\ldots,p_n$ generate a discrete distribution. For a continuous and strictly monotone function $z: I\to\mathbb{R}$ we introduce the following weighted quasi-arithmetic mean

$$M_z := z^{-1}\left(\sum_{j=1}^n p_j z(x_j)\right). \tag{26}$$

We now study the monotonicity of the means (25).

COROLLARY 2. *Let $n\geqslant 2$ be a fixed integer, and let*

$$(g_j(m))_{m\in\mathbb{N}_+},\quad 1\leqslant j\leqslant n$$

be strictly increasing sequences such that

$$\alpha := g_1(1)=\ldots=g_n(1)>0.$$

Let $I\subset\mathbb{R}$ be an interval, $x_j\in I$ $(1\leqslant j\leqslant n)$, let $p_1,\ldots,p_n$ generate a discrete distribution, and let φ, $\psi: I\to\mathbb{R}$ be continuous and strictly monotone functions. Then

(a)

$$M_\varphi = M^g_{\psi,\varphi}(1)\leqslant\ldots\leqslant M^g_{\psi,\varphi}(k)\leqslant M^g_{\psi,\varphi}(k+1)\leqslant\ldots\leqslant M_\psi,\quad k\in\mathbb{N}_+, \tag{27}$$

if either $\psi\circ\varphi^{-1}$ is convex and ψ is increasing or $\psi\circ\varphi^{-1}$ is concave and ψ is decreasing.

(b)

$$M_\varphi = M^g_{\psi,\varphi}(1) \geqslant \ldots \geqslant M^g_{\psi,\varphi}(k) \geqslant M^g_{\psi,\varphi}(k+1) \geqslant \ldots \geqslant M_\psi, \quad k \in \mathbb{N}_+, \tag{28}$$

if either $\psi \circ \varphi^{-1}$ is convex and ψ is decreasing or $\psi \circ \varphi^{-1}$ is concave and ψ is increasing.

(c) If $\psi \circ \varphi^{-1}$ is strictly convex (strictly concave) in (a) and (b), there are at least two different elements between the numbers $x_1, \ldots, x_n$, and $p_1, \ldots, p_n$ are all positive, then all inequalities in (27) and (28) become strict.

Proof. Theorem 1 can be applied to the function $\psi \circ \varphi^{-1}$, if it is convex ($-\psi \circ \varphi^{-1}$, if it is concave) and the n-tuples $(\varphi(x_1), \ldots, \varphi(x_n))$, then upon taking ψ^{-1}, we get (a) and (b). Theorem 2 implies (c). $\square$

3. Preliminary results and the proofs

LEMMA 1. *Let $k \in \mathbb{N}_+$ be an integer, and $(i_1, \ldots, i_n) \in S_{k+1}$ be fixed. Then*

$$\sum_{\{l \in \{1,\ldots,n\} \mid i_l \neq 1\}} \frac{(k-1)!}{(i_1-1)! \ldots (i_{l-1}-1)!\,(i_l-2)!\,(i_{l+1}-1)! \ldots (i_n-1)!}$$

$$= \frac{k!}{(i_1-1)! \ldots (i_n-1)!}.$$

Proof. The lowest common denominator is $(i_1-1)! \ldots (i_n-1)!$. Combined with $\sum_{j=1}^{n} i_j = n+k$ the result follows. $\square$

The next two lemmas will be used in the proof of Theorem 2.

LEMMA 2. *Let $n \in \mathbb{N}_+$ is fixed, and let*

$$\left(g_j(m)\right)_{m \in \mathbb{N}_+}, \quad 1 \leqslant j \leqslant n$$

be strictly increasing sequences such that

$$\alpha := g_1(1) = \ldots = g_n(1) > 0.$$

Then the finite sequences $\left(u_k(i_1, \ldots, i_n)\right)_{(i_1,\ldots,i_n) \in S_k}$ $(k \in \mathbb{N}_+)$ defined by (18) and (19) are all positive.

Proof. The case $k = 1$ follows from (18), and the proof is completed by induction on k, using an easy argument. $\square$

LEMMA 3. *Let $n \geqslant 2$ be an integer. If $\alpha > 0$ and $a_i > \alpha$ $(1 \leqslant i \leqslant n)$, then*

$$\det \begin{pmatrix} a_1 & \alpha & \dots & \alpha \\ \alpha & a_2 & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \dots & a_n \end{pmatrix} > 0. \tag{29}$$

Proof. We prove by induction on n. The result is true for $n = 2$, since

$$\det \begin{pmatrix} a_1 & \alpha \\ \alpha & a_2 \end{pmatrix} = a_1 a_2 - \alpha^2 > 0.$$

Let $n \geqslant 2$ be an integer such that the result holds for every possible parameters, and let $\alpha > 0$ and $a_i > \alpha$ $(1 \leqslant i \leqslant n+1)$. Now we study

$$\det \begin{pmatrix} a_1 & \alpha & \dots & \alpha \\ \alpha & a_2 & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \dots & a_{n+1} \end{pmatrix}. \tag{30}$$

Substracting the second row from the first row, and then expanding (30) along the first row we have that it is

$$(a_1 - \alpha) \det \begin{pmatrix} a_2 & \alpha & \dots & \alpha \\ \alpha & a_3 & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \dots & a_{n+1} \end{pmatrix}$$

$$+ (a_2 - \alpha) \det \begin{pmatrix} \alpha & \alpha & \dots & \alpha \\ \alpha & a_3 & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \dots & a_{n+1} \end{pmatrix}.$$

The induction hypothesis together with $a_1 > \alpha$ imply that the first member in the previous sum is positive. Thus it is enough to show that the second member is also positive. Since $a_2 > \alpha$, this follows from

$$\det \begin{pmatrix} \alpha & \alpha & \dots & \alpha \\ \alpha & a_3 & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \dots & a_{n+1} \end{pmatrix} > 0. \tag{31}$$

To compute this determinant, we substract the first row from all the other rows, and then expand it along the first column, we have that it is

$$\alpha (a_3 - \alpha) \dots (a_{n+1} - \alpha),$$

which is positive by the assumptions. The proof is complete. $\square$

REMARK 2. In fact, we have proved that the matrix in (29) is positive definite.

Proof of Theorem 1. We proceed in three steps.

(i) By using the definition of T_1, (16) and (18), we have

$$T_1 = u_k(1,\ldots,1)\left(\sum_{j=1}^{n} g_j(1)\,p_j\right) f\left(\frac{\sum_{j=1}^{n} g_j(1)\,p_j x_j}{\sum_{j=1}^{n} g_j(1)\,p_j}\right)$$

$$= \frac{1}{\alpha}\alpha\left(\sum_{j=1}^{n} p_j\right) f\left(\frac{\alpha\sum_{j=1}^{n} p_j x_j}{\alpha\sum_{j=1}^{n} p_j}\right) = f\left(\sum_{j=1}^{n} p_j x_j\right).$$

(ii) Next, we show that $T_k \leqslant T_{k+1}$ $(k\in\mathbb{N}_+)$. Fix $k\in\mathbb{N}_+$. It is easy to check that for every $(i_1,\ldots,i_n)\in S_k$

$$\frac{\sum_{j=1}^{n} g_j(i_j)\,p_j x_j}{\sum_{j=1}^{n} g_j(i_j)\,p_j} = \frac{1}{1+\sum_{j=1}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}}\cdot\sum_{l=1}^{n}\left(\frac{g_l(i_l)}{g_l(i_l+1)-g_l(i_l)}\right.$$
$$\left.\times\frac{\sum_{j=1}^{n} g_j(i_j)\,p_j x_j + (g_l(i_l+1)-g_l(i_l))\,p_l x_l}{\sum_{j=1}^{n} g_j(i_j)\,p_j + (g_l(i_l+1)-g_l(i_l))\,p_l}\;\frac{\sum_{j=1}^{n} g_j(i_j)\,p_j + (g_l(i_l+1)-g_l(i_l))\,p_l}{\sum_{j=1}^{n} g_j(i_j)\,p_j}\right) \tag{32}$$

Since the sequences $(g_j(m))_{m\in\mathbb{N}_+}$ $(1\leqslant j\leqslant n)$ are strictly increasing and positive, for every $l=1,\ldots,n$ the numbers

$$\frac{1}{1+\sum_{j=1}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}}\cdot\frac{g_l(i_l)}{g_l(i_l+1)-g_l(i_l)}\cdot\frac{\sum_{j=1}^{n} g_j(i_j)\,p_j + (g_l(i_l+1)-g_l(i_l))\,p_l}{\sum_{j=1}^{n} g_j(i_j)\,p_j} \tag{33}$$

are positive. It can be obtained by an easy calculation that the sum of these numbers is equal to 1. Therefore, since f is convex on C, Theorem A shows that

$$f\left(\frac{\sum_{j=1}^{n} g_j(i_j) p_j x_j}{\sum_{j=1}^{n} g_j(i_j) p_j}\right) \leqslant \frac{1}{1+\sum_{j=1}^{n} \frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}} \cdot \sum_{l=1}^{n} \left(\frac{g_l(i_l)}{g_l(i_l+1)-g_l(i_l)}\right. \tag{34}$$

$$\left.\times \frac{\sum_{j=1}^{n} g_j(i_j) p_j + (g_l(i_l+1)-g_l(i_l)) p_l}{\sum_{j=1}^{n} g_j(i_j) p_j} f\left(\frac{\sum_{j=1}^{n} g_j(i_j) p_j x_j + (g_l(i_l+1)-g_l(i_l)) p_l x_l}{\sum_{j=1}^{n} g_j(i_j) p_j + (g_l(i_l+1)-g_l(i_l)) p_l}\right)\right),$$

and thus

$$T_k = \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_k(i_1,\ldots,i_n) \left(\sum_{j=1}^{n} g_j(i_j) p_j\right) f\left(\frac{\sum_{j=1}^{n} g_j(i_j) p_j x_j}{\sum_{j=1}^{n} g_j(i_j) p_j}\right)$$

$$\leqslant \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} \left(u_k(i_1,\ldots,i_n) \frac{1}{1+\sum_{j=1}^{n} \frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}} \right. \tag{35}$$

$$\times \sum_{l=1}^{n} \frac{g_l(i_l)}{g_l(i_l+1)-g_l(i_l)} \cdot \left(\sum_{j=1}^{n} g_j(i_j) p_j + (g_l(i_l+1)-g_l(i_l)) p_l\right)$$

$$\left.\times f\left(\frac{\sum_{j=1}^{n} g_j(i_j) p_j x_j + (g_l(i_l+1)-g_l(i_l)) p_l x_l}{\sum_{j=1}^{n} g_j(i_j) p_j + (g_l(i_l+1)-g_l(i_l)) p_l}\right)\right).$$

By interchanging the order of summation on the right hand side, we have

$$T_k \leqslant \sum_{l=1}^{n} \left(\sum_{\substack{i_1+\ldots i_{l-1}+(i_l+1)+i_{l+1}+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_k(i_1,\ldots,i_n) \cdot \frac{1}{1+\sum_{j=1}^{n} \frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}} \right.$$

$$\times \frac{g_l(i_l)}{g_l(i_l+1)-g_l(i_l)} \cdot \left(\sum_{j=1}^{n} g_j(i_j) p_j + (g_l(i_l+1)-g_l(i_l)) p_l\right)$$

$$\times f\left(\frac{\sum_{j=1}^{n} g_j(i_j)\, p_j x_j + (g_l(i_l+1) - g_l(i_l))\, p_l x_l}{\sum_{j=1}^{n} g_j(i_j)\, p_j + (g_l(i_l+1) - g_l(i_l))\, p_l}\right)\Bigg)$$

$$= \sum_{l=1}^{n}\left(\sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n;\quad i_l\neq 1}} u_k(i_1,\ldots,i_{l-1},i_l-1,i_{l+1},\ldots,i_n)\right.$$

$$\times \frac{1}{1+\frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)}+\sum\limits_{\substack{j=1\\ j\neq l}}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}} \cdot \frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)} \tag{36}$$

$$\left.\times\left(\sum_{j=1}^{n} g_j(i_j)\, p_j\right) f\left(\frac{\sum_{j=1}^{n} g_j(i_j)\, p_j x_j}{\sum_{j=1}^{n} g_j(i_j)\, p_j}\right)\right).$$

In the previous expression the multiplicity of an element $(i_1,\ldots,i_n)\in S_{k+1}$ is the cardinality of the set $\{l\in\{1,\ldots,n\}\mid i_l\neq 1\}$, and therefore (36) can be written as

$$\sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n;}}\left(\sum_{\{l\in\{1,\ldots,n\}\mid i_l\neq 1\}} \frac{1}{1+\frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)}+\sum\limits_{\substack{j=1\\ j\neq l}}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}}\right.$$

$$\left.\times\frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)} u_k(i_1,\ldots,i_{l-1},i_l-1,i_{l+1},\ldots,i_n)\right)$$

$$\times\left(\sum_{j=1}^{n} g_j(i_j)\, p_j\right) f\left(\frac{\sum_{j=1}^{n} g_j(i_j)\, p_j x_j}{\sum_{j=1}^{n} g_j(i_j)\, p_j}\right),$$

from which, in view of (19),

$$T_k \leqslant \sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n;}} u_{k+1}(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} g_j(i_j)\, p_j\right) f\left(\frac{\sum_{j=1}^{n} g_j(i_j)\, p_j x_j}{\sum_{j=1}^{n} g_j(i_j)\, p_j}\right) = T_{k+1}.$$

(iii) Finally, we show that

$$T_k \leqslant \sum_{j=1}^{n} p_j f(x_j), \quad k\in\mathbb{N}_+.$$

It follows from Theorem A that

$$T_k = \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} g_j(i_j)\,p_j\right) f\left(\frac{\sum_{j=1}^{n} g_j(i_j)\,p_j x_j}{\sum_{j=1}^{n} g_j(i_j)\,p_j}\right)$$
$$\leqslant \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} \left(u_k(i_1,\ldots,i_n)\sum_{j=1}^{n} g_j(i_j)\,p_j f(x_j)\right), \quad k\in\mathbb{N}_+. \tag{37}$$

and this tells us that it is enough to observe that for any $j=1,\ldots,n$

$$\sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_k(i_1,\ldots,i_n)\,g_j(i_j)=1, \quad k\in\mathbb{N}_+. \tag{38}$$

For a fixed $j\in\{1,\ldots,n\}$, we prove this by induction on k. We can obviously suppose that $j=1$. By (16) and (18)

$$u_1(1,\ldots,1)\,g_1(1)=1,$$

thus (38) is true for $k=1$. Suppose then that k is a positive integer for which (38) holds. Then

$$\sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_{k+1}(i_1,\ldots,i_n)\,g_1(i_1) = u_{k+1}(k+1,1,\ldots,1)\,g_1(k+1)$$
$$+\sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n;\quad i_1\leqslant k}} u_{k+1}(i_1,\ldots,i_n)\,g_1(i_1).$$

With the help of (19) this yields

$$\sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_{k+1}(i_1,\ldots,i_n)\,g_1(i_1)$$
$$= \sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} \left(\sum_{\{l\in\{1,\ldots,n\}\mid i_l\neq 1\}} \frac{1}{1+\frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)}+\sum\limits_{\substack{j=1\\ j\neq l}}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}}\right.$$
$$\left.\times\frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)}u_k(i_1,\ldots,i_{l-1},i_l-1,i_{l+1},\ldots,i_n)\right) g_1(i_1).$$

A simple calculation confirms that the previous sum can be rearranged in the following form

$$\begin{aligned}
&\sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_{k+1}(i_1,\ldots,i_n)\, g_1(i_1) \\
&= \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} \left(u_k(i_1,\ldots,i_n)\cdot \frac{1}{1+\sum\limits_{j=1}^{n}\frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}} g_1(i_1) \right. \\
&\quad \left. \times \left(\frac{g_1(i_1+1)}{g_1(i_1+1)-g_1(i_1)} + \sum_{j=2}^{n} \frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)} \right) \right).
\end{aligned}$$

Combined with

$$\frac{g_1(i_1+1)}{g_1(i_1+1)-g_1(i_1)} = 1 + \frac{g_1(i_1)}{g_1(i_1+1)-g_1(i_1)}$$

this gives

$$\sum_{\substack{i_1+\ldots+i_n=n+k \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_{k+1}(i_1,\ldots,i_n)\, g_1(i_1) = \sum_{\substack{i_1+\ldots+i_n=n+k-1 \\ i_j\in\mathbb{N}_+;\quad 1\leqslant j\leqslant n}} u_k(i_1,\ldots,i_n)\, g_1(i_1),$$

and hence we can apply the induction hypothesis. The proof is complete. □

Proof of Theorem 3. Since $(1,\ldots,1)$ is the only element of S_1, we have from (22) that

$$u_1(1,\ldots,1) = \frac{1}{\alpha},$$

which is exactly (18). We make the inductive assumption that (22) holds for some $k\in\mathbb{N}_+$. Then, by (19)

$$\begin{aligned}
u_{k+1}(i_1,\ldots,i_n) = &\sum_{\{l\in\{1,\ldots,n\}\mid i_l\neq 1\}} \left(\frac{1}{1+\frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)} + \sum\limits_{\substack{j=1\\ j\neq l}}^{n} \frac{g_j(i_j)}{g_j(i_j+1)-g_j(i_j)}} \right. \\
&\times \frac{g_l(i_l-1)}{g_l(i_l)-g_l(i_l-1)} \frac{1}{\alpha} \prod_{j=1}^{k-1} \frac{1}{1+c(j)} \\
&\times \frac{(k-1)!}{(i_1-1)!\ldots(i_{l-1}-1)!\,(i_l-2)!\,(i_{l+1}-1)!\ldots(i_n-1)!} \\
&\left. \times \prod_{l=1}^{i_l-2} \frac{g_l(m)}{g_l(m+1)-g_l(m)} \cdot \prod_{\substack{j=1\\ j\neq l}}^{n} \left(\prod_{m=1}^{i_j-1} \frac{g_j(m)}{g_j(m+1)-g_j(m)} \right) \right),
\end{aligned}$$

and therefore (21) and Lemma 1 insure that

$$u_{k+1}(i_1,\ldots,i_n) = \frac{1}{\alpha}\prod_{j=1}^{k}\frac{1}{1+c(j)}\cdot\prod_{j=1}^{n}\left(\prod_{m=1}^{i_j-1}\frac{g_j(m)}{g_j(m+1)-g_j(m)}\right)$$
$$\times\sum_{\{l\in\{1,\ldots,n\}\mid i_l\neq 1\}}\left(\frac{(k-1)!}{(i_1-1)!\ldots(i_{l-1}-1)!\,(i_l-2)!\,(i_{l+1}-1)!\ldots(i_n-1)!}\right)$$
$$=\frac{1}{\alpha}\prod_{j=1}^{k}\frac{1}{1+c(j)}\cdot\prod_{j=1}^{n}\left(\prod_{m=1}^{i_j-1}\frac{g_j(m)}{g_j(m+1)-g_j(m)}\right)$$
$$\times\frac{k!}{(i_1-1)!\ldots(i_n-1)!},\quad (i_1,\ldots,i_n)\in S_{k+1}.$$

The proof is completed. □

Proof of Theorem 2. We prove first that

$$T_k < T_{k+1},\quad k\in\mathbb{N}_+. \tag{39}$$

Fix $k\in\mathbb{N}_+$. By Lemma 2

$$u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n}g_j(i_j)\,p_j\right)>0,\quad (i_1,\ldots,i_n)\in S_k,$$

and therefore the estimate of T_k in (35) leads to the next: it is enough to prove that there exists a fix $(i_1,\ldots,i_n)\in S_k$ for which strict inequality holds in (34). Since the numbers (33) are all positive, it follows from Theorem A (b) and (32) that we will be ready, if for some $(i_1,\ldots,i_n)\in S_k$ there are at least two different between the vectors

$$\frac{\sum_{j=1}^{n}g_j(i_j)\,p_jx_j+(g_l(i_l+1)-g_l(i_l))\,p_lx_l}{\sum_{j=1}^{n}g_j(i_j)\,p_j+(g_l(i_l+1)-g_l(i_l))\,p_l},\quad 1\leqslant l\leqslant n. \tag{40}$$

To this end, assume that the vectors (40) are all equal for a fixed $(i_1,\ldots,i_n)\in S_k$, that is

$$\frac{\sum_{j=1}^{n}g_j(i_j)\,p_jx_j+(g_l(i_l+1)-g_l(i_l))\,p_lx_l}{\sum_{j=1}^{n}g_j(i_j)\,p_j+(g_l(i_l+1)-g_l(i_l))\,p_l}=a,\quad 1\leqslant l\leqslant n. \tag{41}$$

It is easy to see that $x_1=\ldots=x_n=a$ is a solution of this linear system, hence we have a contradiction if there is only one solution. Thus we have to prove that the matrix of

(41) is invertible, or equivalently, the determinant of this matrix

$$\frac{\prod_{l=1}^{n} p_l}{\prod_{l=1}^{n}\left(\sum_{j=1}^{n} g_j(i_j)\,p_j + \left(g_l(i_l+1) - g_l(i_l)\right)p_l\right)}$$

$$\times\det\begin{pmatrix} g_1(i_1+1) & g_2(i_2) & \dots & g_n(i_n) \\ g_1(i_1) & g_2(i_2+1) & \dots & g_n(i_n) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(i_1) & g_2(i_2) & \dots & g_n(i_n+1) \end{pmatrix}$$

is not zero. We show this for

$$(k,1,\dots,1) \in S_k,$$

thus we study the determinant

$$\det\begin{pmatrix} g_1(k+1) & g_2(1) & \dots & g_n(1) \\ g_1(k) & g_2(2) & \dots & g_n(1) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(k) & g_2(1) & \dots & g_n(2) \end{pmatrix}. \tag{42}$$

If $n=2$, then

$$\det\begin{pmatrix} g_1(k+1) & g_2(1) \\ g_1(k) & g_2(2) \end{pmatrix} = g_1(k+1)\,g_2(2) - g_1(k)\,g_2(1),$$

which is positive, since g_1 and g_2 are strictly increasing. Suppose $n \geqslant 3$. We have from the properties of determinants that (42) is

$$\det\begin{pmatrix} g_1(k) & \alpha & \dots & \alpha \\ g_1(k) & g_2(2) & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ g_1(k) & \alpha & \dots & g_n(2) \end{pmatrix}$$

$$+\det\begin{pmatrix} g_1(k+1) - g_1(k) & \alpha & \dots & \alpha \\ 0 & g_2(2) & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha & \dots & g_n(2) \end{pmatrix}$$

The second determinant has the value

$$\left(g_1(k+1) - g_1(k)\right)\det\begin{pmatrix} g_2(2) & \alpha & \dots & \alpha \\ \alpha & g_3(2) & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \dots & g_n(2) \end{pmatrix},$$

which is positive by Lemma 3, and by $g_1(k+1) > g_1(k)$. Further, a repetition of the argument applied to the computation of the determinant (31) shows that the first determinant has the value

$$g_1(k)(g_2(2)-\alpha)\dots(g_n(2)-\alpha),$$

which is also positive. (39) has now been proved. The last thing to be shown is that

$$T_k < \sum_{j=1}^{n} p_j f(x_j), \quad k \in \mathbb{N}_+.$$

This comes from (37) by applying Theorem A (b) and Lemma 2. The proof is complete. □

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