Journal of
Mathematical
Inequalities
Volume 9, Number 4 (2015), 1143–1156 doi:10.7153/jmi-09-87

NEW ESTIMATIONS OF THE REMAINDER IN THREE–POINT QUADRATURE FORMULAE OF EULER TYPE

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(*Communicated by M. Matić*)

Abstract. We derive some new bounds for the general three-point quadrature formulae of Euler type using some inequalities for the Chebyshev functional. As special cases, we provide some new error estimates for Euler Simpson formula, dual Euler Simpson formula and Euler Maclaurin formula. Also, applications for Euler Bullen-Simpson formula are obtained.

1. Introduction

The well known Chebyshev functional is defined by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(s)\,g(s)\,\mathrm{d}s-\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\cdot\frac{1}{b-a}\int_a^b g(s)\,\mathrm{d}s,$$

where $f,g:[a,b]\to\mathbf{R}$ are two real functions such that $f,g,f\cdot g\in L^1[a,b]$.

Many researchers have investigated the Chebyshev functional and inequalities related to the Chebyshev functional (see [8], [9], [10] and the references cited therein).

In paper [4] P. Cerone and S. S. Dragomir proved the following Grüss type inequalities:

THEOREM 1. *Let $f,g:[a,b]\to\mathbf{R}$ be two absolutely continuous functions on $[a,b]$ with*

$$(\cdot-a)(b-\cdot)\left(f'\right)^2,\quad(\cdot-a)(b-\cdot)\left(g'\right)^2\in L^1[a,b],$$

then

$$|T(f,g)|\leqslant\frac{1}{\sqrt{2}}[T(f,f)]^{1/2}\frac{1}{\sqrt{b-a}}\left[\int_a^b(s-a)(b-s)\left(g'(s)\right)^2\mathrm{d}s\right]^{1/2}\tag{1}$$

$$\leqslant\frac{1}{2(b-a)}\left[\int_a^b(s-a)(b-s)\left(f'(s)\right)^2\mathrm{d}s\right]^{1/2}\left[\int_a^b(s-a)(b-s)\left(g'(s)\right)^2\mathrm{d}s\right]^{1/2}.$$

The constants $1/\sqrt{2}$ and $1/2$ are the best possible.

Mathematics subject classification (2010): 26D15, 26D20, 26D99.
Keywords and phrases: Chebyshev functional, Three-point quadrature, Euler Simpson formula, dual Euler Simpson formula, Euler Maclaurin formula, Euler Bullen-Simpson formula.
This work has been fully supported by Croatian Science Foundation under the project 5435.

THEOREM 2. *Assume that* $g:[a,b]\to\mathbf{R}$ *is monotonic nondecreasing on* $[a,b]$ *and* $f:[a,b]\to\mathbf{R}$ *is absolutely continuous with* $f'\in L^{\infty}[a,b]$, *then*

$$|T(f,g)|\leqslant\frac{1}{2(b-a)}\|f'\|_{\infty}\cdot\int_a^b(s-a)(b-s)\,\mathrm{d}g(s). \tag{2}$$

The constant $1/2$ *is the best possible.*

Here and hereafter the symbol $B_k(s)$ denotes the Bernoulli polynomials, $B_k=B_k(0)$ the Bernoulli numbers, and $B_k^*(s),\ k\geqslant 0$, periodic functions of period 1 defined by the condition

$$B_k^*(s+1)=B_k^*(s),\ s\in\mathbb{R},$$

and related to the Bernoulli polynomials as follow

$$B_k^*(s)=B_k(s),\ \ 0\leqslant s<1.$$

The Bernoulli polynomials $B_k(s),\ k\geqslant 0$, are uniquely determined by the following identities

$$B_k'(s)=kB_{k-1}(s),\ \ k\geqslant 1;\ \ B_0(s)=1,\ \ B_k(s+1)-B_k(s)=ks^{k-1},\ \ k\geqslant 0.$$

Further, $B_0^*(s)=1$, $B_1^*(s)$ is a discontinuous function with a jump of -1 at each integer and for $k\geqslant 2$, $B_k^*(s)$ are continuous functions. We get

$$B_k^{*\prime}(s)=kB_{k-1}^*(s),\ \ k\geqslant 1 \tag{3}$$

for every $s\in\mathbb{R}$ when $k\geqslant 3$, and for every $s\in\mathbb{R}\setminus\mathbb{Z}$ when $k=1,2$. Also, for $k\geqslant 1$ the following equations hold:

$$\begin{aligned}
B_{2k}\left(\frac{1}{2}\right)&=-\left(1-2^{1-2k}\right)B_{2k}\\
B_{2k}\left(\frac{1}{4}\right)&=-2^{-2k}\left(1-2^{1-2k}\right)B_{2k}\\
B_{2k}\left(\frac{1}{3}\right)&=-2^{-1}\left(1-3^{1-2k}\right)B_{2k}\\
B_{2k}\left(\frac{1}{6}\right)&=2^{-1}\left(1-2^{1-2k}\right)\left(1-3^{1-2k}\right)B_{2k}
\end{aligned}$$

More about Bernoulli polynomials, Bernoulli numbers, and periodic functions B_k^* can be found in [1].

In this paper we give some new bounds for the general three-point quadrature formulae of Euler type using Theorem 1, Theorem 2 and the general three-point quadrature formulae recently published in [5]. We use the above results to get the error estimates for Euler Simpson formula, dual Euler Simpson formula and Euler Maclaurin formula. Also, the corresponding error estimates for Euler Bullen-Simpson formula are derived.

More about quadrature formulae and error estimations (from the point of view of inequality theory) can be found in monographs [2] and [6].

2. Applications for the general three-point formulae of Euler type

Let $x \in [0,1/2)$ and $f:[0,1] \to \mathbb{R}$ be such that $f^{(2n+1)}$ is continuous function of bounded variation on $[0,1]$ for some $n \geqslant 0$. In paper [5] the authors proved the following formula:

$$\int_0^1 f(s)\mathrm{d}s - w(x)f(x) - (1-2w(x))f\left(\frac{1}{2}\right) - w(x)f(1-x) + T_{2n}(x) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(x,s)\mathrm{d}f^{(2n+1)}(s), \tag{4}$$

where

$$T_{2n}(x) = \sum_{k=2}^{2n} \frac{1}{k!} G_k(x,0)\,[f^{(k-1)}(1) - f^{(k-1)}(0)], \tag{5}$$

$$G_k(x,s) = w(x)\left[B_k^*(x-s) + B_k^*(1-x-s)\right] + (1-2w(x))B_k^*\left(\frac{1}{2}-s\right),\ \ k \geqslant 1 \tag{6}$$

$$F_k(x,s) = G_k(x,s) - G_k(x,0),\ \ k \geqslant 2 \tag{7}$$

and $s \in \mathbb{R}$.

Using the properties of Bernoulli polynomials, it is easy to see that

$$G_k(x,1-s) = (-1)^k G_k(x,s),\ \ s \in [0,1]$$

$$\frac{\partial G_k(x,s)}{\partial s} = -kG_{k-1}(x,s)$$

and $G_{2k-1}(x,0) = 0$, for $k \geqslant 2$ and for any choice of the weight w. In general $G_{2k}(x,0) \neq 0$.

To obtain from (4) a quadrature formula with the maximum degree of exactness (which is equal to 3) we have to impose a condition $G_2(x,0) = 0$. This condition gives:

$$w(x) = \frac{1}{6(2x-1)^2},\quad x \in \left[0,\frac{1}{2}\right). \tag{8}$$

Now, formula (4) becomes:

$$\int_0^1 f(s)\mathrm{d}s - Q\left(x,\frac{1}{2},1-x\right) + T_{2n}^{Q3}(x) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{Q3}(x,s)\mathrm{d}f^{(2n+1)}(s), \tag{9}$$

where

$$Q\left(x,\frac{1}{2},1-x\right) = \frac{1}{6(2x-1)^2}\left[f(x) + 24B_2(x)f\left(\frac{1}{2}\right) + f(1-x)\right],$$

$$T_{2n}^{Q3}(x) = \sum_{k=2}^{n} \frac{1}{(2k)!} G_{2k}^{Q3}(x,0)\,[f^{(2k-1)}(1) - f^{(2k-1)}(0)],$$

$$G_k^{Q3}(x,s) = \frac{1}{6(2x-1)^2}\left[B_k^*(x-s) + 24B_2(x)\cdot B_k^*\left(\frac{1}{2}-s\right) + B_k^*(1-x-s)\right],$$

$$F_k^{Q3}(x,s) = G_k^{Q3}(x,s) - G_k^{Q3}(x,0). \tag{10}$$

Assuming $f^{(2n-1)}$ is continuous function of bounded variation on $[0,1]$ for some $n \geqslant 1$, then the following identity holds:

$$\int_0^1 f(s)\mathrm{d}s - Q\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{Q3}(x) = \frac{1}{(2n)!}\int_0^1 G_{2n}^{Q3}(x,s)\mathrm{d}f^{(2n-1)}(s), \tag{11}$$

while assuming $f^{(2n)}$ is continuous function of bounded variation on $[0,1]$ for some $n \geqslant 0$ the following representation holds:

$$\int_0^1 f(s)\mathrm{d}s - Q\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{Q3}(x) = \frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^{Q3}(x,s)\mathrm{d}f^{(2n)}(s). \tag{12}$$

I. Franjić, J. Pečarić and I. Perić [5] proved the following lemma:

LEMMA 1. *For $x \in \{0\} \cup [1/6, 1/2)$ and $k \geqslant 1$, $G_{2k+1}^{Q3}(x,s)$ has no zeros in variable s on the interval $(0, 1/2)$. The sign of this function is determined by*

$$(-1)^k G_{2k+1}^{Q3}(x,s) > 0, \text{ for } x \in [1/6, 1/2)$$

and

$$(-1)^{k+1} G_{2k+1}^{Q3}(x,s) > 0, \text{ for } x = 0.$$

Now, we obtain some new bounds for the remainders in the general three-point formulae of Euler type.

THEOREM 3. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(2n)}$ is absolutely continuous function and $\left(f^{(2n+1)}\right)^2 \in L^1[0,1]$ for some $n \geqslant 1$ and $x \in [0, 1/2)$. Then*

$$\int_0^1 f(s)\,\mathrm{d}s - Q\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{Q3}(x) = K_{2n}^{Q3}(f) \tag{13}$$

and the remainder $K_{2n}^{Q3}(f)$ satisfies the estimation

$$\begin{aligned} &|K_{2n}^{Q3}(f)| \\ &\leqslant \frac{1}{6(2x-1)^2}\left[\frac{-1}{(4n)!}\left(B_{4n} + B_{4n}(1-2x) + 48B_2(x)B_{4n}\left(x+\frac{1}{2}\right) + 288B_2^2(x)B_{4n}\right)\right]^{1/2} \\ &\times \left[\int_0^1 s(1-s)\left(f^{(2n+1)}(s)\right)^2 \mathrm{d}s\right]^{1/2}. \end{aligned} \tag{14}$$

For $f : [0,1] \to \mathbf{R}$ such that $f^{(2n+1)}$ is absolutely continuous function for some $n \geqslant 0$ and $x \in [0, 1/2)$, the following equality holds

$$\int_0^1 f(s)\,\mathrm{d}s - Q\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{Q3}(x) = K_{2n+1}^{Q3}(f) \tag{15}$$

and the remainder $K_{2n+1}^{Q3}(f)$ satisfies the estimation

$$\begin{aligned}&\left|K_{2n+1}^{Q3}(f)\right| \\ &\leqslant \frac{1}{6(2x-1)^2}\left[\frac{1}{(4n+2)!}\left(B_{4n+2}+B_{4n+2}(1-2x)+48B_2(x)B_{4n+2}\left(x+\frac{1}{2}\right)\right.\right. \\ &\left.\left.+288B_2^2(x)B_{4n+2}\right)\right]^{1/2}\cdot\left[\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2 \mathrm{d}s\right]^{1/2}.\end{aligned} \tag{16}$$

Proof. Applying Theorem 1 with G_k^{Q3} in place of f and $f^{(k)}$ in place of g we get

$$\begin{aligned}&\left|\int_0^1 G_k^{Q3}(x,s)f^{(k)}(s)\mathrm{d}s-\int_0^1 G_k^{Q3}(x,s)\mathrm{d}s\cdot\int_0^1 f^{(k)}(s)\mathrm{d}s\right| \\ &\leqslant \frac{1}{\sqrt{2}}\left[T\left(G_k^{Q3}(x,\cdot),G_k^{Q3}(x,\cdot)\right)\right]^{1/2}\left[\int_0^1 s(1-s)\left(f^{(k+1)}(s)\right)^2\mathrm{d}s\right]^{1/2},\end{aligned} \tag{17}$$

where

$$T\left(G_k^{Q3}(x,\cdot),G_k^{Q3}(x,\cdot)\right)=\int_0^1\left(G_k^{Q3}(x,s)\right)^2\mathrm{d}s-\left(\int_0^1 G_k^{Q3}(x,s)\mathrm{d}s\right)^2.$$

By elementary calculations, using properties of Bernoulli polynomials $B_k(s)$ and periodic function B_k^* we obtain

$$\int_0^1 G_k^{Q3}(x,s)\mathrm{d}s=0. \tag{18}$$

Using integration by parts we have

$$\begin{aligned}&\int_0^1\left(G_k^{Q3}(x,s)\right)^2\mathrm{d}s \\ &=(-1)^{k-1}\frac{k(k-1)\cdots 2}{(k+1)(k+2)\cdots(2k-1)}\left[-\frac{1}{2k}G_{2k}^{Q3}(x,s)G_1(x,s)\Big|_0^1+\frac{1}{2k}\int_0^1 G_{2k}^{Q3}(x,s)\mathrm{d}G_1(x,s)\right] \\ &=\frac{(-1)^{k-1}(k!)^2}{6(2x-1)^2(2k)!}\left[-6(2x-1)^2\int_0^1 G_{2k}^{Q3}(x,s)\mathrm{d}s\right. \\ &\left.+G_{2k}^{Q3}(x,x)+24B_2(x)G_{2k}^{Q3}\left(x,\frac{1}{2}\right)+G_{2k}^{Q3}(x,1-x)\right] \\ &=\frac{(-1)^{k-1}(k!)^2}{36(2x-1)^4(2k)!}\left[2B_{2k}+2B_{2k}(1-2x)+96B_2(x)B_{2k}\left(x+\frac{1}{2}\right)+576B_2^2(x)B_{2k}\right].\end{aligned}$$

If we put $k=2n$ using (11) and (17), we obtain representation (13) and bound (14). For $k=2n+1$ by (12) and (17), representation (15) and estimate (16) follow. □

REMARK 1. Using (10) and (18) we obtain

$$\int_0^1 F_k^{Q3}(x,s)\,\mathrm{d}s = \int_0^1 G_k^{Q3}(x,s)\,\mathrm{d}s - \int_0^1 G_k^{Q3}(x,0)\,\mathrm{d}s = -G_k^{Q3}(x,0)$$

and

$$\int_0^1 \left(F_k^{Q3}(x,s)\right)^2 \mathrm{d}s = \int_0^1 \left(G_k^{Q3}(x,s)\right)^2 \mathrm{d}s - 2G_k^{Q3}(x,0)\int_0^1 G_k^{Q3}(x,s)\,\mathrm{d}s + \left(G_k^{Q3}(x,0)\right)^2.$$

Further, if we put $k = 2n+2$ in the proof of Theorem 3, using (9) similar as in (17) (with $n \leftrightarrow n+1$), we get representation (13) and bound (14).

COROLLARY 1. *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n+1)}$ is absolutely continuous for some $n \geqslant 1$, $\left(f^{(2n+2)}\right)^2 \in L^1[0,1]$ and $f^{(2n+1)} \geqslant 0$ on $[0,1]$. Then for $x \in [1/6, 1/2)$*

$$\begin{aligned} 0 &\leqslant (-1)^n \left\{ \int_0^1 f(s)\mathrm{d}s - Q\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{Q3}(x) \right\} \qquad (19)\\ &\leqslant \frac{1}{6(2x-1)^2}\left[\frac{1}{(4n+2)!}\left(B_{4n+2} + B_{4n+2}(1-2x) + 48B_2(x)B_{4n+2}\left(x+\frac{1}{2}\right)\right.\right.\\ &\quad \left.\left. +288B_2^2(x)B_{4n+2}\right)\right]^{1/2} \cdot \left[\int_0^1 s(1-s)\left(f^{(2n+2)}(t)\right)^2 \mathrm{d}s\right]^{1/2}, \end{aligned}$$

and for $x = 0$

$$\begin{aligned} 0 &\leqslant (-1)^{n+1}\left\{\int_0^1 f(s)\mathrm{d}s - Q\left(0, \frac{1}{2}, 1\right) + T_{2n}^{Q3}(0)\right\} \qquad (20)\\ &\leqslant \frac{1}{3}\left[\frac{1+2^{1-4n}}{2(4n+2)!}B_{4n+2}\right]^{1/2} \cdot \left[\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2 \mathrm{d}s\right]^{1/2}, \end{aligned}$$

where $T_0^{Q3}(0) = T_2^{Q3}(0) = 0$ and

$$T_{2n}^{Q3}(0) = \sum_{k=2}^{n} \frac{1}{3(2k)!}\left(2^{2-2k} - 1\right) B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \qquad (21)$$

Proof. We use Lemma 1, representation (15) and inequality (16) to obtain inequalities (19) and (20). □

As special cases of Theorem 3 for $x = 0, 1/4, 1/6$ we derive inequalities related to Euler Simpson formula, dual Euler Simpson formula and Euler Maclaurin formula, respectively.

COROLLARY 2. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n\geqslant 1$ and $\left(f^{(2n+1)}\right)^2\in L^1[0,1]$. Then the following inequality holds*

$$\left|\int_0^1 f(s)\mathrm{d}s-\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]+T_{2n}^{Q3}(0)\right| \tag{22}$$
$$\leqslant \frac{1}{3}\left[-\frac{1+2^{3-4n}}{2(4n)!}B_{4n}\right]^{1/2}\cdot\left(\int_0^1 s(1-s)\left(f^{(2n+1)}(s)\right)^2\mathrm{d}s\right)^{1/2}.$$

If $f^{(2n+1)}$ is absolutely continuous for some $n\geqslant 0$ and $\left(f^{(2n+2)}\right)^2\in L^1[0,1]$ then

$$\left|\int_0^1 f(s)\mathrm{d}s-\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]+T_{2n}^{Q3}(0)\right| \tag{23}$$
$$\leqslant \frac{1}{3}\left[\frac{1+2^{1-4n}}{2(4n+2)!}B_{4n+2}\right]^{1/2}\cdot\left(\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2\mathrm{d}s\right)^{1/2},$$

where $T_{2n}^{Q3}(0)$ is define as (21).

REMARK 2. Specially, if f' is absolutely continuous and $(f'')^2\in L^1[0,1]$ then for $n=0$ in Corollary 2 we obtain

$$\left|\int_0^1 f(s)\mathrm{d}s-\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]\right|$$
$$\leqslant \frac{1}{6\sqrt{2}}\cdot\left(\int_0^1 s(1-s)\left(f''(s)\right)^2\mathrm{d}s\right)^{1/2}.$$

Further, if f'' is absolutely continuous and $(f''')^2\in L^1[0,1]$ then for $n=1$ in Corollary 2 we get

$$\left|\int_0^1 f(s)\mathrm{d}s-\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]\right|$$
$$\leqslant \frac{1}{24\sqrt{15}}\cdot\left(\int_0^1 s(1-s)\left(f'''(s)\right)^2\mathrm{d}s\right)^{1/2}.$$

COROLLARY 3. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n\geqslant 1$ and $\left(f^{(2n+1)}\right)^2\in L^1[0,1]$. Then the following inequality holds*

$$\left|\int_0^1 f(s)\mathrm{d}s-\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right]+T_{2n}^{Q3}\left(\frac{1}{4}\right)\right| \tag{24}$$
$$\leqslant \frac{1}{3}\left[-\frac{1}{2(4n)!}\left(1+3\cdot 2^{3-4n}-2^{4-8n}\right)B_{4n}\right]^{1/2}\cdot\left(\int_0^1 s(1-s)\left(f^{(2n+1)}(s)\right)^2\mathrm{d}s\right)^{1/2}.$$

If $f^{(2n+1)}$ *is absolutely continuous for some* $n \geqslant 0$ *and* $\left(f^{(2n+2)}\right)^2 \in L^1[0,1]$ *then*

$$\left|\int_0^1 f(s)\mathrm{d}s - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] + T_{2n}^{Q3}\left(\frac{1}{4}\right)\right| \tag{25}$$
$$\leqslant \frac{1}{3}\left[\frac{1}{2(4n+2)!}\left(1+3\cdot 2^{1-4n} - 2^{-8n}\right)B_{4n+2}\right]^{1/2} \cdot \left(\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2 \mathrm{d}s\right)^{1/2},$$

where $T_0^{Q3}\left(\frac{1}{4}\right) = T_2^{Q3}\left(\frac{1}{4}\right) = 0$ *and*

$$T_{2n}^{Q3}\left(\frac{1}{4}\right) = \sum_{k=2}^{n} \frac{1}{3(2k)!}\left(2^{3-4k} - 3\cdot 2^{1-2k} + 1\right)B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

REMARK 3. If f' is absolutely continuous and $(f'')^2 \in L^1[0,1]$ then for $n=0$ in Corollary 3 we obtain

$$\left|\int_0^1 f(s)\mathrm{d}s - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right]\right|$$
$$\leqslant \frac{1}{6}\cdot\left(\int_0^1 s(1-s)\left(f''(s)\right)^2 \mathrm{d}s\right)^{1/2}.$$

If f'' is absolutely continuous and $(f''')^2 \in L^1[0,1]$ then

$$\left|\int_0^1 f(s)\mathrm{d}s - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right]\right|$$
$$\leqslant \frac{1}{48}\sqrt{\frac{13}{30}}\cdot\left(\int_0^1 s(1-s)\left(f'''(s)\right)^2 \mathrm{d}s\right)^{1/2}.$$

COROLLARY 4. *Let* $f:[0,1]\to\mathbb{R}$ *be such that* $f^{(2n)}$ *is absolutely continuous for some* $n \geqslant 1$ *and* $\left(f^{(2n+1)}\right)^2 \in L^1[0,1]$. *Then*

$$\left|\int_0^1 f(s)\mathrm{d}s - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] + T_{2n}^{Q3}\left(\frac{1}{6}\right)\right| \tag{26}$$
$$\leqslant \frac{1}{8}\left[-\frac{1}{2(4n)!}\left(1+7\cdot 3^{2-4n}\right)B_{4n}\right]^{1/2} \cdot \left(\int_0^1 s(1-s)\left(f^{(2n+1)}(s)\right)^2 \mathrm{d}s\right)^{1/2}.$$

If $f^{(2n+1)}$ *is absolutely continuous for some* $n \geqslant 0$ *and* $\left(f^{(2n+2)}\right)^2 \in L^1[0,1]$ *then*

$$\left|\int_0^1 f(s)\mathrm{d}s - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] + T_{2n}^{Q3}\left(\frac{1}{6}\right)\right| \tag{27}$$
$$\leqslant \frac{1}{8}\left[\frac{1}{2(4n+2)!}\left(1+7\cdot 3^{-4n}\right)B_{4n+2}\right]^{1/2} \cdot \left(\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2 \mathrm{d}s\right)^{1/2},$$

where $T_0^{Q3}\left(\frac{1}{6}\right)=T_2^{Q3}\left(\frac{1}{6}\right)=0$ *and*

$$T_{2n}^{Q3}\left(\frac{1}{6}\right)=\sum_{k=2}^{n}\frac{1}{8(2k)!}\left(1-2^{1-2k}\right)\left(1-3^{2-2k}\right)B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

REMARK 4. If f' is absolutely continuous and $(f'')^2\in L^1[0,1]$ then for $n=0$ in Corollary 4 we deduce

$$\left|\int_0^1 f(s)\mathrm{d}s-\frac{1}{8}\left[3f\left(\frac{1}{6}\right)+2f\left(\frac{1}{2}\right)+3f\left(\frac{5}{6}\right)\right]\right| \leqslant \frac{1}{8\sqrt{3}}\cdot\left(\int_0^1 s(1-s)\left(f''(s)\right)^2\mathrm{d}s\right)^{1/2}.$$

Further, if f'' is absolutely continuous and $(f''')^2\in L^1[0,1]$ then for $n=1$ in Corollary 4 we obtain

$$\left|\int_0^1 f(s)\mathrm{d}s-\frac{1}{8}\left[3f\left(\frac{1}{6}\right)+2f\left(\frac{1}{2}\right)+3f\left(\frac{5}{6}\right)\right]\right| \leqslant \frac{1}{72\sqrt{10}}\cdot\left(\int_0^1 s(1-s)\left(f'''(s)\right)^2\mathrm{d}s\right)^{1/2}.$$

In the next theorem we will use the following notation

$$\left[f^{(k)};0,1\right]=f^{(k)}(1)-f^{(k)}(0).$$

THEOREM 4. *Let* $f:[0,1]\to\mathbf{R}$ *be such that* $f^{(2n)}$ *is absolutely continuous function for some* $n\geqslant 1$, $f^{(2n+1)}\geqslant 0$ *on* $[0,1]$ *and* $x\in[0,1/2)$. *Then representation (13) holds and the remainder* $K_{2n}^{Q3}(f)$ *satisfies the estimation*

$$|K_{2n}^{Q3}(f)| \leqslant \frac{1}{(2n-1)!}\left\|G_{2n-1}^{Q3}(x,s)\right\|_\infty\left\{\frac{f^{(2n-1)}(0)+f^{(2n-1)}(1)}{2}-\left[f^{(2n-2)};0,1\right]\right\}. \tag{28}$$

If $f^{(2n+1)}$ *is absolutely continuous function, for some* $n\geqslant 0$, $f^{(2n+2)}\geqslant 0$ *on* $[0,1]$ *and* $x\in[0,1/2)$, *then representation (15) holds and the remainder* $K_{2n+1}^{Q3}(f)$ *satisfies the estimation*

$$|K_{2n+1}^{Q3}(f)|\leqslant\frac{1}{(2n)!}\left\|G_{2n}^{Q3}(x,s)\right\|_\infty\left\{\frac{f^{(2n)}(0)+f^{(2n)}(1)}{2}-\left[f^{(2n-1)};0,1\right]\right\}. \tag{29}$$

Proof. Applying Theorem 2 with G_{2n}^{Q3} in place of f and $f^{(2n)}$ in place of g we deduce

$$\left|\int_0^1 G_{2n}^{Q3}(x,s)f^{(2n)}(s)\mathrm{d}s-\int_0^1 G_{2n}^{Q3}(x,s)\mathrm{d}s\cdot\int_0^1 f^{(2n)}(s)\mathrm{d}s\right| \leqslant\frac{2n}{2}\left\|G_{2n-1}^{Q3}(x,s)\right\|_\infty\int_0^1 s(1-s)f^{(2n+1)}(s)\,\mathrm{d}s. \tag{30}$$

Further,

$$\int_0^1 s(1-s) f^{(2n+1)}(s)\,\mathrm{d}s = \int_0^1 (2s-1) f^{(2n)}(s)\,\mathrm{d}s$$
$$= f^{(2n-1)}(1) + f^{(2n-1)}(0) - 2\left[f^{(2n-2)}(1) - f^{(2n-2)}(0)\right].$$

Hence, using representation (13) and inequality (30), we get estimate (28). Similarly, using identity (15) we obtain inequality (29). □

3. Applications for Euler Bullen-Simpson formula

P. S. Bullen in [3] proved that, if $f:[0,1]\to\mathbf{R}$ is a 4-convex function, then the dual Simpson quadrature rule is more accurate than the Simpson quadrature rule, that is

$$0 \leqslant \int_0^1 f(s)\mathrm{d}s - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right]$$
$$\leqslant \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(s)\mathrm{d}s. \tag{31}$$

In [7] the authors established some generalizations of inequality (31) for a class of $(2r)$-convex functions and obtained some estimates for the absolute value of difference between the absolute value of error in the dual Simpson quadrature rule and the absolute value of error in the Simpson quadrature rule. Let us define

$$D(0,1) = \frac{1}{12}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right].$$

It is suitable for our purposes to rewrite inequality (31) in the form

$$\int_0^1 f(s)\mathrm{d}s \leqslant D(0,1).$$

In literature this inequality is known as the Bullen-Simpson inequality.

We consider the sequences of functions $(G_k(s))_{k\geqslant 1}$ and $(F_k(s))_{k\geqslant 1}$ defined by

$$G_k(s) = B_k^*(1-s) + 2B_k^*\left(\frac{1}{4}-s\right) + B_k^*\left(\frac{1}{2}-s\right) + 2B_k^*\left(\frac{3}{4}-s\right), \quad s\in\mathbb{R}$$

and

$$F_k(s) = G_k(s) - \tilde{B}_k, \; s\in\mathbb{R} \tag{32}$$

where

$$\tilde{B}_k = B_k + 2B_k\left(\frac{1}{4}\right) + B_k\left(\frac{1}{2}\right) + 2B_k\left(\frac{3}{4}\right).$$

For any function $f:[0,1]\to\mathbb{R}$ such that $f^{(n-1)}$ exists on $[0,1]$ for some $n\geqslant 1$ we define $T_0(f)=T_1(f)=0$ and for $2\leqslant m\leqslant \lfloor n/2\rfloor$

$$T_m(f)=\frac{1}{3}\sum_{k=2}^{m}\frac{1}{(2k)!}2^{-2k}(1-2^{2-2k})B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]. \tag{33}$$

In the next lemma from [7] the authors established the Euler Bullen-Simpson formulae.

LEMMA 2. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n\geqslant 1$. Then*

$$\int_0^1 f(s)\mathrm{d}s=D(0,1)+T_p(f)+R_n(f), \tag{34}$$

and

$$\int_0^1 f(s)\mathrm{d}s=D(0,1)+T_r(f)+\hat{R}_n(f), \tag{35}$$

where

$$R_n(f)=\frac{1}{6\,(n!)}\int_0^1 G_n(s)\,\mathrm{d}f^{(n-1)}(s),$$

$$\hat{R}_n(f)=\frac{1}{6\,(n!)}\int_0^1 F_n(s)\,\mathrm{d}f^{(n-1)}(s),$$

$p=\lfloor n/2\rfloor$ *and* $r=\lfloor (n-1)/2\rfloor$.

Using Theorem 1 for identity (34) we obtain the following representation of Euler Bullen-Simpson formula and a related Grüss type inequality.

THEOREM 5. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $\left(f^{(n+1)}\right)^2\in L^1[0,1]$ for some $n\geqslant 1$ and $p=\lfloor n/2\rfloor$. Then*

$$\int_0^1 f(s)\mathrm{d}s-D(0,1)-T_p(f)=K_n(f), \tag{36}$$

and the remainder $K_n(f)$ satisfies the estimation

$$|K_n(f)|\leqslant\frac{1}{6}\left[\frac{(-1)^{n-1}}{(2n)!}2^{1-2n}\left(1+2^{3-2n}\right)B_{2n}\right]^{1/2}\times\left[\int_0^1 s(1-s)\left(f^{(n+1)}(s)\right)^2\mathrm{d}s\right]^{1/2}. \tag{37}$$

Proof. Applying Theorem 1 for $f \to G_n,\ g \to f^{(n)}$, we deduce

$$\left| \int_0^1 G_n(s) f^{(n)}(s) \mathrm{d}s - \int_0^1 G_n(s) \, \mathrm{d}s \cdot \int_0^1 f^{(n)}(s) \mathrm{d}s \right| \leqslant \frac{1}{\sqrt{2}} \left[T(G_n(\cdot), G_n(\cdot)) \right]^{1/2} \cdot \left[\int_0^1 s(1-s) \left(f^{(n+1)}(s) \right)^2 \mathrm{d}s \right]^{1/2}, \tag{38}$$

where

$$T(G_n(\cdot), G_n(\cdot)) = \int_0^1 (G_n(s))^2 \, \mathrm{d}s - \left[\int_0^1 G_n(s) \, \mathrm{d}s \right]^2.$$

By elementary calculations we get $\int_0^1 G_n(s) \, ds = 0$ and using integration by parts we obtain

$$\begin{aligned} \int_0^1 (G_n(s))^2 \, \mathrm{d}s &= (-1)^{n-1} \frac{n(n-1)\ldots 2}{(n+1)(n+2)\ldots(2n-1)} \left[\int_0^1 G_1(s) G_{2n-1}(s) \mathrm{d}s \right] \\ &= (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[-6 \int_0^1 G_{2n}(s) \mathrm{d}s + 2G_{2n}(0) + 4G_{2n}\left(\frac{1}{4}\right) \right] \\ &= (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[10B_{2n} + 16B_{2n}\left(\frac{1}{4}\right) + 10B_{2n}\left(\frac{1}{2}\right) \right]. \end{aligned}$$

Using (34) and (38), we deduce representation (36) and bound (37). □

REMARK 5. Because of (32) we obtain

$$\int_0^1 F_k(s) \, \mathrm{d}s = \int_0^1 G_k(s) \, \mathrm{d}s - \int_0^1 \tilde{B}_k \mathrm{d}s = -\tilde{B}_k,$$

and also

$$\int_0^1 (F_k(s))^2 \, \mathrm{d}s = \int_0^1 (G_k(s))^2 \, \mathrm{d}s - 2\tilde{B}_k \int_0^1 G_k(s) \, \mathrm{d}s + \tilde{B}_k^2.$$

So, using (35) similar as in (38), we deduce representation (36) and inequality (37), too.

The following Grüss type inequality also holds.

THEOREM 6. *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \geqslant 0$ on $[0,1]$. Then representation (36) holds and the remainder $K_n(f)$ satisfies the bound*

$$|K_n(f)| \leqslant \frac{1}{6(n-1)!} ||G_{n-1}(s)||_\infty \left\{ \frac{f^{(n-1)}(0) + f^{(n-1)}(1)}{2} - \left[f^{(n-2)}; 0, 1 \right] \right\}. \tag{39}$$

Proof. Applying Theorem 2 for $f \to G_n,\ g \to f^{(n)}$, we obtain

$$\begin{aligned}&\left|\int_0^1 G_n(s)\, f^{(n)}(s)\mathrm{d}s - \int_0^1 G_n(s)\,\mathrm{d}s \cdot \int_0^1 f^{(n)}(s)\mathrm{d}s\right| \\ \leqslant\ & \frac{n}{2}\,||G_{n-1}(s)||_\infty \left(\int_0^1 s(1-s) f^{(n+1)}(s)\mathrm{d}s\right). \qquad (40)\end{aligned}$$

So, similarly as in Theorem 4, using equality (36) and inequality (40), we deduce (39). □

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(Received July 30, 2014)

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