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QUASI–ARITHMETIC MEANS AND SUBQUADRACITY

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In honour of Professor Josip Pečarić,
on the occasion of publishing more than 1000 papers

(*Communicated by S. Varošanec*)

Abstract. The inequalities derived in this article for quasi-arithmetic means and power means are related to subquadratic and superquadratic functions.

1. Introduction

The inequalities we peresent in this paper are obtained by using basic properties of the quasi-arithmetic mean (see [5, p. 215])

$$M_f(\mathbf{x},\boldsymbol{\lambda}) = f^{-1}\left(\sum_{r=1}^{n}\lambda_r f(x_r)\right), \tag{1.1}$$
$$\sum_{r=1}^{n}\lambda_r = 1, \qquad \lambda_r \geqslant 0,\ x_{r,} \geqslant 0,\ r = 1,\dots,n\,,$$

and by investigating and consequently by using the properties of the quasi-mean

$$W_f(\mathbf{x},\boldsymbol{\lambda}) = f^{-1}\left(f\left(\sum_{r=1}^{n}\lambda_r x_r\right) + \sum_{r=1}^{n}\lambda_r f\left(\left|x_r - \sum_{i=1}^{n}\lambda_i x_i\right|\right)\right) \tag{1.2}$$
$$\sum_{r=1}^{n}\lambda_r = 1, \qquad \lambda_r \geqslant 0,\ x_{r,} \geqslant 0,\ r = 1,\dots,n\,.$$

Here $\mathbf{x} = (x_1,\dots,x_n)$, $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n)$, and the functions f in (1.1) and in (1.2) are strictly increasing convex functions satisfying $f(0) = 0$. These convex functions obviously include the power functions $f(x) = x^p$, $p \geqslant 1$, $x \geqslant 0$. In this case we denote

$$W_p(\mathbf{x},\boldsymbol{\lambda}) = \left(\left(\sum_{r=1}^{n}\lambda_r x_r\right)^p + \sum_{r=1}^{n}\lambda_r\left(\left|x_r - \sum_{i=1}^{n}\lambda_i x_i\right|\right)^p\right)^{\frac{1}{p}} \tag{1.3}$$
$$\sum_{r=1}^{n}\lambda_r = 1, \qquad \lambda_r \geqslant 0,\ x_r \geqslant 0,\ r = 1,\dots,n\,.$$

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When $f(x)=x^2$, $x \geqslant 0$, the two means $M_2(\mathbf{x},\boldsymbol{\lambda})$ and $W_2(\mathbf{x},\lambda)$ coincide. This leads to results concerning the question: what can we say about the difference $M_f(\mathbf{x},\boldsymbol{\lambda}) - W_f(\mathbf{x},\boldsymbol{\lambda})$, in particular when this difference grows with respect to f.

The inequalities we get for $W_f(\mathbf{x},\boldsymbol{\lambda})$ are used to obtain results related to superquadratic and subquadratic functions. The superquadratic and subquadratic functions satisfy Jensen's type inequalities, and lately are dealt with extensively, see for instance two of the latest papers on this subject [4] and [7]. This paper adds to this subject too.

We start with quoting a definition and some results from [1], a 2009 short survey on superquadracity which are relevant for our discussion here (see also references in [1]).

DEFINITION A. A function f defined on an interval $I=[0,b)$, $0<b\leqslant\infty$ is said to be subquadratric if for each x in I, there exists a real number $C(x)$ such that

$$f(y)-f(x)\leqslant f(|y-x|)+C(x)(y-x) \tag{1.4}$$

for all $y\in I$. The function f is superquadratic if $-f$ is subquadratic.

REMARK A. From (1.4) it easily follows that $f(0)\geqslant 0$ for subquadratic functions. Therefore if f is subquadratic and increasing, then f is nonnegative.

From (1.4) it is also easy to verify that:

LEMMA A. *Let f be defined on $[0,b)$, $0<b\leqslant\infty$. Let $\sum_{r=1}^{n}\lambda_r=1$, $\lambda_r\geqslant 0$, $x_r\in[0,b)$, $r=1,\ldots,n$ then,*

$$\sum_{r=1}^{n}\lambda_r f(x_r)\leqslant f\left(\sum_{i=1}^{n}\lambda_i x_i\right)+\sum_{r=1}^{n}\lambda_r f\left(\left|x_r-\sum_{i=1}^{n}\lambda_i x_i\right|\right) \tag{1.5}$$

if and only if f is subquadratic. The reverse of inequality (1.5) holds if and only if f is superquadratic .

LEMMA B. *Suppose that $f:[0,b)\to\mathbb{R}$ is continuously differentiable and $f(0)\leqslant 0$. If f' is superadditive or $\frac{f'(x)}{x}$ is nondecreasing, then f is superquadratic.*

REMARK B. The following functions are examples of subquadratic increasing functions which are also convex (see [1]).

$$\begin{array}{lll}
a) & f(x)=x^p, \quad 1\leqslant p\leqslant 2, \quad x\geqslant 0, \\
b) & f(x)=(1+x^p)^{\frac{1}{p}}, \quad 1\leqslant p, \quad x\geqslant 0, \\
c) & f(x)=(1+x^p)^{\frac{1}{p}}-1, \quad 1\leqslant p\leqslant 2, \quad x\geqslant 0, \\
d) & f(x)=3x^2-2x^2\log(x), \quad 0\leqslant x\leqslant 1.
\end{array}$$

When we deal with the properties of $M_f(\mathbf{x},\boldsymbol{\lambda})-W_f(\mathbf{x},\boldsymbol{\lambda})$, we use Theorem A below about $M_f(\mathbf{x},\boldsymbol{\lambda})$, (see [3, Theorem 1]), and a theorem we prove about $W_f(\mathbf{x},\boldsymbol{\lambda})$ which is similar to Theorem A:

THEOREM A. *Let $x_i \in [A,B]$, $i=1,\dots,n$ be real numbers. Let F_k, $k=1,\dots,m$ be one to one functions defined on $[A,B]$ where $F_k : [A,B] \to [C,D]$, $k=1,\dots,m$ and let $f_1 = F_1$, $f_{k+1} = F_{k+1} \circ F_k^{-1}$, $k=1,\dots,m-1$ be convex increasing functions.*

Let $F_0(x) = x$, $x \in [A,B]$, and $0 \leqslant \lambda_i$, $i=1,\dots,n$, $\sum_{i=1}^{n} \lambda_i = 1$; then we get the following comparison between the quasi-arithmetic mean $M_{F_k}(\mathbf{x},\boldsymbol{\lambda})$ and the quasi-arithmetic mean $M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda})$:

$$M_{F_m}(\mathbf{x},\boldsymbol{\lambda}) \geqslant,\dots,\geqslant M_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \geqslant M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \geqslant,\dots,\geqslant M_1(\mathbf{x},\boldsymbol{\lambda}), \quad k=1,\dots,m.$$

Equality holds iff $F_k \circ F_{k-1}^{-1} = L \cdot x + R$, where L, R are constant, or if all λ_i, $i=1,\dots,n$ are equal.

We use also the following theorem that appeared in [6] and as a special case of Jensen-Steffensen Inequality in [2, Theorem 2], to prove the analog of Theorem A about $W_f(\mathbf{x},\boldsymbol{\lambda})$:

THEOREM B. *Let $f : I \to \mathbb{R}$, where I is any interval in $\mathbb{R}$, and let $[a,b] \subseteq I$, $a<b$. Let $\mathbf{z} = (z_1,\cdots,z_n)$ be a monotonic n-tuple in $[a,b]^n$ and $\mathbf{v} = (v_1,\cdots,v_n)$ a real n-tuple such that $v_i \neq 0$, $i=1,\dots,n$ and $0 \leqslant V_j \leqslant V_n$, $j=1,\dots,n$, $V_n > 0$, $V_j = \sum_{i=1}^{j} v_i$. If f is convex on I, then*

$$f\left(a+b-\frac{1}{V_n}\sum_{i=1}^{n} v_i z_i\right) \leqslant f(a)+f(b)-\frac{1}{V_n}\sum_{i=1}^{n} v_i f(z_i).$$

2. The quasi-mean $\mathbf{W}_f(\mathbf{x},\boldsymbol{\lambda})$

DEFINITION 1. Let a strictly increasing convex function f be defined on $[0,b)$, $0<b\leqslant\infty$, and let $f(0)=0$. For such f we define the quasi-mean $W_f(x_1,x_2)$ as

$$W_f(x_1,x_2) = f^{-1}\left(f\left(\frac{x_1+x_2}{2}\right)+f\left(\left|\frac{x_1-x_2}{2}\right|\right)\right). \tag{2.1}$$

In the special case that $f(x) = x^p$, $W_p(x_1,x_2)$ is defined as

$$W_p(x_1,x_2) = \left(\left(\frac{x_1+x_2}{2}\right)^p+\left(\left|\frac{x_1-x_2}{2}\right|\right)^p\right)^{\frac{1}{p}} \tag{2.2}$$

for $x \geqslant 0$, $p \geqslant 1$.

LEMMA 1. *Under the conditions of Definition 1 on f, $W_f(x_1,x_2)$ is symmetric and satisfies:*

a) $W_f(x,x) = x$, $x \geqslant 0$,

b) $x_1 \leqslant W_f(x_1,x_2) \leqslant x_2$, $0 \leqslant x_1 \leqslant x_2 \leqslant b$.

When $f(x) = x^p$, $p \geqslant 1$, $W_p(x_1,x_2)$ satisfies

c) $W_p(\lambda x_1,\lambda x_2) = \lambda W_p(x_1,x_2)$, $\lambda \geqslant 0$, $0 \leqslant x_1$, $x_2 \leqslant b$.

Proof. Properties a) and c) are obvious. To prove that b) holds, that is that

$$x_1 \leqslant f^{-1}\left(f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2-x_1}{2}\right)\right) \leqslant x_2, \quad 0 \leqslant x_1 \leqslant x_2 \leqslant b,$$

holds, we use the change of variables $y_1 = \frac{x_2-x_1}{2}$, $y_2 = \frac{x_1+x_2}{2}$ and we have to prove that

$$f(y_2-y_1) \leqslant f(y_1)+f(y_2) \leqslant f(y_1+y_2),$$

which holds because of the convexity of f and because $f(0)=0$.

This completes the proof of the lemma. □

We now extend $W_f(x_1,x_2)$ and $W_p(x_1,x_2)$ as defined in (2.1) and (2.2) above to $W_f(\mathbf{x},\boldsymbol{\lambda})$ and $W_p(\mathbf{x},\boldsymbol{\lambda})$. Motivated by Lemma A the extension is explained here:

First, instead of $\frac{x_1+x_2}{2}$ that can be seen as a mid point of x_1 and x_2 we replace it with the general average $\lambda_1 x_1 + \lambda_2 x_2$ $(\lambda_1,\lambda_2 \geqslant 0,\ \lambda_1+\lambda_2 = 1)$ of x_1 and x_2.

Now we look at $f\left(\left|\frac{x_1-x_2}{2}\right|\right)$ as $\frac{1}{2}f\left(\left|x_1 - \frac{x_1+x_2}{2}\right|\right) + \frac{1}{2}f\left(\left|x_2 - \frac{x_1+x_2}{2}\right|\right)$ and replace it with $\lambda_1 f(|x_1-(\lambda_1x_1+\lambda_2x_2)|) + \lambda_2 f(|x_2-(\lambda_1x_1+\lambda_2x_2)|)$.

We denote $\mathbf{x} = (x_1,x_2)$, and $\boldsymbol{\lambda} = (\lambda_1,\lambda_2)$ and

$$W_f(\mathbf{x},\boldsymbol{\lambda}) = f^{-1}\left(f(\lambda_1x_1+\lambda_2x_2)+\lambda_1 f\left(\left|x_1-\sum_{i=1}^{2}\lambda_i x_i\right|\right)+\lambda_2 f\left(\left|x_2-\sum_{i=1}^{2}\lambda_i x_i\right|\right)\right)$$

which coincide with $W_f(x_1,x_2)$ for $\lambda_1 = \lambda_2 = \frac{1}{2}$.

From here, the extension to the general case of n variables is

$$W_f(\mathbf{x},\boldsymbol{\lambda}) = f^{-1}\left(f\left(\sum_{r=1}^{n}\lambda_r x_r\right)+\sum_{r=1}^{n}\lambda_r f\left(\left|x_r-\sum_{i=1}^{n}\lambda_i x_i\right|\right)\right)$$
$$\sum_{r=1}^{n}\lambda_r = 1,\ \lambda_r \geqslant 0,\ x_r \geqslant 0,\ r = 1,\dots,n\ .$$

The same motivation leads to the extension of $W_p(x_1,x_2)$ to $W_p(\mathbf{x},\boldsymbol{\lambda})$.

In the general case when $\boldsymbol{\lambda} \neq \left(\frac{1}{2},\frac{1}{2}\right)$, the results we get are under the restrictions $\left|x_r - \sum_{i=1}^{n}\lambda_i x_i\right| \leqslant \sum_{i=1}^{n}\lambda_i x_i,\ r = 1,\dots,n.$

REMARK 1. Two sufficient conditions for

$$\left|x_r-\sum_{i=1}^{n}\lambda_i x_i\right| \leqslant \sum_{i=1}^{n}\lambda_i x_i, \quad x_r,\lambda_r \geqslant 0, \quad r=1,\dots,n, \quad \sum_{i=1}^{n}\lambda_i = 1, \tag{2.3}$$

which is equivalent to $0 \leqslant x_i \leqslant 2\sum_{j=1}^{n}\lambda_j x_j,\ i = 1,\dots,n$ to hold are:

1. $0 < a \leqslant x_i \leqslant 2a,\ i = 1,\dots,n$,

and

2. $\lambda_j \geqslant \frac{1}{2}$ when $x_j = \max\limits_{i=1,\dots,n} x_i,\ x_i \geqslant 0,\ \lambda_i \geqslant 0,\ i = 1,\dots,n.$

In particular if $n=2$, $\lambda_1 = \lambda_2 = \frac{1}{2}$, then (2.3) holds always for $x_r \geqslant 0,\ r = 1,2$, and in this case there are no restrictions on $x_r \geqslant 0,\ r = 1,2$, when discussing $W_f(x_1,x_2)$ as defined in (2.1).

Here is our main theorem on $W_f(\mathbf{x},\boldsymbol{\lambda})$ which is a comparison theorem between $W_{F_k}(\mathbf{x},\lambda)$ and $W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda})$:

THEOREM 1. *Let $x_i \in [A,B]$, $i=1,\ldots,n$ be real numbers, $0 \leqslant A < B < b \leqslant \infty$ and let (2.3) be satisfied. Let $F_0(x)=x$, $F_k : [0,b) \to [0,d)$, $k=1,\ldots,m$ be strictly increasing functions, $F_k(0)=0$, $k=0,1,\ldots,m$, and let $f_1=F_1$, $f_{k+1}=F_{k+1}\circ F_k^{-1}$, $k=1,\ldots,m-1$ be convex increasing functions. Then:*

$$W_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \leqslant W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}), \qquad \boldsymbol{\lambda}>0, \qquad \mathbf{x}\in[0,b)^n, \qquad k=1,\ldots,m. \tag{2.4}$$

If in addition, for a specific k, $\left(F_k^{-1}\right)^2$ is convex then:

$$W_{F_{k+1}}(\mathbf{x},\boldsymbol{\lambda}) = W_2(\mathbf{x},\boldsymbol{\lambda}) \leqslant W_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \leqslant W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}), \tag{2.5}$$

or if for a specific k, $F_{k+1}\circ\sqrt{x}$ is convex then:

$$W_{F_{k+1}}(\mathbf{x},\boldsymbol{\lambda}) = W_2(\mathbf{x},\boldsymbol{\lambda}) \geqslant W_{F_{k+2}}(\mathbf{x},\boldsymbol{\lambda}) \geqslant W_{F_{k+3}}(\mathbf{x},\boldsymbol{\lambda}). \tag{2.6}$$

Proof. In order to show that under the conditions on F_k, $k=0,1,\ldots,m$, inequality (2.4) holds, that means that

$$\begin{aligned} &F_k^{-1}\left(F_k\left(\sum_{r=1}^n \lambda_r x_r\right)+\sum_{r=1}^n \lambda_r F_k\left(\left|x_r-\sum_{i=1}^n \lambda_i x_i\right|\right)\right) \\ &\leqslant F_{k-1}^{-1}\left(F_{k-1}\left(\sum_{r=1}^n \lambda_r x_r\right)+\sum_{r=1}^n \lambda_r F_{k-1}\left(\left|x_r-\sum_{i=1}^n \lambda_i x_i\right|\right)\right) \end{aligned} \tag{2.7}$$

holds, we denote:

$$\begin{aligned} F_{k-1}\left(\sum_{j=1}^n \lambda_j x_j\right) &= y_{n+1}, \\ F_{k-1}\left(\left|x_i-\sum_{j=1}^n \lambda_j x_j\right|\right) &= y_i, \qquad i=1,\ldots,n. \end{aligned} \tag{2.8}$$

Then from (2.7) and (2.8) as F_k is increasing and $f_k=F_k\circ F_{k-1}^{-1}$, we get that we have to prove the inequality

$$f_k(y_{n+1})+\sum_{i=1}^n \lambda_i f_k(y_i) \leqslant f_k\left(y_{n+1}+\sum_{j=1}^n \lambda_j y_j\right), \qquad f_k=F_k\circ F_{k-1}^{-1}. \tag{2.9}$$

As it is given that $F_k(0)=0$ and F_k, $k=0,1,\ldots,m$, are strictly increasing, we get that $f_k(0)=0$, and because (2.3) holds and F_{k-1} is increasing we get that $0\leqslant y_i\leqslant y_{n+1}$, $i=1,\ldots,n$.

Rewriting (2.9) and using $f_k(0)=0$, it remains to prove that

$$f_k(y_{n+1}) = f_k\left(0+\left(y_{n+1}+\sum_{j=1}^{n}\lambda_j y_j\right)-\sum_{j=1}^{n}\lambda_j y_j\right) \tag{2.10}$$
$$\leqslant f_k(0)+f_k\left(y_{n+1}+\sum_{j=1}^{n}\lambda_j y_j\right)-\sum_{i=1}^{n}\lambda_i f_k(y_i).$$

With no loss of generality we may assume that $y_i \leqslant y_{i+1}$, $i=1,\ldots,n-1$. From (2.3) and because F_{k-1} is increasing we get that $y_{n+1} \geqslant y_i$, $i=1,\ldots,n$ and hence as F_{k-1} is non-negative $y_{n+1}+\sum_{j=1}^{n}\lambda_j y_j \geqslant y_i$, $i=1,\ldots,n$. Then, using Theorem B with $a=0$, $b=y_{n+1}+\sum_{j=1}^{n}\lambda_j y_j$, $\lambda_i=\frac{v_i}{V_n}$, $z_i=y_i$, $i=1,\ldots,n$ for the covex function f_k we get that (2.10) holds and therefore it is proved that (2.4) holds.

The first inequality in (2.5) holds if $\left(F_k^{-1}\right)^2$ is convex because in this case we may denote $F_{k+1}(x)=x^2$ and therefore it follows from (2.4). Similarly, the first inequality in (2.6) holds because again we may denote $F_{k+1}(x)=x^2$.

This completes the proof of the theorem. $\square$

3. Subquadracity, superquadracity, and $\mathbf{M}_f(\mathbf{x},\boldsymbol{\lambda})-\mathbf{W}_f(\mathbf{x},\boldsymbol{\lambda})$

Subquadratic functions satisfy (1.5). With some restrictions on x_r, λ_r, $r=1,\ldots,n$ there, we get some new inequalities related to subquadratic and superquadratic functions.

Using Lemma A and Remark 1, the following lemma is immediate.

LEMMA 2. *Let f be an increasing subquadratic function on $[0,b)$, $0<b\leqslant\infty$, and let (2.3) be given. Then,*

$$\sum_{r=1}^{n}\lambda_r f(x_r) \leqslant 2f\left(\sum_{r=1}^{n}\lambda_r x_r\right). \tag{3.1}$$

In particular, if $0<a\leqslant x_r\leqslant 2a$, $r=1,\ldots,n$, or if $\lambda_j\geqslant\frac{1}{2}$ when $x_j=\max_{i=1,\ldots,n} x_i$, $x_i\geqslant 0$, $0\leqslant\lambda_i\leqslant 1$, $i=1,\ldots,n$, then (2.3) holds and therefore (3.1) holds.

In the special case $n=2$, $\lambda_1=\lambda_2=\frac{1}{2}$, for an increasing subquadratic functions we get that for $x_i\geqslant 0$, $i=1,2$

$$\frac{1}{2}\left(f(x_1)+f(x_2)\right)\leqslant 2f\left(\frac{x_1+x_2}{2}\right)$$

holds.

EXAMPLE 1. The function $f(x)=3x^2-2x^2\log(x)$, $0\leqslant x\leqslant 1$, (see Case d in remark B) is an increasing subquadratic function which is also convex. Therefore when (2.3) is satisfied the inequalities

$$f\left(\sum_{r=1}^{n}\lambda_r x_r\right)\leqslant\sum_{r=1}^{n}\lambda_r f(x_r)\leqslant 2f\left(\sum_{r=1}^{n}\lambda_r x_r\right)$$

hold, and when $n=2,\ \lambda_1=\lambda_2=\frac{1}{2}$, the inequalities

$$f\left(\frac{x_1+x_2}{2}\right)\leqslant\frac{1}{2}\left(f\left(x_1\right)+f\left(x_2\right)\right)\leqslant 2f\left(\frac{x_1+x_2}{2}\right)$$

hold.

Replacing in (1.5) f with $g\circ f^{-1}$, x_i with $f\left(x_i\right)$, $i=1,\ldots,n$ we get that for a subquadratic function $g\circ f^{-1}$ on $[0,b)$ and for strictly increasing functions f and g when $x_i\in[0,b)$, $\lambda_i\geqslant 0$, $i=1,\ldots,n$, $\sum_{i=1}^{n}\lambda_i=1$ the following inequality

$$g^{-1}\left(\sum\lambda_i g\left(x_i\right)\right)\leqslant g^{-1}\left[g\circ f^{-1}\left(\sum\lambda_i f\left(x_i\right)\right)+\sum\lambda_i g\circ f^{-1}\left(\left|f\left(x_i\right)-\sum\lambda_j f\left(x_j\right)\right|\right)\right] \tag{3.2}$$

holds. Equality holds for $g\circ f^{-1}(x)=x^2$.

As the results in Theorem 2 and Theorem 3 below are obtained under the restriction stated in (2.3) we add the following definition:

DEFINITION 2. A function f defined on an interval $I=[0,b)$, $0<b\leqslant\infty$ is called *semi-subquadratic (semi-superquadratic)* function if the inequality

$$\sum_{r=1}^{n}\lambda_r f\left(x_r\right)\underset{(\geqslant)}{\leqslant} f\left(\sum_{i=1}^{n}\lambda_i x_i\right)+\sum_{r=1}^{n}\lambda_r f\left(\left|x_r-\sum_{i=1}^{n}\lambda_i x_i\right|\right)$$

holds under the restriction stated in (2.3).

It is obvious that subquadratic (superquadratic) functions are also semi-subquadratic (semi-superquadratic).

In the case that $n=2,\ \lambda_1=\lambda_2=\frac{1}{2}$ subquadracity (superquadracity) is the same as semi-subquadracity (semi-superquadracity).

The following theorem combines the inequalities that $W_f\left(\mathbf{x},\boldsymbol{\lambda}\right)$ satisfies as proved in Theorem 1, the properties of $M_f\left(\mathbf{x},\boldsymbol{\lambda}\right)$ as stated in Theorem A and the basic properties of superquadratic, semi-superquadratic, subquadratic and semi-subquadratic functions to get additional inequalities related to these functions and in particular to power functions. Especially, we see that the difference $M_f\left(\mathbf{x},\boldsymbol{\lambda}\right)-W_f\left(\mathbf{x},\boldsymbol{\lambda}\right)$ increases as the distance between $f(x)$ and x^2 grows.

THEOREM 2. *Let (2.3) be given. Let F_k, $k=1,\ldots,m$, be nonnegative strictly increasing functions on $[0,b)$ satisfying $F_k(0)=0$, $k=1,\ldots,m$, and $F_0(x)=x$, $x\in[0,b)$. Let $f_k=F_k\circ F_{k-1}^{-1}$, $k=1,\ldots,m$, be convex functions. Let $\boldsymbol{\lambda}>0$, and $\mathbf{x}\in[0,b)^n$.*

Case I: If F_k is subquadratic or semi-subquadratic function, then F_{k-1} is semi-subquadratic too,

$$M_1\left(\mathbf{x},\boldsymbol{\lambda}\right)\leqslant M_{F_{k-1}}\left(\mathbf{x},\boldsymbol{\lambda}\right)\leqslant M_{F_k}\left(\mathbf{x},\lambda\right) \tag{3.3}$$

$$\leqslant W_{F_k}\left(\mathbf{x},\boldsymbol{\lambda}\right)\leqslant W_{F_{k-1}}\left(\mathbf{x},\boldsymbol{\lambda}\right)\leqslant F_{k-1}^{-1}\left(2F_{k-1}\left(\sum_{j=1}^{n}\lambda_j x_j\right)\right),$$

and

$$M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) - W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_{F_k}(\mathbf{x},\lambda) - W_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \leqslant 0$$

hold. In particular if $\left(F_k^{-1}\right)^2$ *is convex then:*

$$\begin{aligned} M_1(\mathbf{x},\boldsymbol{\lambda}) &\leqslant M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_{F_k}(\mathbf{x},\lambda) \leqslant M_2(\mathbf{x},\boldsymbol{\lambda}) = W_2(\mathbf{x},\boldsymbol{\lambda}) \\ &\leqslant W_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \leqslant W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \leqslant F_{k-1}^{-1}\left(2F_{k-1}\left(\sum_{i=1}^{n}\lambda_i x_i\right)\right). \end{aligned}$$

and

$$M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) - W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_{F_k}(\mathbf{x},\lambda) - W_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_2(\mathbf{x},\boldsymbol{\lambda}) - W_2(\mathbf{x},\boldsymbol{\lambda}) = 0$$

holds where $W_2(\mathbf{x},\boldsymbol{\lambda})$ *is as defined in (1.3).*

Case II: If F_{k-1} *is superquadratic or semi-superquadratic functions then* F_k *is semi-superquadratic too and*

$$M_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \geqslant M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \geqslant W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \geqslant W_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \geqslant,\ldots,\geqslant M_1(\mathbf{x},\boldsymbol{\lambda}).$$

In particular if $F_{k-1}\circ\sqrt{x}$ *is convex then:*

$$\begin{aligned} M_{F_k}(\mathbf{x},\boldsymbol{\lambda}) &\geqslant M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \geqslant M_2(\mathbf{x},\lambda) = W_2(\mathbf{x},\boldsymbol{\lambda}) \\ &\geqslant W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \geqslant W_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \geqslant M_1(\mathbf{x},\boldsymbol{\lambda}), \end{aligned}$$

and

$$0 = M_2(\mathbf{x},\boldsymbol{\lambda}) - W_2(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) - W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_{F_k}(\mathbf{x},\boldsymbol{\lambda}) - W_{F_k}(\mathbf{x},\boldsymbol{\lambda})$$

REMARK 2. If $F_{k-1}\circ\sqrt{x}$ is differentiable, nonnegative, convex increasing and $F_{k-1}(0)=0$ then F_{k-1} according to Lemma B is superquadratic.

Proof. Case I: From (1.5) for the subquadratic function F_k, $k=1,\ldots,m$, that is also nonnegative, strictly increasing functions we get that

$$M_{F_k}(\mathbf{x},\boldsymbol{\lambda}) \leqslant W_{F_k}(\mathbf{x},\boldsymbol{\lambda}), \quad k=1,\ldots,m. \tag{3.4}$$

As $f_k = F_k\circ F_{k-1}^{-1}$, $k=1,\ldots,m$, is also convex, therefore by Theorem A we get that

$$M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_{F_k}(\mathbf{x},\boldsymbol{\lambda}). \tag{3.5}$$

According to Theorem 1, under the conditions on F_k, $k=1,\ldots,m$ (2.4) holds.

Therefore from inequalities (3.4), (3.5) and (2.4) we get that (3.3) holds. From (3.4), (3.5) and (2.4) we get also that $M_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda}) \leqslant W_{F_{k-1}}(\mathbf{x},\boldsymbol{\lambda})$, which means that F_{k-1} is semi-subquadratic too. This completes the proof of Case I. The other cases follow similarly. □

From inequality (3.2) and (2.3) we get as a special case of Theorem 2, the following theorem for the power functions x^p, $x>0$, $p\geqslant 1$. We add a direct proof which is short and not dependent on the proof of the general case, and show that the difference $M_s(\mathbf{x},\boldsymbol{\lambda}) - W_s(\mathbf{x},\boldsymbol{\lambda})$ increases with s when $s\geqslant 1$.

THEOREM 3. *Let*

$$x_i \geqslant 0, \quad \lambda_i \geqslant 0 \quad x_i^r \leqslant 2\sum_{j=1}^{n}\lambda_j x_j^r, \quad i=1,\dots,n, \quad \sum_{i=1}^{n}\lambda_i = 1. \tag{3.6}$$

If

$$0 < r \leqslant s \leqslant t \leqslant 2r, \tag{3.7}$$

then

$$\begin{aligned} M_r(\mathbf{x},\boldsymbol{\lambda}) &\leqslant M_s(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_t(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_{2r}(\mathbf{x},\boldsymbol{\lambda}) \\ &= \left[\left(\sum_{j=1}^{n}\lambda_j x_j^r\right)^2 + \sum_{i=1}^{n}\lambda_i\left|x_i^r - \sum_{j=1}^{n}\lambda_j x_j^r\right|^2\right]^{1/2r} \\ &\leqslant \left[\left(\sum_{j=1}^{n}\lambda_j x_j^r\right)^{t/r} + \sum_{i=1}^{n}\lambda_i\left|x_i^r - \sum_{j=1}^{n}\lambda_j x_j^r\right|^{t/r}\right]^{1/t} \\ &\leqslant \left[\left(\sum_{j=1}^{n}\lambda_j x_j^r\right)^{s/r} + \sum_{i=1}^{n}\lambda_i\left|x_i^r - \sum_{j=1}^{n}\lambda_j x_j^r\right|^{s/r}\right]^{1/s} \\ &\leqslant M_r(\mathbf{x},\boldsymbol{\lambda})\,2^{1/s}. \end{aligned} \tag{3.8}$$

In case that $r=1$, *and*

$$1 \leqslant s \leqslant t \leqslant 2, \tag{3.9}$$

then

$$\begin{aligned} &M_1(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_s(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_t(\mathbf{x},\boldsymbol{\lambda}) \leqslant M_2(\mathbf{x},\boldsymbol{\lambda}) \\ &= W_2(\mathbf{x},\boldsymbol{\lambda}) \leqslant W_t(\mathbf{x},\boldsymbol{\lambda}) \leqslant W_s(\mathbf{x},\boldsymbol{\lambda}) \leqslant 2^{\frac{1}{s}} M_1(\mathbf{x},\boldsymbol{\lambda}). \end{aligned} \tag{3.10}$$

If

$$0 < 2r \leqslant t \leqslant s, \tag{3.11}$$

then

$$\begin{aligned} M_s(\mathbf{x},\boldsymbol{\lambda}) &\geqslant M_t(\mathbf{x},\boldsymbol{\lambda}) \geqslant M_{2r}(\mathbf{x},\boldsymbol{\lambda}) \\ &= \left[\left(\sum_{j=1}^{n}\lambda_j x_j^r\right)^2 + \sum_{i=1}^{n}\lambda_i\left|x_i^2 - \sum_{j=1}^{n}\lambda_j x_j^r\right|^2\right]^{1/2r} \\ &\geqslant \left[\left(\sum_{j=1}^{n}\lambda_j x_j^r\right)^{t/r} + \sum_{i=1}^{n}\lambda_i\left|x_i^r - \sum_{j=1}^{n}\lambda_j x_j^r\right|^{t/r}\right]^{1/t} \\ &\geqslant \left[\left(\sum_{j=1}^{n}\lambda_j x_j^r\right)^{s/r} + \sum_{i=1}^{n}\lambda_i\left|x_i^r - \sum_{j=1}^{n}\lambda_j x_j^r\right|^{s/r}\right]^{1/s} \\ &\geqslant M_r(\mathbf{x},\boldsymbol{\lambda}). \end{aligned} \tag{3.12}$$

In case that $r=1$,

$$s \geqslant t \geqslant 2, \tag{3.13}$$

then

$$\begin{aligned} M_s(\mathbf{x},\boldsymbol{\lambda}) &\geqslant M_t(\mathbf{x},\boldsymbol{\lambda}) \geqslant M_2(\mathbf{x},\boldsymbol{\lambda}) \\ &= W_2(\mathbf{x},\boldsymbol{\lambda}) \geqslant W_t(\mathbf{x},\boldsymbol{\lambda}) \geqslant W_s(\mathbf{x},\boldsymbol{\lambda}) \geqslant M_1(\mathbf{x},\boldsymbol{\lambda}). \end{aligned} \tag{3.14}$$

Moreover when $r=1$, *the difference* $M_s(\mathbf{x},\boldsymbol{\lambda}) - W_s(\mathbf{x},\boldsymbol{\lambda})$ *increases when* $s \geqslant 1$, *is negative when* $1 \leqslant s < 2$, *positive when* $s > 2$ *and is equal to zero when* $s=2$.

Proof. The first three inequalities in (3.8) are well known properties of power means when (3.7) holds (see [5], [3]) and are special cases of Theorem A for power functions.

As the functions $x^{\frac{s}{r}}$ and $x^{\frac{t}{r}}$ $1 \leqslant \frac{s}{r}, \frac{t}{r} \leqslant 2$, are subquadratic, it follows from (1.5) and (3.2) that

$$\left(\sum_{i=1}^{n} \lambda_i x_i^s\right)^{1/s} \leqslant \left(\sum_{i=1}^{n} \lambda_i x_i^r\right)^{1/r} \left[1 + \sum_{i=1}^{n} \lambda_i \left|\frac{x_i^r - \sum_{j=1}^{n} \lambda_j x_j^r}{\sum_{j=1}^{n} \lambda_j x_j^r}\right|^{s/r}\right]^{1/s} \tag{3.15}$$

$$\left(\sum_{i=1}^{n} \lambda_i x_i^t\right)^{1/t} \leqslant \left(\sum_{i=1}^{n} \lambda_i x_i^r\right)^{1/r} \left[1 + \sum_{i=1}^{n} \lambda_i \left|\frac{x_i^r - \sum_{j=1}^{n} \lambda_j x_j^r}{\sum_{j=1}^{n} \lambda_j x_j^r}\right|^{t/r}\right]^{1/t} \tag{3.16}$$

and

$$\left(\sum_{i=1}^{n} \lambda_i x_i^{2r}\right)^{1/2r} = \left(\sum_{i=1}^{n} \lambda_i x_i^r\right)^{1/r} \left[1 + \sum_{i=1}^{n} \lambda_i \left|\frac{x_i^r - \sum_{j=1}^{n} \lambda_j x_j^r}{\sum_{j=1}^{n} \lambda_j x_j^r}\right|^{2}\right]^{1/2r}. \tag{3.17}$$

However, as the function $\left(1+\sum_{i=1}^{n} \lambda_i z_i^m\right)^{1/m}$, $\lambda_i \geqslant 0$, $i=1,\ldots,n$, $\sum_{i=1}^{n} \lambda_i = 1$, is a decreasing function of m when $0 \leqslant z_i \leqslant 1$, $i=1,\ldots,n$ and $m>0$ we get from (3.6) and (3.7) that the right handside inequalities in (3.8) holds and together with (3.15), (3.16), and (3.17) all the inequalities in (3.8) hold. Therefore also when (3.9) is satisfied (3.10) holds. Similarly, we get that when (3.11) is satisfied, (3.12) holds, and therefore also when (3.13) is satisfied, (3.14) holds. The fact that the difference $M_s(\mathbf{x},\boldsymbol{\lambda}) - W_s(\mathbf{x},\boldsymbol{\lambda})$ increases when $s \geqslant 1$, is negative when $1 \leqslant s < 2$, positive when $s>2$ and is equal to zero when $s=2$ follows directly from (3.9) and (3.10) and from (3.13) and (3.14).

The proof of the theorem is completed. □

EXAMPLE 2. Choosing $F_1(x) = x^{(a+b)/a}$, $F_2(x) = x^{a+b}$, $x \geqslant 0$, $1 \leqslant a \leqslant a+b$, $\lambda_i \geqslant 0$, $i=1,\ldots,n$, $\sum_{i=1}^{n} \lambda_i = 1$ we get that $f_2(x) = F_2\left(F_1^{-1}(x)\right) = x^a$ is a convex function and as well as F_1.

Mereover, if $\frac{a+b}{a} \geqslant 2$, F_1 and F_2 are superquadratic, and we get that when (2.3) holds then according to Theorem 2

$$\begin{aligned}\left(\sum_{i=1}^{n}\lambda_i x_i^{a+b}\right)^{1/(a+b)} &\geqslant \left(\sum_{i=1}^{n}\lambda_i x_i^{(a+b)/a}\right)^{a/(a+b)}\\ &\geqslant \left(\left(\sum_{j=1}^{n}\lambda_j x_j\right)^{(a+b)/a}+\sum_{i=1}^{n}\lambda_i\left(\left|x_i-\sum_{j=1}^{n}\lambda_j x_j\right|\right)^{(a+b)/a}\right)^{a/(a+b)}\\ &\geqslant \left(\left(\sum_{j=1}^{n}\lambda_j x_j\right)^{a+b}+\sum_{i=1}^{n}\lambda_i\left(\left|x_i-\sum_{j=1}^{n}\lambda_j x_j\right|\right)^{a+b}\right)^{1/(a+b)}.\end{aligned}$$

If $a+b \leqslant 2$, F_1 and F_2 are subquadratic and we get that when (2.3) holds

$$\begin{aligned}\left(\sum_{j=1}^{n}\lambda_j x_j\right)^{a+b} &\leqslant \left(\sum_{i=1}^{n}\lambda_i x_i^{(a+b)/a}\right)^{a} \leqslant \sum_{j=1}^{n}\lambda_j x_j^{a+b}\\ &\leqslant \left(\sum_{j=1}^{n}\lambda_j x_j\right)^{a+b}+\sum_{i=1}^{n}\lambda_i\left(\left|x_i-\sum_{j=1}^{n}\lambda_j x_j\right|\right)^{a+b}\\ &\leqslant \left(\sum_{j=1}^{n}\lambda_j x_j\right)^{(a+b)/a}+\sum_{i=1}^{n}\lambda_i\left(\left|x_i-\sum_{j=1}^{n}\lambda_j x_j\right|\right)^{(a+b)/a},\end{aligned}$$

is satisfied.

REMARK 3. We proved in this paper inequalities related to the difference $M_p(\mathbf{x},\boldsymbol{\lambda}) - W_p(\mathbf{x},\boldsymbol{\lambda})$ for various $p \geqslant 1$. These results can be extended to the differences between $M_p(\mathbf{x},\boldsymbol{\lambda})$ and $W_q(\mathbf{x},\boldsymbol{\lambda})$ for $p \neq q$.

It is well known that the original Jensen's inequality is easily generalized when the discrete terms and sums are replaced by integrals and more generally by linear functionals. It may be also possible to get in a similar way, generalized theorems of all the theorems proved in this paper.

The present author aims to investigate these subjects in a forthcoming paper.

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