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STABILITY OF A PEXIDER TYPE FUNCTIONAL EQUATION RELATED TO DISTANCE MEASURES

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Abstract. This work aims to study of the stability of two generalizations of the functional equation $f(pr,qs)+f(ps,qr)=f(p,q)f(r,s)$, namely (i) $f(pr,qs)+g(ps,qr)=h(p,q)h(r,s)$, and (ii) $f(pr,qs)+g(ps,qr)=h(p,q)k(r,s)$ for all $p,q,r,s\in G$, where G is a commutative semigroup. Thus this work is a continuation of our earlier works [15] and [16], and the functional equations studied here arise in the characterizations of symmetrically compositive sum form distance measures.

1. Introduction

Let I denote the open unit interval $(0,1)$ and $J=(0,1]$. Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Let $(G,\cdot)$ be a commutative semigroup.

In [3], Chung, Kannappan, Ng and Sahoo characterized all symmetrically compositive sum form distance measures with a measurable generating function. The following functional equation

$$f(pr,qs)+f(ps,qr)=f(p,q)f(r,s) \tag{FE}$$

holding for all $p,q,r,s\in I$ was instrumental in their characterization. Among other results, they proved the following result giving the general solution of the functional equation (FE). *Suppose $f:I^2\to\mathbb{R}$ satisfies (FE) for all $p,q,r,s\in I$. Then $f(p,q)=M_1(p)M_2(q)+M_1(q)M_2(p)$, where $M_1,M_2:\mathbb{R}\to\mathbb{C}$ are multiplicative functions. Further, either M_1 and M_2 are both real or M_2 is the complex conjugate of M_1. The converse is also true.*

The stability of the functional equation (FE) and four generalizations of (FE) namely,

$$f(pr,qs)+f(ps,qr)=f(p,q)g(r,s) \tag{FE_{fg}}$$

$$f(pr,qs)+f(ps,qr)=g(p,q)f(r,s) \tag{FE_{gf}}$$

$$f(pr,qs)+f(ps,qr)=g(p,q)g(r,s) \tag{FE_{gg}}$$

$$f(pr,qs)+f(ps,qr)=g(p,q)h(r,s) \tag{FE_{gh}}$$

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for all $p,q,r,s \in G$, were studied in [15] and [16]. In this paper, we study the stability of two more generalizations of (FE), namely

$$f(pr,qs)+g(ps,qr)=h(p,q)\,h(r,s) \qquad (FE_{fghh})$$

$$f(pr,qs)+g(ps,qr)=h(p,q)\,k(r,s) \qquad (FE_{fghk})$$

for all $p,q,r,s \in G$. The functional equation (FE_{fghk}) was studied in [7]. The authors of the paper [7], Kannappan, Sahoo and Chung, found all complex-valued functions without any regularity condition when the functional equation of (FE_{fghk}) holds for all $p,q,r,s \in J$. They have found 14 sets of solutions of this functional equation and it is too many to list here. The complex-valued solution of the previous equation (FE_{fghh}) can be easily determined from their results when (FE_{fghh}) holds for all $p,q,r,s \in J$. For other functional equations similar to (FE), the interested reader should refer to [6], [8], [20], [18] and [19]. It should be noted that many well known functional equations like d'Alembert functional equation, Wilson functional equation, Jensen functional equation can be obtained from the functional equation (FE_{fghk}). For instance, letting $r=s=1$ in (FE_{fghk}), one obtains the equation

$$f(p,q)+g(p,q)=k(1,1)\,h(p,q) \quad \forall\, p,q \in J. \tag{1}$$

When $f(p,q)=\psi(p+q)$, $g(p,q)=\psi(p-q)$, and $k(1,1)\,h(p,q)=2\,\psi(p)\,\psi(q)$, then the equation (1) yields the well known d'Alembert functional equation. Similarly, when $f(p,q)=\psi(p+q)$, $g(p,q)=\psi(p-q)$, and $k(1,1)\,h(p,q)=\psi(p)\,\phi(q)$, then (1) yields the Wilson functional equation. Letting $f(p,q)=\psi(p+q)$, $g(p,q)=\psi(p-q)$, and $k(1,1)\,h(p,q)=2\,\psi(p)$ it is easy to see that (1) reduces to Jensen functional equation.

For stability of related functional equations, see papers ([9], [10], [11], [12], [13], [14] and [17]). The superstability for some functional equations of the compositive function form on two variables is found in [1]. The book [4] is an excellent source for reference on stability of functional equations.

2. Stability of functional equation (FE_{fghh})

In this section, we establish the stability of the functional equation (FE_{fghh}).

THEOREM 1. *Let $\phi : G^2 \to \mathbb{R}$ be any nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-h(p,q)\,h(r,s)| \leqslant \phi(r,s), \tag{2}$$

$|h(p,q)-f(p,q)| \leqslant M$, and $|h(p,q)-g(p,q)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either h is bounded or h is a solution of the functional equation (FE), that is

$$h(pr,qs)+h(ps,qr)=h(p,q)h(r,s), \quad \forall\, p,q,r,s \in G. \tag{3}$$

Proof. Let h be an unbounded solution of the inequality (2). Then we can choose a sequence $\{(x_n,y_n)\,|\,n\in\mathbb{N}\}$ in G^2 such that $0\neq|h(x_n,y_n)|\to\infty$ as $n\to\infty$.

Replacing p by x_n and q by y_n in (2), we have

$$|f(x_nr,y_ns)+g(x_ns,y_nr)-h(x_n,y_n)h(r,s)|\leqslant\phi(r,s), \tag{4}$$

which is

$$\left|\frac{f(x_nr,y_ns)+g(x_ns,y_nr)}{h(x_n,y_n)}-h(r,s)\right|\leqslant\frac{\phi(r,s)}{|h(x_n,y_n)|}. \tag{5}$$

Taking the limit as $n\to\infty$, we obtain

$$h(r,s)=\lim_{n\to\infty}\frac{f(x_nr,y_ns)+g(x_ns,y_nr)}{h(x_n,y_n)} \tag{6}$$

Replacing p by x_nr, q by y_ns, r by p, and s by q in (2), we have

$$|f(x_npr,y_nqs)+g(x_nqr,y_nps)-h(x_nr,y_ns)h(p,q)|\leqslant\phi(p,q), \tag{7}$$

which is

$$\left|\frac{f(x_npr,y_nqs)+g(x_nqr,y_nps)}{h(x_n,y_n)}-\frac{h(x_nr,y_ns)}{h(x_n,y_n)}h(p,q)\right|\leqslant\frac{\phi(p,q)}{|h(x_n,y_n)|}. \tag{8}$$

Replacing p by x_ns, q by y_nr, r by p, and s by q in (2) and proceeding as above, we have

$$|f(x_nps,y_nqr)+g(x_nqs,y_npr)-h(x_ns,y_nr)h(p,q)|\leqslant\phi(p,q). \tag{9}$$

From the last inequality (9), we obtain

$$\left|\frac{f(x_nps,y_nqr)+g(x_nqs,y_npr)}{h(x_n,y_n)}-\frac{h(x_ns,y_nr)}{h(x_n,y_n)}h(p,q)\right|\leqslant\frac{\phi(p,q)}{|h(x_n,y_n)|}. \tag{10}$$

Using (6), (8), and (10) and boundedness of $h-f$ and $h-g$, we obtain

$$\begin{aligned}
&h(pr,qs)+h(ps,qr)\\
&=\lim_{n\to\infty}\frac{f(x_npr,y_nqs)+g(x_nqs,y_npr)}{h(x_n,y_n)}+\lim_{n\to\infty}\frac{f(x_nps,y_nqr)+g(x_nqr,y_nps)}{h(x_n,y_n)}\\
&=\lim_{n\to\infty}\frac{h(x_nr,y_ns)+h(x_ns,y_nr)}{h(x_n,y_n)}h(p,q)\\
&=\lim_{n\to\infty}\left[\frac{h(x_nr,y_ns)-f(x_nr,y_ns)+h(x_ns,y_nr)-g(x_ns,y_nr)}{h(x_n,y_n)}+h(r,s)\right]h(p,q)\\
&=h(p,q)\,h(r,s).
\end{aligned}$$

This completes the proof. □

THEOREM 2. *Let $\phi : G^2 \to \mathbb{R}$ be any nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-h(p,q)h(r,s)| \leqslant \phi(p,q), \tag{11}$$

$|h(p,q)-f(p,q)| \leqslant M$, and $|h(p,q)-g(q,p)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either h is bounded or h is a solution of the functional equation (FE).

Proof. Suppose h is unbounded, then we can find a sequence $\{(x_n,y_n)\,|\,n \in \mathbb{N}\}$ in $\mathbb{R}^2$ such that $0 \neq |h(x_n,y_n)| \to \infty$ as $n \to \infty$.

Proceeding similar to the derivation of (5), we obtain

$$h(p,q) = \lim_{n\to\infty} \frac{f(px_n,qy_n)+g(py_n,qx_n)}{h(x_n,y_n)}.$$

The rest of the proof runs similar to the Theorem 1, and we see that h satisfies (FE) for all $p,q,r,s \in G$. This completes the proof of the theorem. □

COROLLARY 1. *Let $\phi : G^2 \to \mathbb{R}$ be any nonzero given function. Let $f,g : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-f(p,q)f(r,s)| \leqslant \phi(r,s), \tag{12}$$

and $|f(p,q)-g(p,q)| \leqslant M$ for all $p,q,r,s \in G$, where M is a nonnegative real constant. Then either f is bounded or f satisfies the equation (FE).

Proof. The proof of the corollary follows by letting $h = f$ in (2) in Theorem 1. □

COROLLARY 2. *Let $\phi : G^2 \to \mathbb{R}$ be any nonzero given function. Let $f,g : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-f(p,q)f(r,s)| \leqslant \phi(p,q), \tag{13}$$

and $|f(p,q)-g(q,p)| \leqslant M$ for all $p,q,r,s \in G$, where M is a nonnegative constant. Then either f is bounded or f satisfies the equation (FE).

Proof. The proof of the corollary follows by letting $h = f$ in Theorem 2. □

COROLLARY 3. *Let $\phi : G^2 \to \mathbb{R}$ be any nonzero given function. Let $f,g : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-g(p,q)g(r,s)| \leqslant \phi(r,s), \tag{14}$$

and $|g(p,q)-f(p,q)| \leqslant M$ for all $p,q,r,s \in G$, where M is a nonnegative real constant. Then either g is bounded or g satisfies the equation (FE).

Proof. The proof of the corollary follows by letting $h = g$ in Theorem 1. $\square$

COROLLARY 4. *Let $\phi : G^2 \to \mathbb{R}$ be any nonzero given function. Let $f, g : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr, qs) + g(ps, qr) - g(p,q)g(r,s)| \leqslant \phi(p,q), \tag{15}$$

$|g(p,q) - f(p,q)| \leqslant M$, and $|g(p,q) - g(q,p)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are nonnegative real constants. Then either g is bounded or g satisfies the equation (FE).

Proof. The proof of the corollary follows by letting $h = g$ in Theorem 2. $\square$

3. Stability of functional equation (FE_{fghk})

In this section, we treat the stability of the functional equation (FE_{fghk}).

THEOREM 3. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f, g, h, k : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr, qs) + g(ps, qr) - h(p,q)k(r,s)| \leqslant \phi(r,s), \tag{16}$$

$|h(p,q) - f(p,q)| \leqslant M$ and $|h(p,q) - g(p,q)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either h is bounded or k is a solution of the equation (FE), that is

$$k(pr, qs) + k(ps, qr) = k(p,q)k(r,s). \tag{17}$$

In addition, if h satisfies the equation (FE), then k and h are the solutions of the equation (FE_{fg}), that is

$$k(pr, qs) + k(ps, qr) = k(p,q)h(r,s). \tag{18}$$

Proof. Let h be an unbounded solution of the inequality (16). Then we can choose a sequence $\{(x_n, y_n) | n \in \mathbb{N}\}$ in G^2 such that $0 \neq |h(x_n, y_n)| \to \infty$ as $n \to \infty$.

Replacing p by x_n and q by y_n in (16), we have

$$|f(x_n r, y_n s) + g(x_n s, y_n r) - h(x_n, y_n)k(r,s)| \leqslant \phi(r,s) \tag{19}$$

which is

$$\left| \frac{f(x_n r, y_n s) + g(x_n s, y_n r)}{h(x_n, y_n)} - k(r,s) \right| \leqslant \frac{\phi(r,s)}{|h(x_n, y_n)|}. \tag{20}$$

Taking the limit as $n \to \infty$, we obtain

$$k(r,s) = \lim_{n \to \infty} \frac{f(x_n r, y_n s) + g(x_n s, y_n r)}{h(x_n, y_n)}. \tag{21}$$

Replacing p by $x_n r$, q by $y_n s$, r by p, and s by q in (16), we have

$$|f(x_n pr, y_n qs) + g(x_n qr, y_n ps) - h(x_n r, y_n s)\,k(p,q)| \leqslant \phi(p,q) \tag{22}$$

which is

$$\left|\frac{f(x_n pr, y_n qs) + g(x_n qr, y_n ps)}{h(x_n, y_n)} - \frac{h(x_n r, y_n s)}{h(x_n, y_n)}\,k(p,q)\right| \leqslant \frac{\phi(p,q)}{|h(x_n, y_n)|}. \tag{23}$$

Similarly, replacing p by $x_n s$, q by $y_n r$, r by p, and s by q in (16) and proceeding as above, we have

$$|f(x_n ps, y_n qr) + g(x_n qs, y_n pr) - h(x_n s, y_n r)\,k(p,q)| \leqslant \phi(p,q). \tag{24}$$

From the last inequality (24), we obtain

$$\left|\frac{f(x_n ps, y_n qr) + g(x_n qs, y_n pr)}{h(x_n, y_n)} - \frac{h(x_n s, y_n r)}{h(x_n, y_n)}\,k(p,q)\right| \leqslant \frac{\phi(p,q)}{|h(x_n, y_n)|}. \tag{25}$$

Using (21), (23), (25) and the fact that $h-f$ and $h-g$ are bounded, we obtain

$$\begin{aligned}
&k(pr,qs)+k(ps,qr)\\
&\quad= \lim_{n\to\infty}\frac{f(x_n pr, y_n qs) + g(x_n qs, y_n pr)}{h(x_n, y_n)} + \lim_{n\to\infty}\frac{f(x_n ps, y_n qr) + g(x_n qr, y_n ps)}{h(x_n, y_n)}\\
&\quad= \lim_{n\to\infty}\frac{h(x_n r, y_n s) + h(x_n s, y_n r)}{h(x_n, y_n)}\,k(p,q)\\
&\quad= \lim_{n\to\infty}\left[\frac{h(x_n r, y_n s) - f(x_n r, y_n s) + h(x_n s, y_n r) - g(x_n s, y_n r)}{h(x_n, y_n)} + k(r,s)\right]k(p,q)\\
&\quad= k(p,q)\,k(r,s).
\end{aligned} \tag{26}$$

Moreover, if h satisfies the equation (FE), then we have from (26) that k and h are the solution of the equation (FE_{fg}), that is

$$k(pr,qs)+k(ps,qr) = k(p,q)\,h(r,s).$$

Now the proof of the theorem is complete. □

THEOREM 4. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h,k : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-h(p,q)k(r,s)| \leqslant \phi(p,q), \tag{27}$$

$|k(p,q)-f(p,q)| \leqslant M$ and $|k(p,q)-g(q,p)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either k is bounded or h satisfies the equation (FE). In addition, if k satisfies the equation (FE), then h and k satisfies the equation (FE_{fg}), that is

$$h(pr,qs)+h(ps,qr) = h(p,q)\,k(r,s). \tag{28}$$

Proof. Let k be unbounded. Then we can choose a sequence $\{(x_n, y_n) \mid n \in \mathbb{N}\}$ in $\mathbb{R}^2$ such that $0 \neq |k(x_n, y_n)| \to \infty$ as $n \to \infty$.

Proceeding similar to the derivation of (21), we obtain

$$h(p,q) = \lim_{n\to\infty} \frac{f(px_n, qy_n) + g(py_n, qx_n)}{k(x_n, y_n)}.$$

Proceeding similar to the proof of Theorem 3, we see that h satisfies (FE) for all $p,q,r,s \in G$. Moreover, if k satisfies (FE), then it is easy to verify that h and k satisfy the functional equation (FE_{gf}), that is (28). The proof of the theorem is now complete. □

The following corollaries are consequences of Theorem 3 and Theorem 4.

COROLLARY 5. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs) + g(ps,qr) - h(p,q)f(r,s)| \leqslant \phi(r,s),$$

$|h(p,q) - f(p,q)| \leqslant M$ and $|h(p,q) - g(p,q)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either h is bounded or f satisfies the equation (FE). In addition, if h satisfies the equation (FE), then f and h satisfy the equation (FE_{fg}), that is

$$f(pr,qs) + f(ps,qr) = f(p,q)h(r,s).$$

COROLLARY 6. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs) + g(ps,qr) - h(p,q)f(r,s)| \leqslant \phi(p,q),$$

and $|f(p,q) - g(q,p)| \leqslant M$ for all $p,q,r,s \in G$, where M is some nonnegative constant. Then either f is bounded or h satisfies the equation (FE). In addition, if f satisfies the equation (FE), then h and f satisfies the equation (FE_{fg}), that is

$$h(pr,qs) + h(ps,qr) = h(p,q)f(r,s).$$

COROLLARY 7. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs) + g(ps,qr) - f(p,q)h(r,s)| \leqslant \phi(r,s),$$

and $|f(p,q) - g(p,q)| \leqslant M$ for all $p,q,r,s \in G$, where M is some nonnegative constant. Then either f is bounded or h satisfies the equation (FE). In addition, if f satisfies the equation (FE), then h and f satisfies the equation (FE_{fg}), that is

$$h(pr,qs) + h(ps,qr) = h(p,q)f(r,s).$$

COROLLARY 8. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-f(p,q)h(r,s)| \leqslant \phi(p,q),$$

$|h(p,q)-f(p,q)| \leqslant M$ and $|h(p,q)-g(q,p)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either h is bounded or f satisfies the equation (FE). In addition, if h satisfies the equation (FE), then f and h satisfies the equation (FE_{fg}), that is

$$f(pr,qs)+f(ps,qr) = f(p,q)\,h(r,s).$$

COROLLARY 9. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-h(p,q)g(r,s)| \leqslant \phi(r,s),$$

$|h(p,q)-f(p,q)| \leqslant M$ and $|h(p,q)-g(p,q)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either h is bounded or g satisfies the equation (FE). In addition, if h satisfies the equation (FE), then g and h satisfies the equation (FE_{fg}), that is

$$g(pr,qs)+g(ps,qr) = g(p,q)\,h(r,s).$$

COROLLARY 10. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-h(p,q)g(r,s)| \leqslant \phi(p,q),$$

$|g(p,q)-f(p,q)| \leqslant M$ and $|g(p,q)-g(q,p)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either g is bounded or h satisfies the equation (FE). In addition, if g satisfies the equation (FE), then h and g satisfies the equation (FE_{fg}), that is

$$h(pr,qs)+h(ps,qr) = h(p,q)\,g(r,s).$$

COROLLARY 11. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-g(p,q)h(r,s)| \leqslant \phi(r,s),$$

and $|g(p,q)-f(p,q)| \leqslant M$ for all $p,q,r,s \in G$, where M is some nonnegative constant. Then either g is bounded or h satisfies the equation (FE). In addition, if g satisfies the equation (FE), then h and g satisfies the equation (FE_{fg}), that is

$$h(pr,qs)+h(ps,qr) = h(p,q)\,g(r,s).$$

COROLLARY 12. *Let $\phi : G^2 \to \mathbb{R}$ be a nonzero given function. Let $f,g,h : G^2 \to \mathbb{R}$ satisfy the functional inequalities*

$$|f(pr,qs)+g(ps,qr)-g(p,q)h(r,s)| \leqslant \phi(p,q),$$

$|h(p,q)-f(p,q)| \leqslant M$ and $|h(p,q)-g(q,p)| \leqslant M'$ for all $p,q,r,s \in G$, where M and M' are some nonnegative constants. Then either h is bounded or g satisfies the equation (FE). In addition, if h satisfies the equation (FE), then g and h satisfies the equation (FE_{fg}), that is

$$g(pr,qs)+g(ps,qr) = g(p,q)h(r,s).$$

If we take $f=g$ in Corollaries 8–12, and assume that h is a given solution of (FE) or it is unbounded, then we obtain a kind of hyperstability result for (FE_{fg}) (with a given h). Some information on hyperstability can be found in [2].

4. Extension of the results to Banach algebra

In this section, we will extend Theorem 3 and Theorem 4 of Section 3 to the semisimple commutative Banach algebra. For simplicity, we will combine the two theorems of Section 3 into the one theorem. Let $(E,\|\cdot\|)$ be a semisimple commutative Banach algebra.

THEOREM 5. *Let $\phi : G^2 \to E$ be a nonzero given function. Let $f,g,h : G^2 \to E$ satisfy the functional inequality*

$$\|f(pr,qs)+g(ps,qr)-h(p,q)k(r,s)\| \leqslant \begin{cases} \phi(r,s) & \forall p,q,r,s \in G \quad (i) \\ \phi(p,q) & \forall p,q,r,s \in G \quad (ii). \end{cases} \tag{29}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$:

(a) In the case (i), if $|h(p,q)-f(p,q)| \leqslant M$, and $|h(p,q)-g(p,q)| \leqslant M'$ for all $p,q \in G$ and some nonnegative constants M,M', then either $x^* \circ h$ is bounded or k satisfies (FE) for all $p,q,r,s \in G$. In addition, if $x^* \circ h$ satisfies the equation (FE), then k and h satisfies the equation (FE_{fg}).

(b) In the case (ii), if $|k(p,q)-f(p,q)| \leqslant M$, and $|k(p,q)-g(q,p)| \leqslant M'$ for all $p,q \in G$ and some nonnegative constants M,M', then either $x^* \circ k$ is bounded or h satisfies (FE) for all $p,q,r,s \in G$. In addition, if $x^* \circ k$ satisfies the equation (FE), then h and k satisfies the equation (FE_{fg}).

Proof. First we show (a). Assume that (i) of (29) holds, and let us fix arbitrarily a linear multiplicative functional $x^* \in E^\star$. As well known we have $\|x^*\| = 1$ hence, for every $x,y \in E$, we have

$$\begin{aligned} \phi(r,s) &\geqslant \|f(pr,qs)+g(ps,qr)-h(p,q)k(r,s)\| \\ &\geqslant \left|x^*\big(f(pr,qs)\big)+x^*\big(g(ps,qr)\big)-x^*\big(h(p,q)\big)x^*\big(k(r,s)\big)\right|, \end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$, $x^* \circ h$ and $x^* \circ k$ yield solutions of inequality (16). Since, by assumption, the superposition $x^* \circ h$ is unbounded, an appeal to Theorem 3 shows that the function $x^* \circ k$ solves the equation (FE). In otherwords, bearing the linear multiplicativity of x^* in mind, for all $p,q,r,s \in G$, the difference

$$\mathscr{D}FE(p,q,r,s) := k(pr,qs) + k(ps,qr) - k(p,q)k(r,s)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathscr{D}FE(p,q,r,s) \in \bigcap \left\{ \ker x^* : x^* \text{ is a multiplicative member of } E^* \right\}$$

for all $p,q,r,s \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is

$$k(pr,qs) + k(ps,qr) - k(p,q)k(r,s) = 0 \qquad \text{for all} \quad p,q,r,s \in G,$$

as claimed. The other case (b) is similar, so its proof will be omitted. This completes the proof the theorem. □

COROLLARY 13. *Let $f,g,h,k : G^2 \to E$ be functions satisfying*

$$\|f(pr,qs) + g(ps,qr) - h(p,q)k(r,s)\| \leqslant \varepsilon \tag{30}$$

for all $p,q,r,s \in G$ and for some $\varepsilon \geqslant 0$. For an arbitrary linear multiplicative functional $x^* \in E^*$:

(a) In the case (i), if $|h(p,q) - f(p,q)| \leqslant M$, and $|h(p,q) - g(p,q)| \leqslant M'$ for all $p,q \in G$ and some nonnegative constants M, M', then either $x^* \circ h$ is bounded or k satisfies (FE) for all $p,q,r,s \in G$. In addition, if $x^* \circ h$ satisfies the equation (FE), then k and h satisfies the equation (FE_{fg}).

(b) In the case (ii), if $|k(p,q) - g(p,q)| \leqslant M$, and $|k(p,q) - f(q,p)| \leqslant M'$ for all $p,q \in A$ and some nonnegative constants M, M', then either $x^* \circ k$ is bounded or h satisfies (FE) for all $p,q,r,s \in G$. In addition, if $x^* \circ k$ satisfies the equation (FE), then h and k satisfies the equation (FE_{fg}).

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