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A REMARK ON SCHUR–CONVEXITY OF THE MEAN OF A CONVEX FUNCTION

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Abstract. In this note the new result and some remarks have been made about proving convexity and Schur-convexity of the mean of a convex function $L:[0,1]\to\mathbb{R}$ associated with the Hermit-Hadamard inequality which is considered in literature [4] and [5]:

$$L(t):=\frac{1}{2(b-a)}\int_a^b[f(ta+(1-t)x)+f(tb+(1-t)x)]dx,$$

where $f:I\subseteq\mathbb{R}\to\mathbb{R}$ and $a,b\in I,\ \ a<b$.

1. Introduction

Let I be an interval with a non-empty interior. Let $\mathbf{x}=(x_1,x_2,..,x_n)$ and $\mathbf{y}=(y_1,y_2,..,y_n)$ in I^n be two n-tuples such that $\mathbf{x}\prec\mathbf{y}$, i.e.

$$\sum_{i=1}^k x_{[i]}\leqslant\sum_{i=1}^k y_{[i]},\ \ k=1,\dots,n-1$$
$$\sum_{i=1}^n x_{[i]}=\sum_{i=1}^n y_{[i]},$$

where $x_{[i]}$ denotes the i th largest component in $\mathbf{x}$.

DEFINITION 1. Function $F:I^n\to\mathbb{R}$ is Schur-convex on I^n if

$$F(x_1,x_2,..,x_n)\leqslant F(y_1,y_2,..,y_n)$$

for each two n-tuples $\mathbf{x}$ and $\mathbf{y}$ such that it holds $\mathbf{x}\prec\mathbf{y}$ on I^n.
Function F is Schur-concave on I^n if and only if $-F$ is Schur-convex.

The next lemma gives us a necessary and sufficient condition for verifying the Schur-convexity property of F as a function of two variables ([6, p.57], see also [8, p. 333]):

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LEMMA A 1. Let $F : I^2 \to \mathbb{R}$ be a continuous function on I^2 and differentiable in interior of I^2. Then F is Schur-convex if and only if it is symmetric and it holds

$$\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)(y-x) \geqslant 0 \tag{1}$$

for all $x, y \in I$, $x \neq y$.

M. Merkle in [7] was inspired by inequalities concerning gamma Γ and digamma Ψ functions and proved the following theorem

THEOREM A 1. The function $(x,y) \to F(x,y)$ defined by

$$F(x,y) = \begin{cases} \frac{\log\Gamma(x) - \log\Gamma(y)}{x-y}, & a,b \in I, x \neq y \\ \Psi(x), & x = y \end{cases}$$

is strictly Schur–concave on $x > 0$, $y > 0$.

N. Elezović and J. Pečarić in [3] generalized this result to the case of integral arithmetic mean by using the Hermit-Hadamard inequality.

THEOREM A 2. Let $I \subseteq \mathbb{R}$ be an interval with a non-empty interior and let f be a continuous function on I. Then

$$F(a,b) = \begin{cases} \frac{1}{b-a}\int_a^b f(x)dx, & a,b \in I, a \neq b \\ f(a), & a = b \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Dragomir in [2] (see also [1, p.108] and [1, p.113]) considered mappings $L : [0,1] \to \mathbb{R}$ and $J : [0,1] \to \mathbb{R}$ associated with the Hermit-Hadamard inequality:

$$L(t) := \frac{1}{2(b-a)} \int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)]dx, \tag{2}$$

$$J(t) := L(1-t), \tag{3}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$. He showed convexity (concavity) of L and J if f is convex (concave) function on I and $t \in [0,1]$: (see also [4])

$$L(t) \leqslant (\geqslant) L(1) = \frac{f(a) + f(b)}{2}, \tag{4}$$

$$J(t) \leqslant (\geqslant) J(0) = \frac{f(a) + f(b)}{2}. \tag{5}$$

Yang in [9] (see also [1, p.147]) considered a similar function $G(t) : [0,1] \to \mathbb{R}$

$$G(t) := \frac{1}{2(b-a)} \int_a^b \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) \right] dx, \tag{6}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$. He showed convexity of G if f is convex function on I.

Huan-Nan Shi in [4] found a similar result as N. Elezović and J. Pečarić's Theorem A2 for the function L:

THEOREM A 3. Let $I \subseteq \mathbb{R}$ be an interval with a non-empty interior and let f be a continuous function on I. For function $P_L(a,b)$ defined on I^2 as

$$P_L(a,b) = \begin{cases} L(t), & a,b \in I, a \neq b, \\ f(a), & a = b, \end{cases}$$

the following holds

(i) for $\frac{1}{2} \leqslant t \leqslant 1$, if f is convex on I, then P_L is Schur-convex on I^2;

(ii) for $0 \leqslant t \leqslant \frac{1}{2}$, if f is concave on I, then P_L is Schur-concave on I^2.

Huan-Nan Shi, Li and Gu in [5] gave a similar result for properties of function P_G defined by function G.

THEOREM A 4. Let $I \subseteq \mathbb{R}$ be an interval with a non-empty interior and let f be a continuous function on I. A function $P_G(a,b)$ is defined on I^2 as

$$P_G(a,b) = \begin{cases} G(t), & a,b \in I, a \neq b, \\ f(a), & a = b. \end{cases}$$

Then, for any $0 \leqslant t \leqslant 1$, if f is convex (concave) on I, then P_G is Schur-convex (Schur-concave) on I^2.

In this note our primary aims are to prove a similar result as in Theorem A2 and to point out that this result implies Theorem A4. Also, we will remark that Theorem A3 is not correct. As the applications of the main result the Schur-convexity of the new defined logarithmic mean will be done.

2. Results

We will consider the possibility of further generalization in the companion mappings.

Let $\alpha : [0,1] \to [0,1]$ be a monotonic nondecreasing continuous function on $[0,1]$. Let $L_\alpha : [0,1] \to \mathbb{R}$ and $J_\alpha : [0,1] \to \mathbb{R}$ be functions defined by

$$L_\alpha(t) := \frac{1}{2(b-a)} \int_a^b [f(\alpha(t)a + (1-\alpha(t))x) + f(\alpha(t)b + (1-\alpha(t))x)]dx. \tag{7}$$

and

$$J_\alpha(t) := L(1-\alpha(t)). \tag{8}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$.

We use the fact that $L_\alpha(t) = L(\alpha(t))$ and $J_\alpha(t) := J(\alpha(t))$ to derive the following lemma.

LEMMA 1. Suppose that f, L, J , α, L_α, J_α are as above. If f is convex (concave) then

(i) L_α is convex (concave) if α is a linear function;

(ii) The following inequality holds for all $t \in [0,1]$

$$L_\alpha(t) \leqslant (\geqslant) \frac{f(a)+f(b)}{2} \tag{9}$$

and

$$J_\alpha(t) \leqslant (\geqslant) \frac{f(a)+f(b)}{2}. \tag{10}$$

Proof. (i) It is obvious by convexity of L and linearity of α. We note that if f is convex and g is linear, then the composition $f \circ g$ is convex.

(ii) By the convexity (concavity) of f one has:

$$L_\alpha(t) \leqslant (\geqslant) \frac{1-\alpha(t)}{(b-a)} \int_a^b f(x)\,dx + \alpha(t) \frac{f(a)+f(b)}{2} \leqslant (\geqslant), \frac{f(a)+f(b)}{2} \tag{11}$$

$$J_\alpha(t) \leqslant (\geqslant) \frac{\alpha(t)}{(b-a)} \int_a^b f(x)\,dx + (1-\alpha(t)) \frac{f(a)+f(b)}{2} \leqslant (\geqslant) \frac{f(a)+f(b)}{2}. \tag{12}$$

REMARK 1. In papers [2] and [9] convexity of functions L and G were considered.

Moreover, it is well known that if $f : I \to \mathbb{R}$ is convex, and $\alpha(t) = At + B$, $A > 0$, $(A < 0)$ then L_α is convex (concave) on $[0,1]$. Convexity of L implies convexity of L_α by fact that $L_\alpha(t) = L(\alpha(t))$.

Since $\alpha(t) = \frac{t+1}{2}$ is increasing on $[0,1]$ Yang's result for convexity of G in [9] is consequence of Dragomir's result in [2] for convexity of L.

Our main result is the next theorem which is similar as N. Elezović and J. Pečarić's Theorem A2 for functions P_α and $\widetilde{P}_\alpha$.

THEOREM 1. Let $I \subseteq \mathbb{R}$ be an interval with a non-empty interior. Let f be a continuous functions on I and $\alpha : [0,1] \to [0,1]$ be a monotonic nondecreasing continuous function on $[0,1]$.

For a function $P_\alpha(a,b)$ defined on I^2 as

$$P_\alpha(a,b) = \begin{cases} L_\alpha(t), & a,b \in I, a \neq b, \\ f(a), & a = b, \end{cases}$$

and for a function $\widetilde{P}_\alpha(a,b)$ defined on I^2 as

$$\widetilde{P}_\alpha(a,b) = \begin{cases} J_\alpha(t), & a,b \in I, a \neq b, \\ f(a), & a = b, \end{cases}$$

the following holds

(i) for α such that $\min_{t\in[0,1]}\alpha(t)\geqslant\frac{1}{2}$, $\max_{t\in[0,1]}\alpha(t)\leqslant 1$, if f is convex (concave) on I, then P_α is Schur-convex (Schur-concave) on I^2;

(ii) for α such that $\min_{t\in[0,1]}\alpha(t)\geqslant 0$, $\max_{t\in[0,1]}\alpha(t)\leqslant\frac{1}{2}$, if f is convex (concave) on I, then $\widetilde{P}_\alpha$ is Schur-convex (Schur-concave) on I^2.

Proof. The property of Schur-convexity of function $P_\alpha(a,b)$ is considered with (a,b) in I^2. Applying Lemma A1 we find out that holds $a\neq b$

$$\begin{aligned}
&(b-a)\Big(\frac{\partial P_\alpha(a,b)}{\partial b}-\frac{\partial P_\alpha(a,b)}{\partial a}\Big)\\
=&-2L_\alpha(t)\\
&+\frac{1}{2(1-\alpha(t))}\{(1-2\alpha(t))[f(\alpha(t)a+(1-\alpha(t))b)+f(\alpha(t)b+(1-\alpha(t))a)]\\
&+f(a)+f(b)\}
\end{aligned}$$

and analogue

$$\begin{aligned}
&(b-a)\Big(\frac{\partial \widetilde{P}_\alpha(a,b)}{\partial b}-\frac{\partial \widetilde{P}_\alpha(a,b)}{\partial a}\Big)\\
=&-2J_\alpha(t)\\
&+\frac{1}{2\alpha(t)}\{(2\alpha(t)-1)[f((1-\alpha(t))a+\alpha(t)b)+f((1-\alpha(t))b+\alpha(t)a)]\\
&+f(a)+f(b)\}.
\end{aligned}$$

(i) Let α such that $\min_{t\in[0,1]}\alpha(t)\geqslant\frac{1}{2}$, $\max_{t\in[0,1]}\alpha(t)\leqslant 1$, then $(1-2\alpha(t))\leqslant 0$ for all $t\in[0,1]$.

According (9) in Lemma 1 if f is convex (concave) function then it holds

$$\begin{aligned}
&(b-a)\Big(\frac{\partial P_\alpha(a,b)}{\partial b}-\frac{\partial P_\alpha(a,b)}{\partial a}\Big)\\
\geqslant&(\leqslant)-2L_\alpha(t)+f(a)+f(b)\\
\geqslant&(\leqslant)0
\end{aligned}$$

and P_α is Schur-convex (Schur-concave) on I^2;

(ii) For α such that $\min_{t\in I}\alpha(t)\geqslant 0$, $\max_{t\in[0,1]}\alpha(t)\leqslant\frac{1}{2}$, then $(1-2\alpha(t))\geqslant 0$ for all $t\in[0,1]$.

According (10) in Lemma 1 if f is convex (concave) function then

$$\begin{aligned}
&(b-a)\Big(\frac{\partial \widetilde{P}_\alpha(a,b)}{\partial b}-\frac{\partial \widetilde{P}_\alpha(a,b)}{\partial a}\Big)\\
\geqslant&(\leqslant)-2L_\alpha(t)+f(a)+f(b)\\
\geqslant&(\leqslant)0
\end{aligned}$$

and $\widetilde{P}_\alpha$ is Schur-convex (Schur-concave) on I^2;

So, for $\alpha(t)=t$ the corrected Huan-Nan's result is as in following corollary:

COROLLARY 1. Let $I \subseteq \mathbb{R}$ be an interval with a non-empty interior and let f be a continuous function on I. For function $P_L(a,b)$ defined on I^2 as

$$P_L(a,b) = \begin{cases} L(t), & a,b \in I, a \neq b, \\ f(a), & a = b, \end{cases}$$

and for function $\widetilde{P}_L(a,b)$ defined on I^2 as

$$\widetilde{P}_L(a,b) = \begin{cases} J(t), & a,b \in I, a \neq b, \\ f(a), & a = b, \end{cases}$$

the following holds

(i) for $\frac{1}{2} \leqslant t \leqslant 1$, if f is convex (concave) on I, then P_L is Schur-convex (Schur-concave) on I^2;

(ii) for $0 \leqslant t \leqslant \frac{1}{2}$, if f is convex (concave) on I, then $\widetilde{P}_L$ is Schur-convex (Schur-concave) on I^2.

REMARK 2. So, the result in Theorem A4 is also consequence of Theorem 1. Function $G(t) = L(\alpha(t))$ where $\alpha(t) = \frac{t+1}{2}$ is function such that case (i) is satisfied.

3. Application

N. Elezović and J. Pečarić in [3], applied Theorem A2 to logarithmic mean $L_r(a,b)$:

COROLLARY A 1. The generalized logarithmic mean defined as follows

$$L_r(a,b) = \left[\frac{b^r - a^r}{r(b-a)}\right]^{\frac{1}{r-1}}, \quad a,b > 0, \tag{13}$$

$$L_1 = \frac{1}{e}\left[\frac{a^a}{b^b}\right]^{\frac{1}{a-b}},$$

$$L_0 = \frac{b-a}{\log b - \log a},$$

$$L(a,a) = a, \tag{14}$$

is Schur-convex for $r > 2$ and Schur-concave for $r < 2$.

As a application of the Theorem 1 is the following result for the means:

THEOREM 2. Let $\alpha : [0,1] \to [0,1)$ be a monotonic nondecreasing continuous function, such that $\min_{t\in[0,1]} \alpha(t) \geqslant \frac{1}{2}$, $\max_{t\in[0,1)} \alpha(t) < 1$. The logarithmic means are defined by

$$L_r(a,b;\alpha) = \left[\frac{(b^r - a^r) - (u_\alpha^r - v_\alpha^r)}{2(b-a)r(1-\alpha(t))}\right]^{\frac{1}{r-1}}, \quad a,b>0,\ a\neq b,$$
$$L_r(a,b;\alpha) = a, \quad a=b,$$
$$L_0(a,b;\alpha) = \left[\frac{\log b - \log a - (\log u_\alpha - \log v_\alpha)}{2(1-\alpha(t))(b-a)}\right]^{-1}, \quad a,b>0,\ a\neq b,$$
$$L_0(a,b;\alpha) = a, \quad a=b,$$

where $u_\alpha = \alpha(t)b + (1-\alpha(t))a$ and $v_\alpha = \alpha(t)a + (1-\alpha(t))b$.

(i) If $r>2$ then $L_r(a,b;\alpha)$ is Schur-convex on $\mathbb{R}_+^2$;

(ii) If $r<2$ then $L_r(a,b;\alpha)$ is Schur-concave on $\mathbb{R}_+^2$.

Proof. Applying Theorem 1 for function $f(x) = x^{r-1}, r\neq 0,$ then for $a\neq b$, we have

$$\begin{aligned}
L_\alpha(t) &:= \frac{1}{2(b-a)}\int_a^b [(\alpha(t)a+(1-\alpha(t))x)^{r-1} + (\alpha(t)b+(1-\alpha(t))x)^{r-1}]dx \\
&= \frac{1}{2(b-a)}\cdot\frac{1}{r(1-\alpha(t))}[(\alpha(t)a+(1-\alpha(t))x)^r\Big|_a^b + (\alpha(t)b+(1-\alpha(t))x)^r\Big|_a^b] \\
&= \frac{(b^r-a^r)+[(\alpha(t)a+(1-\alpha(t))b)^r - (\alpha(t)b+(1-\alpha(t))a)^r]}{2(b-a)r(1-\alpha(t))} \\
&= \frac{(b^r-a^r)-(u_\alpha^r - v_\alpha^r)}{2(b-a)r(1-\alpha(t))}.
\end{aligned}$$

For $f(x) = x^{-1}$, then for $a\neq b$, we have

$$\begin{aligned}
L_\alpha(t) &:= \frac{1}{2(b-a)}\int_a^b [(\alpha(t)a+(1-\alpha(t))x)^{-1} + (\alpha(t)b+(1-\alpha(t))x)^{-1}]dx \\
&= \frac{1}{2(b-a)(1-\alpha(t))}[\log(\alpha(t)a+(1-\alpha(t))x)\Big|_a^b + \log(\alpha(t)b+(1-\alpha(t))x)\Big|_a^b] \\
&= \frac{(\log b - \log a) + [\log(\alpha(t)a+(1-\alpha(t))b) - \log(\alpha(t)b+(1-\alpha(t))a)]}{2(b-a)(1-\alpha(t))} \\
&= \frac{(\log b - \log a) - (\log u_\alpha - \log v_\alpha)}{2(b-a)(1-\alpha(t))}.
\end{aligned}$$

We use results of Marshall and Olkin in [6, p.61] (see also ([8, p.334],) for properties of compositions of Schur-convex and Schur-concave function of the form $\psi(x) = h(\phi(x))$:

(a) if ϕ is Schur-convex and h is increasing (decreasing) then ψ is Schur-convex (Schur-concave);

(b) if ϕ is Schur-concave and h is increasing (decreasing) then ψ is Schur-concave (Schur-convex).

If $r>2$ then f is convex function on $\mathbb{R}_+$, and from Theorem 1 we have $P_\alpha(a,b;\alpha)$ is Schur-convex on $\mathbb{R}^2_+$. Since $h(x)=x^{\frac{1}{r-1}}$ is increasing on $\mathbb{R}_+$, then applying (a) $L_r(a,b;\alpha)=[P_\alpha(a,b)]^{\frac{1}{r-1}}$ is Schur-convex on $\mathbb{R}^2_+$;

If $r<1$, then f is convex function on $\mathbb{R}_+$, and from Theorem 1 we have $P_\alpha(a,b;\alpha)$ is Schur-convex on $\mathbb{R}^2_+$. Since $h(x)=x^{\frac{1}{r-1}}$ is decreasing on $\mathbb{R}_+$, then applying (a) $L_r(a,b;\alpha)=[P_\alpha(a,b)]^{\frac{1}{r-1}}$ is Schur-concave on $\mathbb{R}^2_+$;

If $1<r<2$ then f is concave function on $\mathbb{R}_+$, and from Theorem 1 we have $P_\alpha(a,b;\alpha)$ is Schur-concave on $\mathbb{R}^2_+$. Since $h(x)=x^{\frac{1}{r-1}}$ is increasing on $\mathbb{R}_+$, then applying (b) $L_r(a,b;\alpha)=[P_\alpha(a,b)]^{\frac{1}{r-1}}$ is Schur-concave on $\mathbb{R}^2_+$.

Setting $r\to 1$, it is deduced that $L_r(a,b;\alpha)$ is still Schur-concave on $\mathbb{R}^2_+$.

Analoguous theorem holds for the mean $\widetilde{L}_r(a,b;\alpha)$:

THEOREM 3. Let $\alpha:[0,1]\to(0,1]$ be a monotonic nondecreasing continuous function, such that $\min_{t\in[0,1]}\alpha(t)>0$, $\max_{t\in[0,1]}\alpha(t)\leqslant\frac{1}{2}$. For $a,b\in[0,\infty)$, the logarithmic means are defined by

$$\begin{aligned}
&\widetilde{L}_r(a,b;\alpha)=\left[\frac{(b^r-a^r)-(v_\alpha^r-u_\alpha^r)}{2(b-a)r\alpha(t)}\right]^{\frac{1}{r-1}}, \quad a,b>0,\ a\neq b,\\
&\widetilde{L}_r(a,b;\alpha)=a, \quad a=b,\\
&\widetilde{L}_0(a,b;\alpha)=\left[\frac{\log b-\log a-(\log v_\alpha-\log u_\alpha)}{2\alpha(t)(b-a)}\right]^{-1}, \quad a,b>0,\ a\neq b,\\
&\widetilde{L}_0(a,b;\alpha)=a, \quad a=b,
\end{aligned}$$

where $u_\alpha=(\alpha(t)b+(1-\alpha(t))a$ and $v_\alpha=\alpha(t)a+(1-\alpha(t))b$.

(i) If $r>2$ then $\widetilde{L}_r(a,b;\alpha)$ is Schur-convex on $\mathbb{R}^2_+$;

(ii) If $r<2$ then $\widetilde{L}_r(a,b;\alpha)$ is Schur-concave on $\mathbb{R}^2_+$.

REMARK 3. For $\alpha(t)=\frac{1+t}{2}$, we obtain the same result for Schur-convexity of the means $L_r(a,b;\alpha)$ as Huan-Nan Shi and all in [5].

REMARK 4. For $\alpha(t)=t, t\in[\frac{1}{2},1)$ we obtain result for Schur-convexity of the means $L_r(a,b,t)$. Huan- Nan Shi in [4] gave the similar result for $L_r(a,b,t)$ but this result is not correct as a consequence of Theorem 3.

COROLLARY 2. Let $t\in[\frac{1}{2},1)$, and let

$$\begin{aligned}
&L_r(a,b;t)=\left[\frac{(b^r-a^r)-(u^r-v^r)}{2(b-a)r(1-t))}\right]^{\frac{1}{r-1}}, \quad a,b>0,\ a\neq b,\\
&L_r(a,b;t)=a, \quad a=b,\\
&L_0(a,b;t)=\left[\frac{\log b-\log a-(\log u-\log v)}{2(1-t)(b-a)}\right]^{-1}, \quad a,b>0,\ a\neq b,\\
&L_0(a,b;t)=a, \quad a=b,
\end{aligned}$$

where $u = tb + (1-t)a$ and $v = ta + (1-t)b$. Then

(i) if $r > 2$ then $L_r(a,b;t)$ is Schur-convex on $\mathbb{R}^2_+$;

(ii) if $r < 2$ then $L_r(a,b;t)$ is Schur-concave on $\mathbb{R}^2_+$.

COROLLARY 3. Let $\alpha : [0,1] \to [0,1)$ be a monotonic nondecreasing continuous function, such that $\min_{t\in[0,1]} \alpha(t) \geqslant \frac{1}{2}$, $\max_{t\in[0,1)} \alpha(t) < 1$.

(i) If $r > 2$ then

$$\frac{a+b}{2} \leqslant L_r(a,b;\alpha) \leqslant (a+b)\left[\frac{(\alpha(t))^r - (1-\alpha(t))^r - 1}{r(\alpha(t)-1)}\right]^{\frac{1}{r-1}};$$

(ii) If $r < 2$ then

$$\frac{a+b}{2} \geqslant L_r(a,b;\alpha) \geqslant (a+b)\left[\frac{(\alpha(t))^r - (1-\alpha(t))^r - 1}{r(\alpha(t)-1)}\right]^{\frac{1}{r-1}}.$$

Proof. For 2-tuples $(\frac{a+b}{2}, \frac{a+b}{2})$, (a,b), $(a+b,0)$ hold the relation of majorization:

$$(\frac{a+b}{2}, \frac{a+b}{2}) \prec (a,b) \prec (a+b,0).$$

Using Theorem 2 (i) we have

$$L_r(\frac{a+b}{2}, \frac{a+b}{2};\alpha) \leqslant L_r(a,b;\alpha) \leqslant L_r(a+b,0;\alpha).$$

REMARK 5. For t_0 such that $\alpha(t_0) = \frac{1}{2}$, we obtain result for Schur-convexity of the generalize logarithmic mean

$$L_r(a,b,t_0) = L_r(a,b)$$

as in Corollary A1 [3] (see also [7]).

REMARK 6. According Corollary 3 for $t = t_0$ such that $\alpha(t_0) = \frac{1}{2}$ we got the known inequalities

$$\frac{a+b}{2} \leqslant \left[\frac{b^r - a^r}{(b-a)r}\right]^{\frac{1}{r-1}} \leqslant (a+b)\left[\frac{1}{r}\right]^{\frac{1}{r-1}}, \quad r > 2,$$

$$\frac{a+b}{2} \geqslant \left[\frac{b^r - a^r}{(b-a)r}\right]^{\frac{1}{r-1}} \geqslant (a+b)\left[\frac{1}{r}\right]^{\frac{1}{r-1}}, \quad r < 2,$$

$$\frac{a+b}{2} \geqslant \frac{b-a}{\log b - \log a} \geqslant \frac{a+b}{\log(a+b)}.$$

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