Journal of Mathematical Inequalities
Volume 9, Number 4 (2015), 1195–1205
doi:10.7153/jmi-09-92

SOME INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL AND EULER TWO–POINT FORMULAE

J. PEČARIĆ AND A. VUKELIĆ

(*Communicated by M. Matić*)

Abstract. We use inequalities for the Čebyšev functional in terms of the first derivative (see [3]), for some new bounds for the remainder of general Euler two-point formula and its generalization for Bullen type formula.

1. Introduction

For $k \geqslant 1$ and fixed $x \in [0,1/2]$ define the functions $G_k^x(t)$ and $F_k^x(t)$ as

$$G_k^x(t) = B_k^*(x-t) + B_k^*(1-x-t),\ t \in \mathbb{R}$$

and $F_k^x(t) = G_k^x(t) - \tilde{B}_k(x)$, $t \in \mathbb{R}$, where

$$\tilde{B}_k(x) = B_k(x) + B_k(1-x),\ x \in [0,1/2],\ k \geqslant 1.$$

The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k \geqslant 0$, are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t) = B_k(t),\ 0 \leqslant t < 1 \quad \text{and} \quad B_k^*(t+1) = B_k^*(t),\ t \in \mathbb{R}.$$

The Bernoulli polynomials $B_k(t)$, $k \geqslant 0$ are uniquely determined by the following identities:

$$B_k'(t) = kB_{k-1}(t),\ k \geqslant 1;\ B_0(t) = 1,\ B_k(t+1) - B_k(t) = kt^{k-1},\ k \geqslant 0.$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have that $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. It follows that $B_k(1) = B_k(0) = B_k$ for $k \geqslant 2$, so that $B_k^*(t)$ are continuous functions for $k \geqslant 2$. We get

$$B_k^{*\prime}(t) = kB_{k-1}^*(t),\ k \geqslant 1 \tag{1.1}$$

Mathematics subject classification (2010): 26D15, 26D20, 26D99.
Keywords and phrases: Čebyšev functional, general Euler two-point formula, Bulen type formula, Euler trapezoid formula, Euler midpoint formula, Euler two-point Newton-Cotes formula, Euler two-point Maclaurin formula, Euler bitrapezoid formula.

Paper JMI-09-92

for every $t \in \mathbb{R}$ when $k \geqslant 3$, and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ when $k = 1, 2$.

Especially, we get $\tilde{B}_1(x) = 0$, $\tilde{B}_2(x) = 2x^2 - 2x + 1/3$, $\tilde{B}_3(x) = 0$. Also, for $k \geqslant 2$ we have $\tilde{B}_k(x) = G_k^x(0)$, that is $F_k^x(t) = G_k^x(t) - G_k^x(0)$, $k \geqslant 2$, and $F_1^x(t) = G_1^x(t)$, $t \in \mathbb{R}$. Obviously, $G_k^x(t)$ and $F_k^x(t)$ are periodic functions of period 1 and continuous for $k \geqslant 2$.

Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geqslant 1$. We introduce the following notation for each $x \in [0,1/2]$

$$D(x) = \frac{1}{2}\left[f(x) + f(1-x)\right].$$

Further, we define $\tilde{T}_0(x) = 0$ and, for $1 \leqslant m \leqslant n$, $x \in [0,1/2]$

$$\tilde{T}_m(x) = \frac{1}{2}\left[T_m(x) + T_m(1-x)\right],$$

where

$$T_m(x) = \sum_{k=1}^{m} \frac{B_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right].$$

It is easy to see that

$$\tilde{T}_m(x) = \frac{1}{2}\sum_{k=1}^{m} \frac{\tilde{B}_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]. \tag{1.2}$$

In [5], the authors established the general Euler two-point formulae:

THEOREM 1. *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geqslant 1$. Then for each $x \in [0,1/2]$*

$$\int_0^1 f(t)dt = D(x) - \tilde{T}_n(x) + \tilde{R}_n^1(f) \tag{1.3}$$

and

$$\int_0^1 f(t)dt = D(x) - \tilde{T}_{n-1}(x) + \tilde{R}_n^2(f), \tag{1.4}$$

where

$$\tilde{R}_n^1(f) = \frac{1}{2n!}\int_0^1 G_n^x(t)\,df^{(n-1)}(t), \quad \tilde{R}_n^2(f) = \frac{1}{2n!}\int_0^1 F_n^x(t)\,df^{(n-1)}(t).$$

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{R}$, consider the Čebyšev functional:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt. \tag{1.5}$$

In [3] the authors proved the following theorems:

THEOREM 2. *Let $f,g:[a,b]\to\mathbb{R}$ be two absolutely continuous functions on $[a,b]$ with*

$$(\cdot-a)(b-\cdot)(f')^2,(\cdot-a)(b-\cdot)(g')^2\in L_1[a,b].$$

Then we have the inequality

$$|T(f,g)|\leqslant\frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b(x-a)(b-x)\left[g'(x)\right]^2dx\right)^{\frac{1}{2}}\tag{1.6}$$
$$\leqslant\frac{1}{2(b-a)}\left(\int_a^b(x-a)(b-x)\left[f'(x)\right]^2dx\right)^{\frac{1}{2}}$$
$$\times\left(\int_a^b(x-a)(b-x)\left[g'(x)\right]^2dx\right)^{\frac{1}{2}}.$$

The constant $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}$ are best possible in (1.6).

THEOREM 3. *Assume that $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is absolutely continuous with $f'\in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,g)|\leqslant\frac{1}{2(b-a)}||f'||_\infty\int_a^b(x-a)(b-x)dg(x).\tag{1.7}$$

The constant $\frac{1}{2}$ is best possible.

In [6] the authors gave some estimations of the error for two-point formula (1.3) via pre-Grüss inequality.

In this paper we will use the above theorems to get some new bounds for the remainders of general Euler two-point formula (1.3). Applications for Bullen type formula are also proved. As special cases, some new bounds for Euler trapezoid formula, Euler midpoint formula, Euler two-point Newton-Cotes formula, Euler two-point Maclaurin formula and Euler bitrapezoid formula are considered.

2. Applications for the general Euler two-point formula

Using Theorem 2 for identity (1.3) we get the following Grüss type inequality:

THEOREM 4. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n\geqslant 1$ and $(f^{(n+1)})^2\in L_1[0,1]$. Then for $x\in[0,1/2]$ we have*

$$D(x)-\int_0^1 f(t)dt-\tilde{T}_n(x)=TG_n(f)\tag{2.1}$$

and the remainder $TG_n(f)$ satisfies the estimation

$$|TG_n(f)|\leqslant\frac{1}{2}\left[\frac{(-1)^{n-1}}{(2n)!}(B_{2n}+B_{2n}(1-2x))\right]^{\frac{1}{2}}\cdot\left(\int_0^1 t(1-t)\left[f^{(n+1)}(t)\right]^2dt\right)^{\frac{1}{2}}.\tag{2.2}$$

Proof. If we apply Theorem 2 for $f \to G_n^x$, $g \to f^{(n)}$, we deduce

$$\left| \int_0^1 G_n^x(t) f^{(n)}(t) dt - \int_0^1 G_n^x(t) \cdot \int_0^1 f^{(n)}(t) dt \right| \leqslant \frac{1}{\sqrt{2}} \left[T\left(G_n^x(\cdot), G_n^x(\cdot) \right) \right]^{\frac{1}{2}} \cdot \left(\int_0^1 t(1-t) \left[f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}}, \tag{2.3}$$

where

$$T\left(G_n^x(\cdot), G_n^x(\cdot) \right) = \int_0^1 \left[G_n^x(t) \right]^2 dt - \left[\int_0^1 G_n^x(t)\, dt \right]^2.$$

From [5] we have $\int_0^1 G_n^x(t)\, dt = 0$ and

$$\int_0^1 \left[G_n^x(t) \right]^2 dt = (-1)^{n-1} \frac{2(n!)^2}{(2n)!} (B_{2n} + B_{2n}(1-2x)).$$

Using (1.3) and (2.3), we deduce representation (2.1) and bound (2.2). □

REMARK 1. We have

$$\int_0^1 F_k^x(t)\, dt = \int_0^1 G_k^x(t)\, dt - \int_0^1 \tilde{B}_k(x) dt = -\tilde{B}_k(x),$$

and also

$$\int_0^1 \left[F_k^x(t) \right]^2 dt = \int_0^1 \left[G_k^x(t) \right]^2 dt - 2\tilde{B}_k(x) \int_0^1 G_k(t)\, dt + \tilde{B}_k^2(x).$$

So, using (1.4) similar as in (2.3), we deduce representation (2.1) and bound (2.2), too.

COROLLARY 1. *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2k-1)}$ is absolutely continuous for some $k \geqslant 2$, $(f^{(2k)})^2 \in L_1[0,1]$ and $f^{(2k-1)} \geqslant 0$ on $[0,1]$. Then for $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}} \right]$ we have*

$$0 \leqslant (-1)^{k-1} \left\{ D(x) - \int_0^1 f(t) dt - \tilde{T}_{2k-1}(x) \right\} \leqslant \frac{1}{2} \left[\frac{1}{(4k-2)!} (B_{4k-2} + B_{4k-2}(1-2x)) \right]^{\frac{1}{2}} \cdot \left(\int_0^1 t(1-t) \left[f^{(2k)}(t) \right]^2 dt \right)^{\frac{1}{2}}, \tag{2.4}$$

and for $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2} \right]$

$$0 \leqslant (-1)^{k} \left\{ D(x) - \int_0^1 f(t) dt - \tilde{T}_{2k-1}(x) \right\} \leqslant \frac{1}{2} \left[\frac{1}{(4k-2)!} (B_{4k-2} + B_{4k-2}(1-2x)) \right]^{\frac{1}{2}} \cdot \left(\int_0^1 t(1-t) \left[f^{(2k)}(t) \right]^2 dt \right)^{\frac{1}{2}}. \tag{2.5}$$

Proof. We are using Lemma 2 from [5]. □

If in Theorem 4 we choose $x = 0, 1/2, 1/3, 1/4$ we get inequality related to the trapezoid, the midpoint, the two-point Newton-Cotes and the two point Maclaurin formula of Euler type:

COROLLARY 2. *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geqslant 1$ and $(f^{(n+1)})^2 \in L_1[0,1]$. Then we have*

$$\left| \frac{1}{2}[f(0)+f(1)] - \int_0^1 f(t)dt - \tilde{T}_n(0) \right| \leqslant \frac{1}{2}\left[\frac{2(-1)^{n-1}}{(2n)!}B_{2n}\right]^{\frac{1}{2}} \cdot \left(\int_0^1 t(1-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}, \tag{2.6}$$

where $\tilde{T}_0(0) = 0$ and

$$\tilde{T}_n(0) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

REMARK 2. For $n = 1$ in Corollary 2 we get

$$\left| \frac{1}{2}[f(0)+f(1)] - \int_0^1 f(t)dt \right| \leqslant \frac{1}{2\sqrt{6}} \cdot \left(\int_0^1 t(1-t)\left[f''(t)\right]^2 dt\right)^{\frac{1}{2}}.$$

COROLLARY 3. *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geqslant 1$ and $(f^{(n+1)})^2 \in L_1[0,1]$. Then we have*

$$\left| f\left(\frac{1}{2}\right) - \int_0^1 f(t)dt - \tilde{T}_n\left(\frac{1}{2}\right) \right| \leqslant \frac{1}{2}\left[\frac{2(-1)^{n-1}}{(2n)!}B_{2n}\right]^{\frac{1}{2}} \cdot \left(\int_0^1 t(1-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}, \tag{2.7}$$

where $\tilde{T}_0\left(\frac{1}{2}\right) = 0$ and

$$\tilde{T}_n\left(\frac{1}{2}\right) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(2^{1-2k}-1)B_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

REMARK 3. For $n = 1$ in Corollary 3 we get

$$\left| f\left(\frac{1}{2}\right) - \int_0^1 f(t)dt \right| \leqslant \frac{1}{2\sqrt{6}} \cdot \left(\int_0^1 t(1-t)\left[f''(t)\right]^2 dt\right)^{\frac{1}{2}}.$$

COROLLARY 4. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n\geqslant 1$ and $(f^{(n+1)})^2\in L_1[0,1]$. Then we have*

$$\left|\frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]-\int_0^1 f(t)dt-\tilde{T}_n\left(\frac{1}{3}\right)\right| \tag{2.8}$$
$$\leqslant \frac{1}{2}\left[\frac{(-1)^{n-1}}{2(2n)!}(1+3^{1-2n})B_{2n}\right]^{\frac{1}{2}}\cdot\left(\int_0^1 t(1-t)\left[f^{(n+1)}(t)\right]^2dt\right)^{\frac{1}{2}},$$

where $\tilde{T}_0\left(\frac{1}{3}\right)=0$ and

$$\tilde{T}_n\left(\frac{1}{3}\right)=\frac{1}{2}\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(3^{1-2k}-1)B_{2k}}{(2k)!}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

REMARK 4. For $n=1$ in Corollary 4 we get

$$\left|\frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]-\int_0^1 f(t)dt\right|\leqslant\frac{1}{6\sqrt{2}}\cdot\left(\int_0^1 t(1-t)\left[f''(t)\right]^2dt\right)^{\frac{1}{2}}.$$

COROLLARY 5. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2m)}$ is absolutely continuous for some $m\geqslant 1$ and $(f^{(2m+1)})^2\in L_1[0,1]$. Then we have*

$$\left|\frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right]-\int_0^1 f(t)dt-T_{2m}\left(\frac{1}{4}\right)\right| \tag{2.9}$$
$$\leqslant \frac{1}{2}\left[-\frac{1}{(4m)!}2^{1-4m}B_{4m}\right]^{\frac{1}{2}}\cdot\left(\int_0^1 t(1-t)\left[f^{(2m+1)}(t)\right]^2dt\right)^{\frac{1}{2}},$$

where $\tilde{T}_0\left(\frac{1}{4}\right)=0$ and

$$\tilde{T}_{2m}\left(\frac{1}{4}\right)=\sum_{k=1}^{m}\frac{2^{-2k}(2^{1-2k}-1)B_{2k}}{(2k)!}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

The following Grüss type inequality also holds.

THEOREM 5. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)}\geqslant 0$ on $[0,1]$. Then we have representation (2.1) and remainder $TG_n(f)$ satisfies the bound*

$$|TG_n(f)|\leqslant\frac{1}{2(n-1)!}\left|\left|G_{n-1}^x(t)\right|\right|_\infty\left\{\frac{f^{(n-1)}(0)+f^{(n-1)}(1)}{2}-f^{(n-2)}[0,1]\right\} \tag{2.10}$$

for any $x\in[0,1/2]$ and

$$f^{(n-2)}[0,1]=f^{(n-2)}(1)-f^{(n-2)}(0).$$

Proof. If we apply Theorem 3 for $f \to G_n^x$, $g \to f^{(n)}$, we deduce

$$\left| \int_0^1 G_n^x(t) f^{(n)}(t) dt - \int_0^1 G_n^x(t) dt \cdot \int_0^1 f^{(n)}(t) dt \right| \leqslant \frac{n}{2} \left| \left| G_{n-1}^x(t) \right| \right|_\infty \left(\int_0^1 t(1-t) f^{(n+1)}(t) dt \right). \tag{2.11}$$

Since

$$\int_0^1 t(1-t) f^{(n+1)}(t) dt = \int_0^1 f^{(n)}(t)[2t-1] dt = \left[f^{(n-1)}(1) + f^{(n-1)}(0) \right] - 2\left(f^{(n-2)}(1) - f^{(n-2)}(0) \right),$$

using representation (2.1) and inequality (2.11), we deduce (2.10). □

REMARK 5. From [5] we have that for $n-1 = 2k,\ k \geqslant 2$

$$\left| \left| G_{n-1}^x(t) \right| \right|_\infty = ||G_{2k}^x(t)||_\infty = 2\max\{|B_{2k}(x)|, |B_{2k}(1/2-x)|\}.$$

3. Applications for Bullen type formula

In [4] the authors generalized identities (1.3) and (1.4) by construction a general closed 4-point rule based on Euler-type identities.

For $k \geqslant 1$ and fixed $x \in \left[0, \frac{1}{2}\right]$ we define functions GH_k^x and FH_k^x as

$$GH_k^x(t) = B_k^*(x-t) + B_k^*(1-x-t) + B_k^*(-t) + B_k^*(1-t) = B_k^*(x-t) + B_k^*(1-x-t) + 2B_k^*(-t)$$

and

$$FH_k^x(t) = GH_k^x(t) - \widetilde{BH}_k(x), \tag{3.1}$$

for all $t \in \mathbb{R}$, where

$$\widetilde{BH}_k(x) = B_k(x) + B_k(1-x) + B_k(0) + B_k(1) = \left[1 + (-1)^k\right] [B_k(x) + B_k].$$

Now let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geqslant 1$. We introduce the following notation for each $x \in \left[0, \frac{1}{2}\right]$

$$DH(x) = \frac{1}{4}[f(x) + f(1-x) + f(0) + f(1)].$$

Furthermore, we define

$$\widetilde{TH}_0(x) = 0$$
$$\widetilde{TH}_m(x) = \frac{1}{4}[T_m(x) + T_m(1-x) + T_m(0) + T_m(1)],\ 1 \leqslant m \leqslant n,$$

It can be easily checked that

$$\widetilde{TH}_m(x) = \frac{1}{4}\sum_{k=1}^{m} \frac{\widetilde{BH}_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right].$$

THEOREM 6. *Let* $f:[0,1]\to\mathbb{R}$, *be such that for some* $n\in\mathbb{N}$ *derivative* $f^{(n-1)}$ *is a continuous function of bounded variation on* $[0,1]$. *Then for every* $x\in[0,1/2]$

$$\int_0^1 f(t)\,dt = DH(x) - \widetilde{TH}_n(x) + \widetilde{RH}_n^1(x) \tag{3.2}$$

and

$$\int_0^1 f(t)\,dt = DH(x) - \widetilde{TH}_{n-1}(x) + \widetilde{RH}_n^2(x), \tag{3.3}$$

where

$$\widetilde{RH}_n^1(x) = \frac{1}{4n!}\int_0^1 GH_n^x(t)\,df^{(n-1)}(t)$$

and

$$\widetilde{RH}_n^2(x) = \frac{1}{4n!}\int_0^1 FH_n^x(t)\,df^{(n-1)}(t).$$

Using Theorem 2 for identity (3.2) we get the following Grüss type inequality:

THEOREM 7. *Let* $f:[0,1]\to\mathbb{R}$ *be such that* $f^{(n)}$ *is absolutely continuous for some* $n\geqslant 1$ *and* $(f^{(n+1)})^2\in L_1[0,1]$. *Then for* $x\in[0,1/2]$ *we have*

$$DH(x) - \int_0^1 f(t)dt - \widetilde{TH}_n(x) = TGH_n(f) \tag{3.4}$$

and the remainder $TGH_n(f)$ *satisfies the estimation*

$$|TGH_n(f)| \leqslant \frac{1}{4}\left[\frac{(-1)^{n-1}}{(2n)!}(3B_{2n}+4B_{2n}(x)+B_{2n}(1-2x))\right]^{\frac{1}{2}} \times\left(\int_0^1 t(1-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}. \tag{3.5}$$

Proof. If we apply Theorem 2 for $f\to GH_n^x$, $g\to f^{(n)}$, we deduce

$$\left|\int_0^1 GH_n^x(t)\,f^{(n)}(t)dt - \int_0^1 GH_n^x(t)\,dt\cdot\int_0^1 f^{(n)}(t)dt\right| \leqslant \frac{1}{\sqrt{2}}\left[T\left(GH_n^x(\cdot),GH_n^x(\cdot)\right)\right]^{\frac{1}{2}}\cdot\left(\int_0^1 t(1-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}}, \tag{3.6}$$

where

$$T\left(GH_n^x(\cdot),GH_n^x(\cdot)\right) = \int_0^1 \left[GH_n^x(t)\right]^2 dt - \left[\int_0^1 GH_n^x(t)\,dt\right]^2.$$

By easy calculation we get

$$\int_0^1 GH_n^x(t)\,dt = 0,$$

and using integration by part we have

$$\begin{aligned}
&\int_0^1 (GH_n^x(t))^2\,dt\\
&= (-1)^{n-1}\frac{n(n-1)\dots 2}{(n+1)(n+2)\dots(2n-1)}\left[\int_0^1 GH_1^x(t)GH_{2n-1}^x(t)dt\right]\\
&= (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[-4\int_0^1 GH_{2n}^x(t)dt + 2GH_{2n}^x(0) + GH_{2n}^x(x) + GH_{2n}^x(1-x)\right]\\
&= (-1)^{n-1}\frac{2(n!)^2}{(2n)!}\left[3B_{2n} + 4B_{2n}(x) + B_{2n}(1-2x)\right].
\end{aligned}$$

Using (3.2) and (3.6), we deduce representation (3.4) and bound (3.5). □

REMARK 6. Because of (3.1) we get

$$\int_0^1 FH_k^x(t)\,dt = \int_0^1 GH_k^x(t)\,dt - \int_0^1 \widetilde{BH}_k(x)dt = -\widetilde{BH}_k(x),$$

and also

$$\int_0^1 [FH_k^x(t)]^2\,dt = \int_0^1 [GH_k^x(t)]^2\,dt - 2\widetilde{BH}_k(x)\int_0^1 G_k(t)\,dt + \widetilde{BH}_k^2(x).$$

So, using (3.3) similar as in (3.6), we deduce representation (3.4) and bound (3.5), too.

REMARK 7. If in Theorem 7 we choose $x = 0$ we get inequality related to the trapezoid formula (see Corollary 2).

If in Theorem 7 we choose $x = 1/2$ we get inequality related to the Euler bitrapezoid formula:

COROLLARY 6. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geqslant 1$ and $(f^{(n+1)})^2 \in L_1[0,1]$. Then we have*

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right] - \int_0^1 f(t)dt - \widetilde{TH}_n\left(\frac{1}{2}\right)\right| \qquad (3.7)\\
&\leqslant \frac{1}{2}\left[\frac{(-1)^{n-1}}{(2n)!}2^{1-2n}B_{2n}\right]^{\frac{1}{2}}\cdot\left(\int_0^1 t(1-t)\left[f^{(n+1)}(t)\right]^2 dt\right)^{\frac{1}{2}},
\end{aligned}$$

where $\widetilde{TH}_0\left(\frac{1}{2}\right) = 0$ and

$$\widetilde{TH}_n\left(\frac{1}{2}\right) = \sum_{k=1}^{\lfloor n/2\rfloor}\frac{2^{-2k}B_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

REMARK 8. For $n=1$ in Corollary 6 we get

$$\left|\frac{1}{4}\left[f(0)+2f\left(\frac{1}{2}\right)+f(1)\right]-\int_0^1 f(t)dt\right| \leqslant \frac{1}{4\sqrt{6}}\cdot\left(\int_0^1 t(1-t)\left[f''(t)\right]^2 dt\right)^{\frac{1}{2}}.$$

The following Grüss type inequality also holds.

THEOREM 8. *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)}\geqslant 0$ on $[0,1]$. Then we have representation (3.4) and remainder $TGH_n(f)$ satisfies the bound*

$$|TGH_n(f)| \leqslant \frac{1}{4(n-1)!}\left|\left|GH_{n-1}^x(t)\right|\right|_\infty \left\{\frac{f^{(n-1)}(0)+f^{(n-1)}(1)}{2}-f^{(n-2)}[0,1]\right\} \tag{3.8}$$

for any $x\in[0,1/2]$ and

$$f^{(n-2)}[0,1]=f^{(n-2)}(1)-f^{(n-2)}(0).$$

Proof. If we apply Theorem 3 for $f\to GH_n^x$, $g\to f^{(n)}$, we deduce

$$\begin{aligned}&\left|\int_0^1 GH_n^x(t)f^{(n)}(t)dt-\int_0^1 GH_n^x(t)dt\cdot\int_0^1 f^{(n)}(t)dt\right|\\ &\leqslant \frac{n}{2}\left|\left|GH_{n-1}^x(t)\right|\right|_\infty\left(\int_0^1 t(1-t)f^{(n+1)}(t)dt\right).\end{aligned} \tag{3.9}$$

Using representation (3.4) and inequality (3.9), we deduce (3.8). □

REMARK 9. From [4] we have that for $n-1=2k$, $k\geqslant 2$

$$\left|\left|GH_{n-1}^x(t)\right|\right|_\infty = ||GH_{2k}^x(t)||_\infty = 2\max\left\{|B_{2k}(x)+B_{2k}|,|B_{2k}(1/2-x)+B_{2k}(1/2)|\right\}.$$

Acknowledgements. This work has been fully supported by Croatian Science Foundation under the project 5435.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of mathematical functions with formulae, graphs and mathematical tables*, National Bureau of Standards, Applied Math. Series 55, 4th printing, Washington 1965.

[2] I. S. BEREZIN AND N. P. ZHIDKOV, *Computing methods*, Vol. I, Pergamon Press, Oxford, 1965.

[3] P. CERONE AND S. S. DRAGOMIR, *Some new bounds for the Čebyšev functional in terms of the first derivative and applications*, J. Math. Ineq. **8** (1) (2014), 159–170.

[4] M. KLARIČIĆ BAKULA, J. PEČARIĆ, *Generalized Hadamard's inequalities based on general Euler 4-point formulae*, ANZIAM J. **48** (2007), 1–18.

[5] J. PEČARIĆ, I. PERIĆ, A. VUKELIĆ, *On general Euler two-point formulae*, ANZIAM J. **46** (2005), 555–574.

[6] J. PEČARIĆ, A. VUKELIĆ, *Estimations of the error for two-point formula via pre-Grüss inequality*, Gen. Math. **13** (2) (2005), 95–104.

(Received October 20, 2014)

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb, Croatia
e-mail: pecaric@element.hr

A. Vukelić
Faculty of Food Technology and Biotechnology
University of Zagreb
Pierottijeva 6, 10000 Zagreb, Croatia
e-mail: avukelic@pbf.hr