SOME INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL AND EULER TWO–POINT FORMULAE

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Abstract. We use inequalities for the Čebyšev functional in terms of the first derivative (see [3]), for some new bounds for the remainder of general Euler two-point formula and its generalization for Bullen type formula.

1. Introduction

For \( k \geq 1 \) and fixed \( x \in [0, 1/2] \) define the functions \( G_k^x(t) \) and \( F_k^x(t) \) as

\[
G_k^x(t) = B_k^x(x-t) + B_k^x(1-x-t), \; t \in \mathbb{R}
\]

and \( F_k^x(t) = G_k^x(t) - \tilde{B}_k(x) \), \( t \in \mathbb{R} \), where

\[
\tilde{B}_k(x) = B_k(x) + B_k(1-x), \; x \in [0, 1/2], \; k \geq 1.
\]

The functions \( B_k(t) \) are the Bernoulli polynomials, \( B_k = B_k(0) \) are the Bernoulli numbers, and \( B_k^x(t), \; k \geq 0, \) are periodic functions of period 1, related to the Bernoulli polynomials as

\[
B_k^x(t) = B_k(t), \; 0 \leq t < 1 \quad \text{and} \quad B_k^x(t+1) = B_k^x(t), \; t \in \mathbb{R}.
\]

The Bernoulli polynomials \( B_k(t), \; k \geq 0 \) are uniquely determined by the following identities:

\[
B_k'(t) = kB_{k-1}(t), \; k \geq 1; \; B_0(t) = 1, \; B_k(t+1) - B_k(t) = kt^{k-1}, \; k \geq 0.
\]

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have that \( B_0^x(t) = 1 \) and \( B_1^x(t) \) is a discontinuous function with a jump of \(-1\) at each integer. It follows that \( B_k^x(1) = B_k^x(0) = B_k \) for \( k \geq 2 \), so that \( B_k^x(t) \) are continuous functions for \( k \geq 2 \). We get

\[
B_k^{xt} = kB_{k-1}^x(t), \; k \geq 1
\]
for every $t \in \mathbb{R}$ when $k \geq 3$, and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ when $k = 1, 2$.

Especially, we get $\tilde{B}_1(x) = 0$, $\tilde{B}_2(x) = 2x^2 - 2x + 1/3$, $\tilde{B}_3(x) = 0$. Also, for $k \geq 2$ we have $\tilde{B}_k(x) = G_k^1(0)$, that is $F_k^1(t) = G_k^1(t)$, $k \geq 2$, and $F_k^1(t) = G_k^1(t)$, $t \in \mathbb{R}$. Obviously, $G_k^1(t)$ and $F_k^1(t)$ are periodic functions of period 1 and continuous for $k \geq 2$.

Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0, 1]$ for some $n \geq 1$. We introduce the following notation for each $x \in [0, 1/2]$

$$D(x) = \frac{1}{2} [f(x) + f(1-x)].$$

Further, we define $\tilde{T}_0(x) = 0$ and, for $1 \leq m \leq n$, $x \in [0, 1/2]$

$$\tilde{T}_m(x) = \frac{1}{2} [T_m(x) + T_m(1-x)],$$

where

$$T_m(x) = \sum_{k=1}^{m} \frac{B_k(x)}{k!} \left[ f^{(k-1)}(1) - f^{(k-1)}(0) \right].$$

It is easy to see that

$$\tilde{T}_m(x) = \frac{1}{2} \sum_{k=1}^{m} \frac{\tilde{B}_k(x)}{k!} \left[ f^{(k-1)}(1) - f^{(k-1)}(0) \right]. \quad (1.2)$$

In [5], the authors established the general Euler two-point formulae:

**Theorem 1.** Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0, 1]$, for some $n \geq 1$. Then for each $x \in [0, 1/2]$

$$\int_{0}^{1} f(t) dt = D(x) - \tilde{T}_n(x) + \tilde{R}_n^1(f) \quad (1.3)$$

and

$$\int_{0}^{1} f(t) dt = D(x) - \tilde{T}_{n-1}(x) + \tilde{R}_n^2(f), \quad (1.4)$$

where

$$\tilde{R}_n^1(f) = \frac{1}{2n!} \int_{0}^{1} G_n^x(t) f^{(n-1)}(t) dt, \quad \tilde{R}_n^2(f) = \frac{1}{2n!} \int_{0}^{1} F_n^x(t) f^{(n-1)}(t) dt.$$
Theorem 2. Let \( f, g : [a, b] \to \mathbb{R} \) be two absolutely continuous functions on \( [a, b] \) with
\[
(\cdot - a)(b - \cdot)(f')^2, (\cdot - a)(b - \cdot)(g')^2 \in \mathcal{L}_1[a, b].
\]
Then we have the inequality
\[
|T(f, g)| \leq \frac{1}{\sqrt{2}} \left[ T(f, f) \right]^\frac{1}{2} \frac{1}{\sqrt{b - a}} \left( \int_a^b (x - a)(b - x) [g'(x)]^2 \, dx \right) \cdot \left( \int_a^b (x - a)(b - x) [f'(x)]^2 \, dx \right)^\frac{1}{2}.
\]
(1.6)
The constant \( \frac{1}{\sqrt{2}} \) and \( \frac{1}{2} \) are best possible in (1.6).

Theorem 3. Assume that \( g : [a, b] \to \mathbb{R} \) is monotonic nondecreasing on \( [a, b] \) and \( f : [a, b] \to \mathbb{R} \) is absolutely continuous with \( f' \in \mathcal{L}_\infty[a, b] \). Then we have the inequality
\[
|T(f, g)| \leq \frac{1}{2(b - a)} ||f'||_{\infty} \int_a^b (x - a)(b - x)dg(x).
\]
(1.7)
The constant \( \frac{1}{2} \) is best possible.

In [6] the authors gave some estimations of the error for two-point formula (1.3) via pre-Grüss inequality.

In this paper we will use the above theorems to get some new bounds for the remainders of general Euler two-point formula (1.3). Applications for Bullen type formula are also proved. As special cases, some new bounds for Euler trapezoid formula, Euler midpoint formula, Euler two-point Newton-Cotes formula, Euler two-point Maclaurin formula and Euler bitrapezoid formula are considered.

2. Applications for the general Euler two-point formula

Using Theorem 2 for identity (1.3) we get the following Grüss type inequality:

Theorem 4. Let \( f : [0, 1] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous for some \( n \geq 1 \) and \( (f^{(n+1)})^2 \in \mathcal{L}_1[0, 1] \). Then for \( x \in [0, 1/2] \) we have
\[
D(x) - \int_0^1 f(t) \, dt - \tilde{T}_n(x) = T G_n(f)
\]
and the remainder \( T G_n(f) \) satisfies the estimation
\[
|T G_n(f)| \leq \frac{1}{2} \left[ \frac{(-1)^{n-1}}{(2n)!} (B_{2n} + B_{2n}(1 - 2x)) \right]^\frac{1}{2} \cdot \left( \int_0^1 (1 - t) \left[ f^{(n+1)}(t) \right]^2 \, dt \right)^\frac{1}{2}.
\]
(2.2)
Proof. If we apply Theorem 2 for \( f \to G^x_n, \ g \to f^{(n)} \), we deduce

\[
\left| \int_0^1 G^x_n(t) f^{(n)}(t) dt - \int_0^1 G^x_n(t) \cdot \int_0^1 f^{(n)}(t) dt \right| \leq \frac{1}{\sqrt{2}} \left[ T \left( G^x_n(\cdot), G^x_n(\cdot) \right) \right]^{1/2} \cdot \left( \int_0^1 t(1-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{1/2}, \tag{2.3}
\]

where

\[
T \left( G^x_n(\cdot), G^x_n(\cdot) \right) = \int_0^1 \left[ G^x_n(t) \right]^2 dt - \left[ \int_0^1 G^x_n(t) dt \right]^2.
\]

From [5] we have \( \int_0^1 G^x_n(t) dt = 0 \) and

\[
\int_0^1 \left[ G^x_n(t) \right]^2 dt = (-1)^{n-1} \frac{2(n!)^2}{(2n)!} (B_{2n} + B_{2n}(1 - 2x)).
\]

Using (1.3) and (2.3), we deduce representation (2.1) and bound (2.2). \( \square \)

Remark 1. We have

\[
\int_0^1 F^x_k(t) dt = \int_0^1 G^x_k(t) dt - \int_0^1 \tilde{B}_k(\cdot) dt = -\tilde{B}_k(\cdot),
\]

and also

\[
\int_0^1 \left[ F^x_k(t) \right]^2 dt = \int_0^1 \left[ G^x_k(t) \right]^2 dt - 2\tilde{B}_k(\cdot) \int_0^1 G_k(t) dt + \tilde{B}_k^2(\cdot).
\]

So, using (1.4) similar as in (2.3), we deduce representation (2.1) and bound (2.2), too.

Corollary 1. Let \( f : [0, 1] \to \mathbb{R} \) be such that \( f^{(2k-1)} \) is absolutely continuous for some \( k \geq 2 \), \( (f^{(2k)})^2 \in L_1[0, 1] \) and \( f^{(2k-1)} \) \( \geq 0 \) on \( [0, 1] \). Then for \( x \in \left[ 0, \frac{1}{2} - \frac{1}{2\sqrt{3}} \right] \) we have

\[
0 \leq (-1)^{k-1} \left\{ D(x) - \int_0^1 f(t) dt - \tilde{T}_{2k-1}(x) \right\} \tag{2.4}
\]

\[
\leq \frac{1}{2} \left[ \frac{1}{(4k-2)!} (B_{4k-2} + B_{4k-2}(1 - 2x)) \right]^\frac{1}{2} \cdot \left( \int_0^1 t(1-t) \left[ f^{(2k)}(t) \right]^2 dt \right)^\frac{1}{2},
\]

and for \( x \in \left[ \frac{1}{2\sqrt{3}}, \frac{1}{2} \right] \)

\[
0 \leq (-1)^{k} \left\{ D(x) - \int_0^1 f(t) dt - \tilde{T}_{2k-1}(x) \right\} \tag{2.5}
\]

\[
\leq \frac{1}{2} \left[ \frac{1}{(4k-2)!} (B_{4k-2} + B_{4k-2}(1 - 2x)) \right]^\frac{1}{2} \cdot \left( \int_0^1 t(1-t) \left[ f^{(2k)}(t) \right]^2 dt \right)^\frac{1}{2}.
\]
Proof. We are using Lemma 2 from [5]. □

If in Theorem 4 we choose \( x = 0, 1/2, 1/3, 1/4 \) we get inequality related to the trapezoid, the midpoint, the two-point Newton-Cotes and the two point Maclaurin formula of Euler type:

**COROLLARY 2.** Let \( f : [0, 1] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous for some \( n \geq 1 \) and \( (f^{(n+1)})^2 \in L_1[0, 1] \). Then we have

\[
\left| \frac{1}{2} \left[ f(0) + f(1) \right] - \int_0^1 f(t) dt - \tilde{T}_n(0) \right| \leq \frac{1}{2} \left[ \frac{2(-1)^{n-1}}{(2n)!} B_{2n} \right]^{\frac{1}{2}} \cdot \left( \int_0^1 t(1-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}},
\]

where \( \tilde{T}_n(0) = 0 \) and

\[
\tilde{T}_n(0) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right].
\]

**REMARK 2.** For \( n = 1 \) in Corollary 2 we get

\[
\left| \frac{1}{2} [f(0) + f(1)] - \int_0^1 f(t) dt \right| \leq \frac{1}{2\sqrt{6}} \cdot \left( \int_0^1 t(1-t) [f''(t)]^2 dt \right)^{\frac{1}{2}}.
\]

**COROLLARY 3.** Let \( f : [0, 1] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous for some \( n \geq 1 \) and \( (f^{(n+1)})^2 \in L_1[0, 1] \). Then we have

\[
\left| f \left( \frac{1}{2} \right) - \int_0^1 f(t) dt - \tilde{T}_n \left( \frac{1}{2} \right) \right| \leq \frac{1}{2} \left[ \frac{2(-1)^{n-1}}{(2n)!} B_{2n} \right]^{\frac{1}{2}} \cdot \left( \int_0^1 t(1-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}},
\]

where \( \tilde{T}_n \left( \frac{1}{2} \right) = 0 \) and

\[
\tilde{T}_n \left( \frac{1}{2} \right) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!} \left( \frac{2^{1-2k} - 1}{(2k)!} \right) \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right].
\]

**REMARK 3.** For \( n = 1 \) in Corollary 3 we get

\[
\left| f \left( \frac{1}{2} \right) - \int_0^1 f(t) dt \right| \leq \frac{1}{2\sqrt{6}} \cdot \left( \int_0^1 t(1-t) [f''(t)]^2 dt \right)^{\frac{1}{2}}.
\]
**Corollary 4.** Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geq 1$ and $(f^{(n+1)})^2 \in L_1[0, 1]$. Then we have

$$
\left| \frac{1}{2} \left[ f \left( \frac{1}{3} \right) + f \left( \frac{2}{3} \right) \right] - \int_0^1 f(t)dt - \tilde{T}_n \left( \frac{1}{3} \right) \right| \leq \frac{1}{2} \left[ \frac{(-1)^{n-1}}{2(2n)!} (1 + 3^{1-2n})B_{2n} \right]^{\frac{1}{2}} \cdot \left( \int_0^1 t(1-t) \left[ f^{(n+1)}(t) \right]^2 dt \right)^{\frac{1}{2}},
$$

where $\tilde{T}_0 \left( \frac{1}{3} \right) = 0$ and

$$
\tilde{T}_n \left( \frac{1}{3} \right) = \frac{1}{2} \sum_{k=1}^{[n/2]} \frac{(3^{1-2k} - 1)B_{2k}}{(2k)!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right].
$$

**Remark 4.** For $n = 1$ in Corollary 4 we get

$$
\left| \frac{1}{2} \left[ f \left( \frac{1}{3} \right) + f \left( \frac{2}{3} \right) \right] - \int_0^1 f(t)dt \right| \leq \frac{1}{6\sqrt{2}} \cdot \left( \int_0^1 t(1-t) \left[ f''(t) \right]^2 dt \right)^{\frac{1}{2}}.
$$

**Corollary 5.** Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(2m)}$ is absolutely continuous for some $m \geq 1$ and $(f^{(2m+1)})^2 \in L_1[0, 1]$. Then we have

$$
\left| \frac{1}{2} \left[ f \left( \frac{1}{4} \right) + f \left( \frac{3}{4} \right) \right] - \int_0^1 f(t)dt - T_{2m} \left( \frac{1}{4} \right) \right| \leq \frac{1}{2} \left[ -\frac{1}{(4m)!} 2^{1-4m}B_{4m} \right]^{\frac{1}{2}} \cdot \left( \int_0^1 t(1-t) \left[ f^{(2m+1)}(t) \right]^2 dt \right)^{\frac{1}{2}},
$$

where $T_0 \left( \frac{1}{4} \right) = 0$ and

$$
T_{2m} \left( \frac{1}{4} \right) = \sum_{k=1}^{m} 2^{-2k} \frac{(2^{1-2k} - 1)B_{2k}}{(2k)!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right].
$$

The following Grüss type inequality also holds.

**Theorem 5.** Let $f : [0, 1] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \geq 0$ on $[0, 1]$. Then we have representation \((2.1)\) and remainder $TG_n(f)$ satisfies the bound

$$
|TG_n(f)| \leq \frac{1}{2(n-1)!} \left\| G_{n-1}^x(t) \right\|_\infty \left\{ \frac{f^{(n-1)}(0) + f^{(n-1)}(1)}{2} - f^{(n-2)}[0, 1] \right\}
$$

for any $x \in [0, 1/2]$ and

$$
f^{(n-2)}[0, 1] = f^{(n-2)}(1) - f^{(n-2)}(0).$$
Proof. If we apply Theorem 3 for $f \to G^x_n$, $g \to f^{(n)}$, we deduce
\[
\left| \int_0^1 G^x_n(t) f^{(n)}(t) dt - \int_0^1 G^x_n(t) dt \cdot \int_0^1 f^{(n)}(t) dt \right| \leq \frac{n}{2} ||G^x_{n-1}(t)||_\infty \left( \int_0^1 (1-t) f^{(n+1)}(t) dt \right).
\] (2.11)

Since
\[
\int_0^1 t(1-t) f^{(n+1)}(t) dt = \int_0^1 f^{(n)}(t)[2t-1]dt \]
\[
= \left[ f^{(n-1)}(1) + f^{(n-1)}(0) \right] - 2 \left( f^{(n-2)}(1) - f^{(n-2)}(0) \right),
\]
using representation (2.1) and inequality (2.11), we deduce (2.10). \qed

Remark 5. From [5] we have that for $n - 1 = 2k$, $k \geq 2$
\[
||G^x_{n-1}(t)||_\infty = ||G^x_{2k}(t)||_\infty = 2 \max \{|B_{2k}(x)|, |B_{2k}(1/2-x)|\}.
\]

3. Applications for Bullen type formula

In [4] the authors generalized identities (1.3) and (1.4) by construction a general closed 4-point rule based on Euler-type identities.

For $k \geq 1$ and fixed $x \in [0, \frac{1}{2}]$ we define functions $GH^x_k$ and $FH^x_k$ as
\[
GH^x_k(t) = B^x_k(x-t) + B^x_k(1-x-t) + B^x_k(-t) + B^x_k(1-t)
\]
\[
= B^x_k(x-t) + B^x_k(1-x-t) + 2B^x_k(-t)
\]
and
\[
FH^x_k(t) = GH^x_k(t) - \widetilde{BH}_k(x),
\] (3.1)
for all $t \in \mathbb{R}$, where
\[
\widetilde{BH}_k(x) = B_k(x) + B_k(1-x) + B_k(0) + B_k(1)
\]
\[
= \left[ 1 + (-1)^k \right] [B_k(x) + B_k].
\]

Now let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geq 1$. We introduce the following notation for each $x \in [0, \frac{1}{2}]$
\[
DH(x) = \frac{1}{4} [f(x) + f(1-x) + f(0) + f(1)].
\]
Furthermore, we define
\[
\widetilde{TH}_0(x) = 0
\]
\[
\widetilde{TH}_m(x) = \frac{1}{4} [T_m(x) + T_m(1-x) + T_m(0) + T_m(1)], 1 \leq m \leq n,
\]
It can be easily checked that

\[ \widetilde{TH}_m(x) = \frac{1}{4} \sum_{k=1}^{m} \frac{BH_k(x)}{k!} \left[ f^{(k-1)}(1) - f^{(k-1)}(0) \right]. \]

**Theorem 6.** Let \( f : [0, 1] \to \mathbb{R} \), be such that for some \( n \in \mathbb{N} \) derivative \( f^{(n-1)} \) is a continuous function of bounded variation on \([0, 1]\). Then for every \( x \in [0, 1/2] \)

\[ \int_0^1 f(t) \, dt = DH(x) - \widetilde{TH}_n(x) + RH^1_n(x) \quad (3.2) \]

and

\[ \int_0^1 f(t) \, dt = DH(x) - \widetilde{TH}_{n-1}(x) + RH^2_n(x), \quad (3.3) \]

where

\[ RH^1_n(x) = \frac{1}{4n!} \int_0^1 GH^n_x(t) \, df^{(n-1)}(t) \]

and

\[ RH^2_n(x) = \frac{1}{4n!} \int_0^1 FH^n_n(t) \, df^{(n-1)}(t). \]

Using Theorem 2 for identity (3.2) we get the following Grüss type inequality:

**Theorem 7.** Let \( f : [0, 1] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous for some \( n \geq 1 \) and \((f^{(n+1)})^2 \in L_1[0, 1]\). Then for \( x \in [0, 1/2] \) we have

\[ DH(x) - \int_0^1 f(t) \, dt - \widetilde{TH}_n(x) = TGH_n(f) \quad (3.4) \]

and the remainder \( TGH_n(f) \) satisfies the estimation

\[ |TGH_n(f)| \leq \frac{1}{4} \left[ \frac{(-1)^{n-1}}{(2n)!} (3B_{2n} + 4B_{2n}(x) + B_{2n}(1 - 2x)) \right]^{1/2} \]

\[ \times \left( \int_0^1 t(1-t) \left[ f^{(n+1)}(t) \right]^2 \, dt \right)^{1/2}. \quad (3.5) \]

**Proof.** If we apply Theorem 2 for \( f \to GH^n_x, \ g \to f^{(n)} \), we deduce

\[ \left| \int_0^1 GH^n_x(t) \, f^{(n)}(t) \, dt - \int_0^1 GH^n_x(t) \, dt \cdot \int_0^1 f^{(n)}(t) \, dt \right| \]

\[ \leq \frac{1}{\sqrt{2}} \left[ T \left(GH^n_x(\cdot), GH^n_x(\cdot)\right) \right]^{1/2} \cdot \left( \int_0^1 t(1-t) \left[ f^{(n+1)}(t) \right]^2 \, dt \right)^{1/2}, \quad (3.6) \]

where

\[ T \left(GH^n_x(\cdot), GH^n_x(\cdot)\right) = \int_0^1 [GH^n_x(t)]^2 \, dt - \left[ \int_0^1 GH^n_x(t) \, dt \right]^2. \]
By easy calculation we get
\[ \int_0^1 GH_n^x(t) \, dt = 0, \]
and using integration by part we have
\begin{align*}
\int_0^1 (GH_n^x(t))^2 \, dt &= (-1)^{n-1} \frac{n(n-1) \ldots 2}{(n+1)(n+2) \ldots (2n-1)} \left[ \int_0^1 GH_1^x(t)GH_{2n-1}^x(t) \, dt \right] \\
&= (-1)^{n-1} \frac{(n!)^2}{(2n)!} \left[ -4 \int_0^1 GH_{2n}^x(t) \, dt + 2GH_{2n}^x(0) + GH_{2n}^x(x) + GH_{2n}^x(1-x) \right] \\
&= (-1)^{n-1} \frac{2(n!)^2}{(2n)!} \left[ 3B_{2n} + 4B_{2n}(x) + B_{2n}(1-2x) \right].
\end{align*}

Using (3.2) and (3.6), we deduce representation (3.4) and bound (3.5). □

**Remark 6.** Because of (3.1) we get
\[ \int_0^1 FH_k^x(t) \, dt = \int_0^1 GH_k^x(t) \, dt - \int_0^1 BH_k(x) \, dt = -BH_k(x), \]
and also
\[ \int_0^1 [FH_k^x(t)]^2 \, dt = \int_0^1 [GH_k^x(t)]^2 \, dt - 2BH_k(x) \int_0^1 G_k(t) \, dt + 2BH^2_k(x). \]
So, using (3.3) similar as in (3.6), we deduce representation (3.4) and bound (3.5), too.

**Remark 7.** If in Theorem 7 we choose \( x = 0 \) we get inequality related to the trapezoid formula (see Corollary 2).

If in Theorem 7 we choose \( x = 1/2 \) we get inequality related to the Euler bitrapezoid formula:

**Corollary 6.** Let \( f : [0, 1] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous for some \( n \geq 1 \) and \( (f^{(n+1)})^2 \in L_1[0, 1] \). Then we have
\begin{align*}
\left| \frac{1}{4} \left[ f(0) + 2f \left( \frac{1}{2} \right) + f(1) \right] - \int_0^1 f(t) \, dt - \overline{TH}_n \left( \frac{1}{2} \right) \right| \leq \frac{1}{2} \left[ \frac{(-1)^{n-1}}{(2n)!} 2^{1-2n}B_{2n} \right]^\frac{1}{2} \cdot \left( \int_0^1 t(1-t) \left[ f^{(n+1)}(t) \right]^2 \, dt \right)^\frac{1}{2},
\end{align*}
where \( \overline{TH}_0 \left( \frac{1}{2} \right) = 0 \) and
\[ \overline{TH}_n \left( \frac{1}{2} \right) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{2^{2k}B_{2k}}{(2k)!} \left[ f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]. \]
REMARK 8. For \( n = 1 \) in Corollary 6 we get
\[
\left| \frac{1}{4} \left[ f(0) + 2f\left( \frac{1}{2} \right) + f(1) \right] - \int_0^1 f(t) \, dt \right| \leq \frac{1}{4\sqrt{6}} \left( \int_0^1 t(1-t) \left[ f''(t) \right]^2 \, dt \right)^{\frac{1}{2}}.
\]

The following Grüss type inequality also holds.

**Theorem 8.** Let \( f : [0,1] \to \mathbb{R} \) be such that \( f^{(n)} \) is absolutely continuous and \( f^{(n+1)} \geq 0 \) on \([0,1]\). Then we have representation (3.4) and remainder \( TGH_n(f) \) satisfies the bound
\[
|TGH_n(f)| \leq \frac{1}{4(n-1)!} \left| |GH_n^{x}(t)| \right|_{\infty} \left\{ \frac{f^{(n-1)}(0) + f^{(n-1)}(1)}{2} - f^{(n-2)}[0,1] \right\}
\]
for any \( x \in [0,1/2] \) and
\[
f^{(n-2)}[0,1] = f^{(n-2)}(1) - f^{(n-2)}(0).
\]

**Proof.** If we apply Theorem 3 for \( f \to GH_n^{x} \), \( g \to f^{(n)} \), we deduce
\[
\left| \int_0^1 GH_n^{x}(t)f^{(n)}(t) \, dt - \int_0^1 GH_n^{x}(t) \, dt \cdot \int_0^1 f^{(n)}(t) \, dt \right| \leq \frac{n}{2} \left| |GH_n^{x-1}(t)| \right|_{\infty} \left( \int_0^1 t(1-t)f^{(n+1)}(t) \, dt \right).
\]
Using representation (3.4) and inequality (3.9), we deduce (3.8). ■

**Remark 9.** From [4] we have that for \( n - 1 = 2k \), \( k \geq 2 \)
\[
\left| |GH_n^{x-1}(t)| \right|_{\infty} = \left| |GH_{2k}^{x}(t)| \right|_{\infty} = 2 \max \left\{ |B_{2k}(x) + B_{2k}|, |B_{2k}(1/2 - x) + B_{2k}(1/2)| \right\}.
\]

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**References**


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