

ASYMPTOTIC EXPANSION OF THE ARITHMETIC–GEOMETRIC MEAN AND RELATED INEQUALITIES

TOMISLAV BURIĆ AND NEVEN ELEZOVIĆ

(Communicated by I. Perić)

Abstract. Asymptotic expansion of the arithmetic–geometric mean is obtained and it is used to analyze inequalities and relations between arithmetic–geometric mean and other classical means.

1. Introduction

Arithmetic–geometric mean is an example of interesting mean obtained by limiting process in the following way. Let s and t be positive numbers. Define $a_0 = s$, $g_0 = t$ and

$$a_{k+1} = \frac{a_k + g_k}{2}, \quad g_{k+1} = \sqrt{a_k g_k}, \quad k \geq 0. \quad (1.1)$$

Then both sequences converge to the same limit $D(s, t)$ which is called arithmetic–geometric mean. We shall use the letter D for this mean. It lies somewhere between A and G , but, as we shall see, closer to the geometric mean.

The D -mean has a known integral representation in terms of the elliptic integral. Namely, the complete elliptic integral of the first kind is defined as

$$K(s, t) = \int_0^{\pi/2} \frac{1}{\sqrt{s^2 \cos^2 \theta + t^2 \sin^2 \theta}} d\theta \quad (1.2)$$

and it is proved in [2] that

$$D(s, t) = \frac{\pi}{2K(s, t)}. \quad (1.3)$$

For more information about this connection, please refer to [2] and references therein.

It is proved that D -mean lies between logarithmic and identric mean (see e.g. [3, 15, 20]):

$$L(s, t) \leq D(s, t) \leq I(s, t) \quad (1.4)$$

Mathematics subject classification (2010): 26E60, 41A60.

Keywords and phrases: Arithmetic–geometric mean, asymptotic expansions, asymptotic inequalities.

This work has been fully supported by Croatian Science Foundation under the project 5435.

where

$$L(s, t) = \frac{t - s}{\log t - \log s}, \quad (1.5)$$

$$I(s, t) = \exp\left(\frac{t \log t - s \log s}{t - s} - 1\right). \quad (1.6)$$

F. Qi and A. Sofo used connection with elliptic integral to establish some properties of D -mean. The main result in [19] is the comparison of this mean and logarithmic mean

THEOREM 1.1. *The best choice of the constant a and b such that inequality*

$$aL(s, t) < D(s, t) < bL(s, t) \quad (1.7)$$

is satisfied for all $s, t > 0$, $s \neq t$ is $a = 1$ and $b = \pi/2$.

In this paper we shall use the new technique of asymptotic expansions to establish various relations between arithmetic-geometric mean and other elementary means. The technique of deriving asymptotic expansion of mean is developed in recent papers [12, 13, 16, 17, 18] and it is successfully used in the comparison of classical means. See cited papers for details.

Namely, the asymptotic expansion of the mean $M(s, t)$ is representation of this mean in the form

$$M(x + s, x + t) = x \sum_{n=0}^{\infty} c_n(s, t) x^{-n}, \quad x \rightarrow \infty. \quad (1.8)$$

Here, $c_n(s, t)$ are homogeneous polynomials of the degree n in variables s and t . It is shown that simpler form of these coefficients is obtained under substitution

$$t = \alpha + \beta, \quad s = \alpha - \beta. \quad (1.9)$$

Therefore, in the rest of the paper we shall use representations of means through variables α and β .

Obviously, expansion of the arithmetic mean in this variables is

$$A(x + s, x + t) = x + \alpha$$

and it has only these two terms. Note that for all classical means it holds $c_0 = 1$ and $c_1 = \alpha$. For an example, asymptotic expansion of the geometric mean is derived in [17] and it reads as

$$G(x + s, x + t) \sim x + \alpha - \frac{1}{2}\beta^2 x^{-1} + \frac{1}{2}\alpha\beta^2 x^{-2} - \frac{1}{8}\beta^2(4\alpha^2 + \beta^2)x^{-3} + \dots$$

In the next chapter, using same technique, we will derive coefficients in asymptotic expansion of the arithmetic-geometric mean. These coefficients will be used in the third

section where we give necessary conditions for various inequalities between D mean and other classical means.

We shall need the following fundamental lemma for the transformation of the asymptotic series. The proof is easy, see e.g. [17].

LEMMA 1.2. *Let function $f(x)$ have asymptotic expansion ($a_0 = 1$)*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \quad x \rightarrow \infty.$$

Then for all real p it holds

$$[f(x)]^p \sim \sum_{n=0}^{\infty} c_n(p) x^{-n},$$

where $c_0 = 1$ and

$$c_n = \frac{1}{n} \sum_{k=1}^n [k(1+p) - n] a_k c_{n-k}. \quad (1.10)$$

In particular, for $p = -1$, the coefficients of the reciprocal value of an asymptotic series are given by:

$$c_n = - \sum_{k=1}^n a_k c_{n-k}. \quad (1.11)$$

We shall also need the following standard result:

LEMMA 1.3. *Let $f(x)$ and $g(x)$ have asymptotic expansions ($a_0 = b_0 = 1$):*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \quad g(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}.$$

Then their product $f(x)g(x)$ and quotient $f(x)/g(x)$ have asymptotic expansions

$$f(x)g(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \quad \frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} d_n x^{-n},$$

where $c_0 = d_0 = 1$ and

$$c_n = \sum_{k=0}^n a_k b_{n-k}, \quad d_n = a_n - \sum_{k=1}^n b_k d_{n-k}. \quad (1.12)$$

2. Asymptotic expansion of arithmetic-geometric mean

Since the arithmetic-geometric mean is defined as a result of the limiting process, we shall follow this process in calculating its expansion into asymptotic series. The idea is as follows.

We will consider the sequence (a_n) as limiting sequence, which converges above to the limit $D(s, t)$. Let us denote $A_n(s, t)$ for the value of n -th iteration. Similarly, let $G_n(s, t)$ is defined through the sequence (g_n) . The A_n and G_n are also means, since they are obtained as the composition of arithmetic and geometric mean to the previous members of these sequences.

Let

$$A_n(s, t) = x \sum_{k=0}^{\infty} a_k^{(n)}(s, t)x^{-k}, \tag{2.1}$$

and

$$G_n(s, t) = x \sum_{k=0}^{\infty} g_k^{(n)}(s, t)x^{-k}, \tag{2.2}$$

be asymptotic expansions of the n -th iteration.

We shall show that functions $a_k^{(n)}$ and $g_k^{(n)}$ converge to the same limit $c_k(t, s)$ when $n \rightarrow \infty$, and it holds

$$D(s, t) = x \sum_{k=0}^{\infty} c_k(s, t)x^{-k}. \tag{2.3}$$

In fact, the following theorem holds true:

THEOREM 2.1. *Let n be arbitrary natural number. Then we have*

$$a_k^{(n)} = g_k^{(n)} \tag{2.4}$$

for all $k \leq 2n$.

Moreover, for fixed k , the sequence $a_k^{(n)}$ is stationary sequence which defines the limiting value c_k . Proof of the theorem follows from the next lemma by mathematical induction.

LEMMA 2.2. *Suppose that coefficients $a_k^{(n)}$ and $g_k^{(n)}$, for arbitrary $n \geq 1$ satisfy*

$$a_k^{(n)} = g_k^{(n)}, \quad k = 0, 1, \dots, K.$$

Then it holds

$$a_k^{(j)} = g_k^{(j)} = a_k^{(n)}, \quad j \geq n + 1, k = 0, \dots, K, \tag{2.5}$$

and

$$a_{K+1}^{(n+1)} = g_{K+1}^{(n+1)}, \quad a_{K+2}^{(n+1)} = g_{K+2}^{(n+1)}. \tag{2.6}$$

Therefore, in each following step, at least two new coefficients coincide and remain equal in the future.

Proof. Let us first prove (2.5). For the arithmetic series the statement is obvious. Let us examine geometric series.

Denote

$$A_{n,k} = \sum_{j=0}^k a_j^{(n)} x^{-j+1}, \quad AR_{n,k} = \sum_{j=k+1}^{\infty} a_j^{(n)} x^{-j+1},$$

and similarly for $G_{n,k}, GR_{n,k}$.

Then

$$G_{n+1} = \sqrt{A_n G_n} = \sqrt{(A_{n,k} + AR_{n,k})(G_{n,k} + GR_{n,k})}.$$

Since $A_{n,k} = G_{n,k}$, this can be written as

$$G_{n+1} = A_{n,k} \sqrt{\left(1 + \frac{AR_{n,k}}{A_{n,k}}\right) \left(1 + \frac{GR_{n,k}}{A_{n,k}}\right)} \tag{2.7}$$

and clearly (2.5) follows.

Now, to prove (2.6), let us find the first two coefficients in the expansion of the quotient $AR_{n,k}/A_{n,k}$. Using Lemma 1.3 (recall that $a_0 = g_0 = 1$), we have

$$\frac{AR_{n,k}}{A_{n,k}} = \frac{a_{k+1}^{(n)}}{x^{k+1}} + \frac{a_{k+2}^{(n)} - a_1^{(n)} a_{k+1}^{(n)}}{x^{k+2}} + \dots$$

In a same way,

$$\frac{GR_{n,k}}{A_{n,k}} = \frac{g_{k+1}^{(n)}}{x^{k+1}} + \frac{g_{k+2}^{(n)} - a_1^{(n)} g_{k+1}^{(n)}}{x^{k+2}} + \dots$$

Now, applying binomial formula to (2.7) it follows

$$G_{n+1} = x \left(1 + \frac{a_1^{(n)}}{x} + \dots\right) \left(1 + \left(\frac{1}{x}\right) \frac{a_{k+1}^{(n)}}{x^{k+1}} + \left(\frac{1}{x}\right) \frac{a_{k+2}^{(n)} - a_1^{(n)} a_{k+1}^{(n)}}{x^{k+2}} + \dots\right) \cdot \left(1 + \left(\frac{1}{x}\right) \frac{g_{k+1}^{(n)}}{x^{k+1}} + \left(\frac{1}{x}\right) \frac{g_{k+2}^{(n)} - a_1^{(n)} g_{k+1}^{(n)}}{x^{k+2}} + \dots\right),$$

and therefore

$$g_{k+1}^{(n+1)} = \frac{a_{k+1}^{(n)} + g_{k+1}^{(n)}}{2} = a_{k+1}^{(n+1)},$$

$$g_{k+2}^{(n+1)} = \frac{a_1^{(n)} a_{k+1}^{(n)}}{2} + \frac{a_1^{(n)} g_{k+1}^{(n)}}{2} + \frac{a_{k+2}^{(n)} - a_1^{(n)} a_{k+1}^{(n)}}{2} + \frac{g_{k+2}^{(n)} - a_1^{(n)} g_{k+1}^{(n)}}{2} = a_{k+2}^{(n+1)}. \quad \square$$

Now we can calculate coefficients c_k of the asymptotic expansion (2.3). In each iteration, coefficients $a_k^{(n)}$ are easily obtained, but for calculating $g_k^{(n)}$ we use Lemma 1.3 and Lemma 1.2 where $p = \frac{1}{2}$.

Here are the first few coefficients in terms of variables α and β :

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \alpha, \\ c_2 &= -\frac{\beta^2}{4}, \\ c_3 &= \frac{\alpha\beta^2}{4}, \\ c_4 &= -\frac{\beta^2}{64}(16\alpha^2 + 5\beta^2), \\ c_5 &= \frac{\alpha\beta^2}{64}(16\alpha^2 + 15\beta^2), \\ c_6 &= -\frac{\beta^2}{256}(64\alpha^4 + 120\alpha^2\beta^2 + 11\beta^4), \\ c_7 &= \frac{\alpha\beta^2}{256}(64\alpha^4 + 200\alpha^2\beta^2 + 55\beta^4). \end{aligned}$$

3. Asymptotic comparison with classical means

In [13, 18], authors developed techniques for comparison of means through their asymptotic expansions.

DEFINITION 3.1. Let M_1 and M_2 be any two means and

$$M_1(x+s, x+t) - M_2(x+s, x+t) = c_k(s, t)x^{-k+1} + O(x^{-k}).$$

If $c_k(s, t) > 0$ for all s and t then we say that mean M_1 is *asymptotically greater* than mean M_2 and write

$$M_1 \succ M_2.$$

Of course, this is equivalent to

$$M_1 \prec M_2.$$

Following theorem holds true:

THEOREM 3.2. *If $M_1 \geq M_2$, then $M_1 \succ M_2$.*

Proof. For x large enough, the sign of the difference $M_1(x+s, x+t) - M_2(x+s, x+t)$ is the same as the sign of the first term in its asymptotic expansion. \square

Therefore, asymptotic inequalities can be considered as a necessary relation between comparable means.

To establish relation of the arithmetic-geometric mean with other classical means, we shall need their asymptotic expansions which can be found in the papers [13, 17, 18].

It is shown in [13] that for the comparison of means, it is sufficient to consider the case $\alpha = 0$. In this case, $c_{2n+1} = 0$ and even coefficients are given in the following table. Letters N, Q, A, I, L, G, and H stand for contraharmonic, quadratic, arithmetic, identric, logarithmic, geometric and harmonic mean, respectively. Coefficients of D mean derived in previous chapter are also put in the table in corresponding place.

M	x	β^2/x	β^4/x^3	β^6/x^5	β^8/x^7
N	1	1	0	0	0
Q	1	$\frac{1}{2}$	$-\frac{1}{8}$	$\frac{1}{16}$	$-\frac{5}{128}$
A	1	0	0	0	0
I	1	$-\frac{1}{6}$	$-\frac{13}{360}$	$-\frac{737}{45360}$	$-\frac{50801}{5443200}$
D	1	$-\frac{1}{4}$	$-\frac{5}{64}$	$-\frac{11}{256}$	$-\frac{469}{16384}$
L	1	$-\frac{1}{3}$	$-\frac{4}{45}$	$-\frac{44}{945}$	$-\frac{428}{14175}$
G	1	$-\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{5}{128}$
H	1	-1	0	0	0

According to the table, D -mean clearly lies between logarithmic and identric mean, which is proved to be true inequality (1.4). And as we mentioned in the beginning of the paper, in comparison with arithmetic and geometric mean, D -mean is somewhere in the middle, but if we observe c_4 , it is closer to the geometric mean.

We will now consider the asymptotic expansion of the linear combination of three means:

$$M_2 - (1 - \mu)M_1 - \mu M_3 \sim c_m(\mu)x^{-m+1} + c_{m+1}(\mu)x^{-m} + \dots \tag{3.1}$$

where c_j equals the corresponding combination of j -th coefficients of means M_i , that is

$$c_j(\mu) = M_2^{(j)} - (1 - \mu)M_1^{(j)} - \mu M_3^{(j)}, \tag{3.2}$$

and c_m denotes the first coefficient that is not zero function of μ .

Since we deal here with comparable means, we shall assume $M_1 \leq M_2 \leq M_3$. Suppose that mean M_2 is greater than convex combination of means M_1 and M_3

$$M_2(s, t) \geq (1 - \mu)M_1(s, t) + \mu M_3(s, t) \tag{3.3}$$

for all values of argument (s, t) , and let μ be such that $c_m(\mu) = 0$. Taking smaller μ will decrease $c_m(\mu)$ to some negative value and the following asymptotic inequality will hold

$$M_2(s, t) < (1 - \mu)M_1(s, t) + \mu M_3(s, t) \tag{3.4}$$

which along with the Theorem 3.2 contradicts the inequality (3.3). Hence, μ such that $c_m(\mu) = 0$ is optimal. Analogous conclusion is made for the reverse inequality in (3.3).

In the next table, optimal parameters μ for the linear combination of the D mean with two other classical means are given.

H	G	L	D	I	A	Q	N	$\times \beta^4/x^3 \geq 0$
$\frac{1}{3}$	-1		$\frac{2}{3}$					$\frac{7}{96}$ +
$\frac{1}{9}$		-1	$\frac{8}{9}$					$\frac{7}{360}$ +
$\frac{1}{10}$			-1	$\frac{9}{10}$				$\frac{73}{1600}$ +
$\frac{1}{4}$			-1		$\frac{3}{4}$			$\frac{5}{64}$ +
$\frac{1}{2}$			-1			$\frac{1}{2}$		$\frac{1}{64}$ +
$\frac{5}{8}$			-1				$\frac{3}{8}$	$\frac{5}{64}$ +
	$\frac{1}{3}$	-1	$\frac{2}{3}$					$\frac{53}{1440}$ +
	$\frac{1}{4}$		-1	$\frac{3}{4}$				$\frac{49}{960}$ +
	$\frac{1}{2}$		-1		$\frac{1}{2}$			$\frac{5}{64}$ +
	$\frac{3}{4}$		-1			$\frac{1}{4}$		$\frac{3}{64}$ +
	$\frac{5}{6}$		-1				$\frac{1}{6}$	$\frac{5}{64}$ +
		$\frac{1}{2}$	-1	$\frac{1}{2}$				$\frac{1}{64}$ +
		$\frac{3}{4}$	-1		$\frac{1}{4}$			$\frac{11}{960}$ +
		$-\frac{9}{10}$	1			$-\frac{1}{10}$		$\frac{23}{1600}$ +
		$-\frac{15}{16}$	1				$-\frac{1}{16}$	$\frac{1}{192}$ +
			$-\frac{2}{3}$	1	$-\frac{1}{3}$			$\frac{23}{1440}$ +
			$-\frac{8}{9}$	1		$-\frac{1}{9}$		$\frac{17}{360}$ +
			$-\frac{14}{15}$	1			$-\frac{1}{15}$	$\frac{53}{1440}$ +
			$-\frac{2}{3}$		1	$-\frac{1}{3}$		$\frac{3}{32}$ +
			$-\frac{4}{5}$		1		$-\frac{1}{5}$	$\frac{17}{450}$ +
			$\frac{2}{5}$			-1	$\frac{3}{5}$	$\frac{3}{32}$ +

The signs of the coefficients are chosen in a such way that the whole combination is asymptotically greater than zero. For an example, the third row should be read as

$$\frac{1}{10}H - D + \frac{9}{10}I \sim \frac{73}{1600}\beta^4x^{-3} + \dots$$

wherefrom it follows

$$\frac{1}{10}H + \frac{9}{10}I \succ D. \tag{3.5}$$

The sign + in the last column means that this inequality is a true one, i.e.

$$\frac{1}{10}H + \frac{9}{10}I \geq D. \tag{3.6}$$

This can be verified by CAS, we used *Mathematica* for this purpose. Since D-mean has explicit form through elliptic integral, proving this inequalities in a traditional way with calculus can be a very tedious job and there aren't many known results. Therefore, our method gives numerous new relations with optimal parameters between arithmetic-geometric mean and other classical means.

4. Asymptotic expansion of the elliptic integral

Through connection (1.3) between arithmetic-geometric mean and elliptic integral, we are able to easily derive asymptotic expansion of the elliptic integral of the first kind.

Using asymptotic expansion of the arithmetic-geometric mean (2.3) and applying Lemma 1.2 for $p = -1$, we obtain following expansion:

$$K(s, t) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{s^2 \cos^2 \theta + t^2 \sin^2 \theta}} \sim \frac{\pi}{2} \sum_{k=0}^{\infty} c_k(s, t) x^{-k+1}, \tag{4.1}$$

where the first few coefficients in terms of variables α and β are $c_0 = 1$ and:

$$\begin{aligned} c_1 &= -\alpha \\ c_2 &= \frac{1}{4}(4\alpha^2 + \beta^2) \\ c_3 &= -\frac{\alpha}{4}(4\alpha^2 + 3\beta^2) \\ c_4 &= \frac{1}{64}(64\alpha^4 + 96\alpha^2\beta^2 + 9\beta^4) \\ c_5 &= -\frac{\alpha}{64}(64\alpha^4 + 160\alpha^2\beta^2 + 45\beta^4) \\ c_6 &= \frac{1}{256}(256\alpha^6 + 960\alpha^4\beta^2 + 540\alpha^2\beta^4 + 25\beta^6) \end{aligned}$$

The complete elliptic integral of the first kind is usually defined through one parameter $0 \leq m \leq 1$ in a way:

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} \tag{4.2}$$

and relation with arithmetic-geometric mean is as follows:

$$K(m) = \frac{\pi/2}{D(1 - \sqrt{m}, 1 + \sqrt{m})}. \tag{4.3}$$

Hence, if we apply expansion (4.1) with $\alpha = 0$, $\beta = \sqrt{m}$ and $x = 1$, we get:

$$K(m) = \frac{\pi}{2} \left(1 + \frac{1}{4}m + \frac{9}{64}m^2 + \frac{25}{256}m^3 + \frac{1225}{16384}m^4 + \frac{3969}{65536}m^5 + \dots \right). \tag{4.4}$$

This is exactly a known expansion for the complete elliptic integral

$$K(m) = \frac{\pi}{2} \sum_{k=0}^{\infty} \left[\frac{(2k-1)!!}{(2k)!!} \right]^2 m^k, \tag{4.5}$$

see [1, p. 591] for details.

Therefore, looking vice versa, asymptotic expansion for the arithmetic-geometric mean can be also derived from the known expansion for the elliptic integral, but the iterative process shown in the second section showed interesting convergence and stationary properties of the coefficients in this expansion.

REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.
- [2] G. E. ANDREWS, R. A. ASKEY, AND R. ROY, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [3] P. BRACKEN, *An arithmetic-geometric mean inequality*, Expo. Math., **19** (2001), 273–279.
- [4] P. S. BULLEN, *Averages still on the move*, Math. Mag., **63** (1990), 250–255.
- [5] P. S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer Academic Publisher, Dordrecht/Boston/London, 2003.
- [6] P. S. BULLEN, D. S. MITRINOVIĆ AND P. M. VASIĆ, *Means and their inequalities*, D Reidel, Dordrecht, 1988.
- [7] T. BURIĆ, N. ELEZOVIĆ, *Bernoulli polynomials and asymptotic expansions of the quotient of gamma functions*, J. Comput. Appl. Math., **235**, 11 (2011), 3315–3331.
- [8] T. BURIĆ, N. ELEZOVIĆ, *New asymptotic expansions of the gamma function and improvements of Stirling's type formulas*, J. Comput. Anal. Appl., **13**, 4 (2011), 785–795.
- [9] T. BURIĆ, N. ELEZOVIĆ, *New asymptotic expansions of the quotient of gamma functions*, Integral Transforms Spec. Funct., **23** (2012), 355–368.
- [10] T. BURIĆ, N. ELEZOVIĆ, *Approximations of the Euler-Mascheroni constant and harmonic numbers*, Appl. Math. Comput., **222** (2013), 604–611.
- [11] T. BURIĆ, N. ELEZOVIĆ, *Asymptotic expansions of the binomial coefficients*, J. Appl. Math. Comput., **46** (2014), 135–145.
- [12] C.-P. CHEN, N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic formulae associated with the Wallis power function and digamma function*, J. Class. Anal. **2** (2013), 151–166.
- [13] N. ELEZOVIĆ, *Asymptotic inequalities and comparison of classical means*, J. Math. Inequal. **9**, 1 (2015), 177–196.
- [14] N. ELEZOVIĆ, *Asymptotic expansions of gamma and related functions, binomial coefficients, inequalities and means*, J. Math. Inequal. **9**, 4 (2015), 1001–1054.
- [15] N. ELEZOVIĆ AND J. PEČARIĆ, *Differential and integral f -means and applications to digamma function*, Math. Inequal. Appl. **3** (2000), 189–196.
- [16] N. ELEZOVIĆ, L. VUKŠIĆ, *Asymptotic expansions of integral means and applications to the ratio of gamma functions*, Appl. Math. Comput. **235** (2014), 187–200.
- [17] N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic expansions of bivariate classical means and related inequalities*, J. Math. Inequal. **8** (2014), 707–724.
- [18] N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic expansions and comparison of bivariate parameter means*, Math. Inequal. Appl. **17** (2014) 1225–1244.
- [19] F. QI, A. SOFO, *An alternative and united proof of a double inequality for bounding the arithmetic-geometric mean*, U.P.B. Sci. Bull. **71** (2009) 69–76.
- [20] M. K. VAMANAMURTHY AND M. VUORINEN, *Inequalities for means*, J. Math. Anal. Appl. **183** (1994), 155–166.

(Received October 7, 2014)

Tomislav Burić, N. Elezović
University of Zagreb
Faculty of Electrical Engineering and Computing
Unska 3, 10000 Zagreb, Croatia
e-mail: tomlslav.buric@fer.hr
e-mail: neven.elez@fer.hr