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MINKOWSKI TYPE INEQUALITY FOR CONVEX FUNCTIONS

LJILJANKA KVESIĆ AND JOSIP PEČARIĆ

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Abstract. We obtain a Minkowski type inequality for convex functions with weights satisfying the Jensen-Steffensen conditions.

1. Introduction

Recently Chunaev [1] obtained Hölder and Minkowski type inequalities with alternating signs. These results were later generalized by Chunaev, Kvesić and Pečarić [2] who obtained the inequalities for more general weights p_k, $k=1,\dots,n$, satisfying the Jensen-Steffensen conditions

$$P_k \geqslant 0, \qquad \text{where } P_k := \sum_{i=1}^{k} p_i, \quad k=1,\dots,n. \tag{1}$$

Notice that the Jensen-Steffensen conditions (1) include both the classical case of nonnegative weights and the case of weights with alternating signs $p_k=(-1)^{k+1}$. The next theorem gives the discrete Minkowski type inequality from [2].

THEOREM 1. *Let $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ be two nonnegative and nonincreasing sequences and let the weights $\{p_k\}_{k=1}^n$ satisfy the Jensen-Steffensen conditions (1). Then*

$$0 \leqslant \frac{\left(\sum_{k=1}^n p_k a_k^p\right)^{1/p} + \left(\sum_{k=1}^n p_k b_k^p\right)^{1/p}}{\left(\sum_{k=1}^n p_k (a_k+b_k)^p\right)^{1/p}} \leqslant 2^{1-1/p}, \quad p \geqslant 1. \tag{2}$$

The constant $2^{1-1/p}$ is best possible. The left hand side of (2) should be read as for all $n \geqslant 2$ there exists no constant depending on p only and bounding the fraction in (2) from below.

The integral version of the Minkowski type inequality from [2] is given in the following theorem.

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THEOREM 2. *Let* $f,g:[a,b]\to\mathbb{R}$ *be two nonnegative and nonincreasing functions and let* $p:[a,b]\to\mathbb{R}$ *be an integrable function such that the function* $P(x):=\int_a^x p(t)\,dt$ *satisfies* $P(x)\geqslant 0$ *for all* $x\in[a,b]$*. Then*

$$0\leqslant\frac{\left(\int_a^b f^p(x)\,dP(x)\right)^{1/p}+\left(\int_a^b g^p(x)\,dP(x)\right)^{1/p}}{\left(\int_a^b (f(x)+g(x))^p\,dP(x)\right)^{1/p}}\leqslant 2^{1-1/p},\quad p\geqslant 1. \tag{3}$$

The constant $2^{1-1/p}$ *is best possible. The left hand side of* (3) *should be read as for all* $n\geqslant 2$ *there exists no constant depending on* p *only and bounding the fraction in* (3) *from below.*

In this paper we will generalize Theorems 1 and 2 by replacing the power function $x\mapsto x^p$ with a more general nondecreasing convex function φ.

2. Main results

In the proof of the discrete version of our results we will use Abel's transformation or summation by parts formula which states that for two sequences $\{c_k\}$ and $\{d_k\}$ the following identity holds

$$\sum_{k=m}^{n} c_k\Delta d_k=c_{n+1}d_{n+1}-c_md_m-\sum_{k=m}^{n} d_{k+1}\Delta c_k, \tag{4}$$

where $\Delta a_k=a_{k+1}-a_k$. The identity is a discrete analogue of the integration by parts formula. In particular, for $m=1$, $d_1=0$ and $p_k=\Delta d_k$, with P_k as in (1), we have $d_{k+1}=\sum_{i=1}^k p_i=P_k$ and Abel's transformation (4) can be rearranged as

$$\sum_{k=1}^{n} p_kc_k=P_nc_n+\sum_{k=1}^{n-1} P_k\left(c_k-c_{k+1}\right). \tag{5}$$

The following is our main result.

THEOREM 3. *Let* $\{a_k\}_{k=1}^n$ *and* $\{b_k\}_{k=1}^n$ *be two nonnegative and nonincreasing sequences and let the weights* $\{p_k\}_{k=1}^n$ *satisfy the Jensen-Steffensen conditions (1). If* $\varphi:I\to\mathbb{R}$ *is an increasing convex function such that* $\varphi(0)\leqslant 0$*, then*

$$\varphi^{-1}\left(\sum_{k=1}^{n} p_k\varphi(a_k)\right)+\varphi^{-1}\left(\sum_{k=1}^{n} p_k\varphi(b_k)\right)\leqslant 2\varphi^{-1}\left(\frac{1}{2}\sum_{k=1}^{n} p_k\varphi(a_k+b_k)\right), \tag{6}$$

where I *is an interval in* $\mathbb{R}$ *such that* $0\in I$ *and all terms in (6) are well defined.*

Proof. Applying Jensen's inequality we have

$$\varphi\left(\frac{1}{2}\varphi^{-1}\left(\sum_{k=1}^{n} p_k\varphi(a_k)\right)+\frac{1}{2}\varphi^{-1}\left(\sum_{k=1}^{n} p_k\varphi(b_k)\right)\right)$$
$$\leqslant \frac{1}{2}\left(\sum_{k=1}^{n} p_k\varphi(a_k)+\sum_{k=1}^{n} p_k\varphi(b_k)\right)=\frac{1}{2}\sum_{k=1}^{n} p_k\left(\varphi(a_k)+\varphi(b_k)\right). \tag{7}$$

Let us now define the function h and sequence $\{c_k\}$ with

$$h(x,y)=\varphi(x+y)-\varphi(x)-\varphi(y), \qquad c_k=h(a_k,b_k). \tag{8}$$

Since a convex function has nondecreasing increments, for $y\geqslant 0$ and $x_1\leqslant x_2$ we have

$$\varphi(x_1+y)-\varphi(x_1)\leqslant\varphi(x_2+y)-\varphi(x_2).$$

Hence, for fixed $y\geqslant 0$, the mapping $x\mapsto h(x,y)$ is nondecreasing, as is, by the same argument, the mapping $y\mapsto h(x,y)$ for fixed $x\geqslant 0$. Therefore

$$c_{k+1}=h(a_{k+1},b_{k+1})\leqslant h(a_k,b_{k+1})\leqslant h(a_k,b_k)=c_k,$$

so $\{c_k\}$ is a nonincreasing sequence. Moreover, since for $x,y\geqslant 0$ we have

$$h(x,y)\geqslant h(x,0)=-\varphi(0)\geqslant 0,$$

the sequence $\{c_k\}$ is nonnegative. By the assumptions of the theorem and properties of $\{c_k\}$, the right-hand side of Abel's transformation formula (5) is nonnegative. Therefore,

$$0\leqslant\sum_{k=1}^{n} p_kc_k=\sum_{k=1}^{n} p_k\varphi(a_k+b_k)-\sum_{k=1}^{n} p_k(\varphi(a_k)+\varphi(b_k)). \tag{9}$$

From (7) and (9) we conclude

$$\varphi\left(\frac{1}{2}\varphi^{-1}\left(\sum_{k=1}^{n} p_k\varphi(a_k)\right)+\frac{1}{2}\varphi^{-1}\left(\sum_{k=1}^{n} p_k\varphi(b_k)\right)\right)\leqslant\frac{1}{2}\sum_{k=1}^{n} p_k\varphi(a_k+b_k).$$

Finally, since the inverse φ^{-1} is an increasing function, by applying φ^{-1} on both sides of the last inequality and then multiplying it by 2 we obtain the desired inequality (6). □

The alternating sign weights $p_k=(-1)^{k+1}$, $k=1,\ldots,n$, satisfy the Jensen-Steffensen conditions (1). Applying Theorem 3 with these weights we immediately obtain the following corollary.

COROLLARY 1. *Let $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ be two nonnegative and nonincreasing sequences. If $\varphi: I\to\mathbb{R}$ is an increasing convex function such that $\varphi(0)\leqslant 0$, then*

$$\varphi^{-1}\left(\sum_{k=1}^{n}(-1)^{k+1}\varphi(a_k)\right)+\varphi^{-1}\left(\sum_{k=1}^{n}(-1)^{k+1}\varphi(b_k)\right)\leqslant 2\varphi^{-1}\left(\frac{1}{2}\sum_{k=1}^{n}(-1)^{k+1}\varphi(a_k+b_k)\right), \tag{10}$$

where I is an interval in $\mathbb{R}$ such that $0\in I$ and all terms in (10) are well defined.

By replacing (monotone) sequences with (monotone) functions and Abel's transformation with integration by parts formula we obtain the integral version of the result. More concretely, the following result holds.

THEOREM 4. *Let $f,g:[a,b]\to\mathbb{R}$ be two nonnegative and nonincreasing functions and let $p:[a,b]\to\mathbb{R}$ be an integrable function such that the function $P(x)=\int_a^x p(t)\,dt$ satisfies $P(x)\geqslant 0$ for all $x\in[a,b]$. If $\varphi:I\to\mathbb{R}$ is an increasing convex function such that $\varphi(0)\leqslant 0$, then*

$$\varphi^{-1}\left(\int_a^b \varphi(f(x))\,dP(x)\right)+\varphi^{-1}\left(\int_a^b \varphi(g(x))\,dP(x)\right) \leqslant 2\varphi^{-1}\left(\frac{1}{2}\int_a^b \varphi(f(x)+g(x))\,dP(x)\right), \tag{11}$$

where I is an interval in $\mathbb{R}$ such that $0\in I$ and all terms in (11) are well defined.

Proof. Analogously as in the proof of Theorem 3 we can show that the function $c(x)=h(f(x),g(x))$, where h is given by (8), is a nonincreasing and nonnegative function. Since P is continuous and c is monotone, the integrals $\int P\,dc$ and $\int c\,dP$ exist and the integration by parts formula holds. Furthermore, since $P(a)=0$ we have

$$\int_a^b c(x)\,dP(x)=c(b)P(b)-\int_a^b P(x)\,dc(x)\geqslant 0, \tag{12}$$

where the last inequality holds due to the assumptions on P and properties of c. The rest of the proof is analogous to the proof of Theorem 3. ☐

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Ljiljanka Kvesić
Faculty of Science and Education
University of Mostar
88 000 Mostar, Bosnia and Herzegovina
e-mail: ljkvesic@gmail.com

Josip Pečarić
Faculty Of Textile Technology
University Of Zagreb
10 000 Zagreb, Croatia
e-mail: pecaric@mahazu.hazu.hr