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FUNCTIONALS RELATED TO THE DEC INEQUALITY

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Abstract. We consider the so-called DEC inequality via theory of isotonic linear functionals. The DEC inequality is a refinement of the well-known Cauchy inequality and its well-known particular cases are the Milne and the Callebaut inequalities. We also investigate properties of some functionals which are arised from the DEC inequality.

1. Introduction

Let us consider the following refinements of the Cauchy inequality: the Milne inequality and the Callebaut inequality. The Callebaut inequality is obtained at 1965 and it states that for positive $a_i, b_i > 0 \ (i = 1, \ldots, n)$ and for $x \in [0,1]$ the following inequality holds, [4]:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \sum_{i=1}^{n} a_i^{1+x} b_i^{1-x} \sum_{i=1}^{n} a_i^{1-x} b_i^{1+x} \leqslant \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2. \tag{1.1}$$

The Milne inequality is the following inequality (see in [8])

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \sum_{i=1}^{n} (a_i^2 + b_i^2) \sum_{i=1}^{n} \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \leqslant \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \tag{1.2}$$

where $a_i, b_i > 0$.

As we can see, the both inequalities are refinements of the well-known Cauchy inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

and have a similar form. In sixties Daykin, Eliezer proposed and Carlitz solved the problem what properties must have a positive function φ which satisfies the generalization of the above-mentioned refinements. Here we give a slightly modified result from [5]. Namely, if $\varphi : (0,\infty) \times (0,\infty) \to (0,\infty)$ is a function with properties

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(D1) $\varphi(ka,kb) = k^2\varphi(a,b)$ for any $a,b,k > 0$

(D2) $\dfrac{b\varphi(a,1)}{a\varphi(b,1)} + \dfrac{a\varphi(b,1)}{b\varphi(a,1)} \leqslant \dfrac{a}{b} + \dfrac{b}{a}$ for any $a,b > 0$

then for $a_i, b_i > 0$ we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \sum_{i=1}^{n} \varphi(a_i,b_i) \sum_{i=1}^{n} \frac{a_i^2 b_i^2}{\varphi(a_i,b_i)} \leqslant \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2. \tag{1.3}$$

In the honour of the above-mentioned mathematicians inequality (1.3) is called the DEC inequality.

Using (D1) we can write a variant of (D2) like:

(D2') $\dfrac{b\varphi(1,a)}{a\varphi(1,b)} + \dfrac{a\varphi(1,b)}{b\varphi(1,a)} \leqslant \dfrac{a}{b} + \dfrac{b}{a}$.

Indeed, the first summand in (D2) can be written like $\dfrac{ba^2\varphi(1,\frac{1}{a})}{ab^2\varphi(1,\frac{1}{b})}$. Replacing a with $\frac{1}{a}$ and b with $\frac{1}{b}$ we get the first summand in (D2').

Let us consider some other natural conditions, given here as (D3), (D4) and (D4').

(D3) The function φ is increasing on the both arguments.

(D4) For $a < b$

$$\frac{\varphi(a,1)}{\varphi(b,1)} \leqslant 1, \quad \frac{\varphi(1,b)}{\varphi(1,a)} \geqslant 1.$$

(D4') For $a < b$

$$\frac{\varphi(1,a)}{\varphi(1,b)} \leqslant 1, \quad \frac{\varphi(b,1)}{\varphi(a,1)} \geqslant 1.$$

LEMMA 1.1. *For a function* $\varphi : (0,\infty)\times(0,\infty) \to (0,\infty)$ *we have D(1) and D(2)* $\Leftrightarrow$ *D(1) and D(3).*

Proof. Let $0 < a < b$. Since for $p > 0$ the inequality $\varphi(a,p) \leqslant \varphi(b,p)$ is equivalent to $\varphi\left(\frac{a}{p},1\right) \leqslant \varphi\left(\frac{b}{p},1\right)$, it is enough to show that $\varphi(a,1)$ is increasing on the argument a. Denote $c = \frac{a}{b}$, $x = \frac{\varphi(a,1)}{\varphi(b,1)}$. Then (D2) looks like

$$\frac{x}{c} + \frac{c}{x} \leqslant c + \frac{1}{c} \text{ or } x^2 - (c^2+1)x + c^2 \leqslant 0 \text{ or } c^2 \leqslant x \leqslant 1.$$

The inequality $x \leqslant 1$ means monotonicity on the first argument. The inequality $c^2 \leqslant x$ can be written like

$$1 \leqslant \frac{b^2\varphi(a,1)}{a^2\varphi(b,1)} = \frac{\varphi(ab,b)}{\varphi(ab,b)} = \frac{\varphi(1,1/a)}{\varphi(1,1/b)}$$

which means monotonicity on the second argument. $\square$

REMARK 1.2. In a similar manner we can prove that D(1) and D(2) $\Leftrightarrow$ D(1) and D(4), (D 1) and D(2') $\Leftrightarrow$ D(1) and D(4') (which also gives D(1) and D(4) $\Leftrightarrow$ D(1) and D(4')). The idea of the proof is similar with Lemma 1.1.

REMARK 1.3. In fact we have a little bit weaker condition for positivity. The condition (D2) must be valid for all non-negative a and b, but after looking at the original proof from Monthly, we only need that (D2) holds for elements of n tuples.

As a consequence of Lemma 1.1 and known results from [5] we get the following result.

COROLLARY 1.4. *If $\varphi : (0,\infty)\times(0,\infty)\to(0,\infty)$ is a positive homogeneous of order 2 function, increasing on every argument, then the DEC inequality (1.3) follows. In particular, it holds if $\varphi(a,b)=M^2(a,b)$, where $M(a,b)$ is a positive homogeneous and increasing on every argument mean.*

REMARK 1.5. Another situation, where DEC inequality appears is if $\varphi(a,b)=1$. In that case (D1) fails, but we still have the validity in the DEC inequality, of course, under the condition that $(a_i^2)_i$ and $(b_i^2)_i$ are oppositely ordered, i.e.

$$(a_i^2-a_j^2)(b_i^2-b_j^2)\leqslant 0$$

for all $i,j=1,\dots,n$. In this case, inequality (1.3) becomes

$$\left(\sum_{i=1}^n a_ib_i\right)^2\leqslant n\sum_{i=1}^n a_i^2b_i^2\leqslant\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2.$$

The first inequality follows from the Cauchy inequality, while the second inequality is a consequence of the classical Chebyshev inequality.

REMARK 1.6. (*The weighted DEC inequality*) Putting in (1.3) $\sqrt{p_i}a_i$ instead of a_i, $\sqrt{p_i}b_i$ instead of b_i, where $p_i>0$, $i=1,2,\dots,n$, and under the assumption that φ is positive homogeneous of order 2, we obtain the weighted DEC inequality:

$$\left(\sum_{i=1}^n p_ia_ib_i\right)^2\leqslant\sum_{i=1}^n p_i\varphi(a_i,b_i)\sum_{i=1}^n p_i\frac{a_i^2b_i^2}{\varphi(a_i,b_i)}\leqslant\sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2. \tag{1.4}$$

In the further text we assume that weights (p_i) are non-negative.

After this introductory section we follow with results about the DEC inequality in the language of isotonic linear functionals and connecting it with various kinds of means. The third section is devoted to investigation of functionals DEC_L and DEC_R in variable p which arise from the DEC inequality. Also, a composite functional is considered. In the fourth section we return to discrete version of the DEC inequality, while the short fifth section contains results for the DEC functional which variable is an index set. In the last section some interesting Hölder type inequalities are given.

2. Functional version of the DEC inequality

In [13] Sitnik investigated a discrete refinement of the Cauchy inequality (1.3) and an integral refinement of the Cauchy inequality is given as

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leqslant \int_a^b \Phi(f(x),g(x))dx \int_a^b \frac{f^2(x)g^2(x)}{\Phi(f(x),g(x))}dx \leqslant \int_a^b f^2(x)dx \int_a^b g^2(x)dx \tag{2.1}$$

for different choices of function Φ. It is natural to consider more general situation in which summation and integration are replaced with an arbitrary linear isotonic functional.

Let E be a non-empty set and L be a class of real-valued functions on E having the properties:

L1. If $f,g \in L$, then $(af+bg) \in L$ for all $a,b \in \mathbf{R}$;

L2. The function $\mathbf{1}$ belongs to L. ($\mathbf{1}(t) = 1$ for $t \in E$).

A functional $A : L \to \mathbf{R}$ is called an isotonic linear functional if

A1. $A(af+bg) = aA(f) + bA(g)$ for $f,g \in L$, $a,b \in \mathbf{R}$;

A2. $f \in L$, $f(t) \geqslant 0$ on E implies $A(f) \geqslant 0$.

A lot of results involving isotonic linear functional are given in monograph [12]. The following theorem contains results which are generalization of refinements given in (1.3) and (2.1).

THEOREM 2.1. *Let A be an isotonic linear functional defined on L. Let φ be a positive real function of two variables and p, f and g be functions such that functions pf^2, pg^2, $p\varphi(f,g)$, $p\frac{f^2g^2}{\varphi(f,g)}$ belong to L.*

(i) Then

$$A^2(pfg) \leqslant A(p\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right). \tag{2.2}$$

(ii) If functions $\dfrac{f^2}{\varphi(f,g)}$, $\dfrac{g^2}{\varphi(f,g)}$ are oppositely ordered, then the following refinement of the Cauchy inequality holds

$$A^2(pfg) \leqslant A(p\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right) \leqslant A(pf^2)A(pg^2). \tag{2.3}$$

(iii) If functions $\dfrac{f^2}{\varphi(f,g)}$, $\dfrac{g^2}{\varphi(f,g)}$ are similarly ordered, then

$$A(p\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right) \geqslant A(pf^2)A(pg^2). \tag{2.4}$$

Proof. (i) Let us mention a functional version of the Cauchy inequality, [12, p. 113]. If p is a non-negative weight and u,v functions such that $pu^2, pv^2, puv \in L$, then

$$A^2(puv) \leqslant A(pu^2)A(pv^2). \tag{2.5}$$

Putting $u = \sqrt{\varphi(f,g)}$, $v = \frac{fg}{\sqrt{\varphi(f,g)}}$ we get (2.2) which is a functional version of the first DEC inequality given in (1.3).

(ii) The proof is based on the functional version of the Chebyshev inequality. Namely, if p is a non-negative weight and functions u and v are oppositely ordered functions such that $p, pu, pv, puv \in L$, then

$$A(p)A(puv) \leqslant A(pu)A(pv). \tag{2.6}$$

Putting in the above-mentioned Chebyshev inequality $p\varphi(f,g)$ instead of p, $u = \frac{f^2}{\varphi(f,g)}$, $v = \frac{g^2}{\varphi(f,g)}$ we get the second inequality in (2.3). The first inequality is proved in (i). Let us point out that (2.3) is a functional version of the DEC inequality (1.3).

(iii) If u and v are similarly ordered, then in the Chebyshev inequality (2.6) an opposite sign is valid and using same substitutions as in (ii) we conclude the statement of (iii). □

REMARK 2.2. As we have already noted, in [13], Sitnik investigated an integral DEC inequality for some choices of function Φ. Some of the considered functions were means. By a mean he called a function $M : [0,\infty) \times [0,\infty) \to [0,\infty)$ with properties:

1) $M(x,x) = x$ for all $x \geqslant 0$.

2) $M(\lambda x, \lambda y) = \lambda M(x,y)$ for $\lambda > 0$ and $x,y \in [0,\infty)$, (positive homogenity).

3) If $x_1 < x_2$, then $M(x_1,y) < M(x_2,y)$ and if $y_1 < y_2$, then $M(x,y_1) < M(x,y_2)$, (monotonicity on both arguments).

4) $M(x,y) = M(y,x)$ for all $x,y \geqslant 0$ (symmetricity).

He stated that inequality (2.1) holds for $\Phi = M^2$ where M is a mean (not neccessary symmetric). Let us show that: if a function M is positive homogeneous and increasing on both arguments, then functions $\frac{f^2}{M^2(f,g)}$, $\frac{g^2}{M^2(f,g)}$ are oppositely ordered, i.e. an assumption of Theorem 2.1(ii) holds.

Consider for instance the case $\frac{g(y)}{f(y)} < \frac{g(x)}{f(x)}$. Since we consider positive f and g, we have $\frac{f(y)}{g(y)} > \frac{f(x)}{g(x)}$ and monotonicity of the function M gives

$$M\left(1, \frac{g(y)}{f(y)}\right) < M\left(1, \frac{g(x)}{f(x)}\right) \quad \text{and} \quad M\left(\frac{f(y)}{g(y)}, 1\right) > M\left(\frac{f(x)}{g(x)}, 1\right)$$

which together with nonnegativity of the mean gives

$$\left[M^2\left(1, \frac{g(x)}{f(x)}\right) - M^2\left(1, \frac{g(y)}{f(y)}\right)\right]\left[M^2\left(\frac{f(x)}{g(x)}, 1\right) - M^2\left(\frac{f(y)}{g(y)}, 1\right)\right] \leqslant 0.$$

Since M is homogeneous, after multiplying with $f^2(x)f^2(y)g^2(x)g^2(y)$ we get

$$\Big(f^2(y)M^2(f(x),g(x))-f^2(x)M^2(f(y),g(y))\Big)$$
$$\times\Big(g^2(y)M^2(f(x),g(x))-g^2(x)M^2(f(y),g(y))\Big)\leqslant 0$$

i.e. functions $\dfrac{f^2}{M^2(f,g)},\dfrac{g^2}{M^2(f,g)}$ are oppositely ordered.

So, if $\varphi=M^2$ where M is a positive homogeneous and increasing on both arguments function, then a functional version of the DEC inequality holds. Let us point out that we do not use the first property of mean ($M(x,x)=x$), only the second and the third properties. If $M(1,1)=C$ by considering $\tilde{M}(x,y)=M(x,y)/C$ we have the first property for $\tilde{M}(x,y)$. On the other hand, note that inequalities, which we get using the mean $M(x,y)$ and the mean $\tilde{M}(x,y)$ are the same.

EXAMPLE 2.3. As it was mentioned in [13], examples of means for which the DEC inequality holds are power means and the Rado means defined as

$$M_\alpha(x,y)=\begin{cases}\left(\frac{x^\alpha+y^\alpha}{2}\right)^{\frac{1}{\alpha}}, & -\infty<\alpha<\infty,\alpha\neq 0\\ \sqrt{xy}, & \alpha=0\\ \min(x,y), & \alpha=-\infty\\ \max(x,y), & \alpha=\infty\end{cases}\qquad\text{the power mean,}$$

$$R_\beta(x,y)=\begin{cases}\left(\frac{x^{\beta+1}+y^{\beta+1}}{(\beta+1)(x-y)}\right)^{\frac{1}{\beta}}, & -\infty<\beta<\infty,\beta\neq 0,-1\\ \frac{y-x}{\log y-\log x}, & \beta=-1\\ \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}, & \beta=0\\ \min(x,y), & \beta=-\infty\\ \max(x,y), & \beta=\infty\end{cases}\qquad\text{the Rado mean.}$$

Besides these examples we mention another type of means for which the DEC inequality holds. Consider now Seiffert type means defined by

$$M(x,y)=\frac{|x-y|}{2f(\frac{|x-y|}{x+y})}$$

with $\lim_{z\to 0}\frac{f(z)}{z}=1$. Recently, Witkowski, [15], proved that $M(x,y)$ is increasing on every argument if and only if the function $\frac{(1+z)f(z)}{z}$ increases and the function $\frac{(1-z)f(z)}{z}$ decreases on $z\in(0,1]$. It is checked that functions $f(z)=\sin z$, $f(z)=\tan z$, $f(z)=\sinh z$, $f(z)=\tanh z$, $f(z)=\log(1+z)$, $f(z)=\arcsin z$, $f(z)=\arctan z$, $f(z)=$

arsinh z, $f(z) = \operatorname{artanh} z$ satisfy this condition and so we get examples of means which are homogeneous, increasing on every argument (and even symmetric as Witkowski showed). If $f(z) = \arcsin z$ this is the mean $P(x,y)$, and if $f(z) = \arctan z$ this is the mean $T(x,y)$, introduced by Seiffert in 1987 and 1995 respectively. In 2003 Neuman and Sándor, ([10]), considered the case of the inverse hyperbolic functions arcsinh and arctanh – means $M(x,y)$ and logarithmic mean $L(x,y)$ respectively. There are Seiffert means which are not monotone increasing on both arguments, for instance the contra-harmonic mean $C(x,y) = \frac{x^2+y^2}{x+y}$.

EXAMPLE 2.4. Let us consider some other examples when φ is not increasing on both arguments, but the DEC inequality still holds. Let A be a functional defined as: $A(f) = \int_0^1 f(x)dx$. Let $\varphi(x,y) = (x-y)^2+1$, $f(x) = x$, $g(x) = x-1$. Then, $\frac{d}{dx}\varphi(x,y) = 2(x-y)$, $\frac{d}{dy}\varphi(x,y) = -2(x-y)$ and φ is not increasing in any arguments. Since $\frac{f^2(x)}{\varphi(f(x),g(x))} = x^2/2$ which is increasing on $[0,1]$ and $\frac{g^2(x)}{\varphi(f(x),g(x))} = (x-1)^2/2$ which is decreasing on $[0,1]$, the assumption of Theorem 2.1 (ii) holds and the DEC inequality holds for this choice of functions f, g, φ.

3. Functionals related to the DEC inequality

Let A be an isotonic linear functional, functions f and g from L, and let φ be a positive function of two variables. By C we denote a cone of weights

$$C = C(A,f,g,\varphi) = \left\{p \in L : p \geqslant 0,\ pf^2,\ pg^2,\ pfg,\ p\varphi(f,g),\ p\frac{f^2g^2}{\varphi(f,g)} \in L\right\}.$$

Let us define functionals DEC_R and DEC_L on C as:

$$DEC_R(p) = A(pf^2)A(pg^2) - A(p\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right),$$

$$DEC_L(p) = A(p\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right) - A^2(pfg).$$

THEOREM 3.1. *Let assumptions of Theorem 2.1 be satisfied.*

(i) If functions $\frac{f^2}{\varphi(f,g)}$, $\frac{g^2}{\varphi(f,g)}$ *are oppositely ordered, then the functional* DEC_R *is superadditive. If those functions are similarly ordered, then* DEC_R *is subadditive.*

(ii) The functional DEC_L *is superadditive on* C.

Proof. (i) Let p and q be two weights from C and let $\frac{f^2}{\varphi(f,g)}$, $\frac{g^2}{\varphi(f,g)}$ be oppositely ordered. Then we have

$$\begin{aligned}
&DEC_R(p+q)-DEC_R(p)-DEC_R(q)\\
&\quad=A((p+q)f^2)A((p+q)g^2)-A((p+q)\varphi(f,g))A\left((p+q)\frac{f^2g^2}{\varphi(f,g)}\right)\\
&\qquad-A(pf^2)A(pg^2)+A(p\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right)\\
&\qquad-A(qf^2)A(qg^2)+A(q\varphi(f,g))A\left(q\frac{f^2g^2}{\varphi(f,g)}\right)\\
&\quad=A(pf^2)A(qg^2)+A(qf^2)A(pg^2)-A(p\varphi(f,g))A\left(q\frac{f^2g^2}{\varphi(f,g)}\right)\\
&\qquad-A(q\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right).
\end{aligned}$$

Functions $\dfrac{f^2}{\varphi(f,g)}$, $\dfrac{g^2}{\varphi(f,g)}$ are oppositely ordered, i.e.

$$\left(\frac{f^2(y)}{\varphi(f(y),g(y))}-\frac{f^2(x)}{\varphi(f(x),g(x))}\right)\left(\frac{g^2(y)}{\varphi(f(y),g(y))}-\frac{g^2(x)}{\varphi(f(x),g(x))}\right)\leqslant 0.$$

Multiplying with $\varphi(f(x),g(x))\varphi(f(y),g(y))$ we get

$$\begin{gathered}
\left(f^2(y)\varphi(f(x),g(x))-f^2(x)\varphi(f(y),g(y))\right)\\
\times\left(\frac{g^2(y)}{\varphi(f(y),g(y))}-\frac{g^2(x)}{\varphi(f(x),g(x))}\right)\leqslant 0\\
\varphi(f(x),g(x))\frac{f^2(y)g^2(y)}{\varphi(f(y),g(y))}+\varphi(f(y),g(y))\frac{f^2(x)g^2(x)}{\varphi(f(x),g(x))}\\
-f^2(x)g^2(y)-f^2(y)g^2(x)\leqslant 0.
\end{gathered}$$

Multiplying the above inequality with $p(x)q(y)$ and acting on it by functional A with respect to x and then, by functional A with respect to y we obtain

$$\begin{gathered}
A(pf^2)A(qg^2)+A(qf^2)A(pg^2)-A(p\varphi(f,g))A\left(q\frac{f^2g^2}{\varphi(f,g)}\right)\\
-A(q\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right)\geqslant 0
\end{gathered}$$

and the proof is established. If the above-mentioned functions are similarly ordered, then sign of inequality is reversed in all above inequalities and we get that functional DEC_R is subadditive.

(ii) Again, we consider the difference $DEC_L(p+q) - DEC_L(p) - DEC_L(q)$.

$$DEC_L(p+q) - DEC_L(p) - DEC_L(q)$$
$$= A(p\varphi(f,g))A\left(q\frac{f^2g^2}{\varphi(f,g)}\right) + A(q\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right) - 2A(pfg)A(qfg).$$

For any x, y we have

$$\left(f(y)g(y)\sqrt{\frac{\varphi(f(x),g(x))}{\varphi(f(y),g(y))}} - f(x)g(x)\sqrt{\frac{\varphi(f(y),g(y))}{\varphi(f(x),g(x))}}\right)^2 \geqslant 0.$$

Multiplying with $p(x)q(y)$ and acting on it by a functional A with respect to x, and then, by functional A with respect to y we obtain that

$$A(p\varphi(f,g))A\left(q\frac{f^2g^2}{\varphi(f,g)}\right) + A(q\varphi(f,g))A\left(p\frac{f^2g^2}{\varphi(f,g)}\right) - 2A(pfg)A(qfg) \geqslant 0$$

and the proof is established. $\square$

Functionals DEC_L and DEC_R are non-negative and superadditive under certain conditions and they are homogeneous of order 2. So, we can use results from [11] to generate new functional. Let us describe some non-classical concepts which we use in the mentioned results.

DEFINITION 3.2. ([14]) Let I and J be intervals in $\mathbf{R}$, $(0,1) \subseteq J$ and let $h : J \to \mathbf{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbf{R}$ is an h*-convex* function, if f is non-negative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leqslant h(\alpha)f(x) + h(1-\alpha)f(y).$$

If the inequality is reversed, f is called an h*-concave* function.

It is evident that this notion generalizes the concepts: of classical convexity (for $h(t) = t$); of s-convexity in the second sense (for $h(t) = t^s$, $s \in (0,1)$), [2]; of P-functions (for $h(t) = 1$) and of Godunova-Levin functions (for $h(t) = t^{-1}$).

It is known, ([14]), that the function $f(x) = x^\lambda$ is s-convex in the second sense if

$$(\lambda \in (-\infty, 0] \cup [1, \infty),\ s \leqslant 1) \ \text{ or } \ (\lambda \in (0,1),\ s \leqslant \lambda).$$

The function $f(x) = x^\lambda$ is s-concave in the second sense if

$$(\lambda \in (0,1),\ s \geqslant 1) \text{ or } (\lambda > 1,\ s \geqslant \lambda).$$

Examples of a function Φ, non-decreasing on $(0,\infty)$, h-concave, but not concave, where $h(x) = x^s$, $s > 1$ are for instance

$$\Phi(x) = \arctan(x^s), \ \ \Phi(x) = \tanh(x^s), \ \ \Phi(x) = \frac{x^s}{1+x^s}.$$

Let us mention also

$$f(x)=\begin{cases} x^s, \ x\in[0,1] \\ x, \ \ x\in(1,b] \end{cases}$$

which is non-decreasing, convex on $(0,1]$, s-concave in the second sense on $[0,\infty)$, [11]. More examples of h-concave functions are given in [11].

THEOREM 3.3. *(i) Let h be a non-negative, submultiplicative function, and $\Phi : [0,\infty)\to[0,\infty)$ be h-concave and non-decreasing. If assumptions of Theorem 2.1 are satisfied and if functions $\frac{f^2}{\varphi(f,g)}, \frac{g^2}{\varphi(f,g)}$ are oppositely ordered, then the functional η defined on $C_1=\{w\in C \,:\, A(w)>0\}$ by*

$$\eta(w)=h(A^2(w))\Phi\left(\frac{DEC_R(w)}{A^2(w)}\right).$$

is superadditive.

(ii) Furthermore, if h is positive homogeneous of order k and if $w,v\in C_1$ and $M\geqslant m>0$ such that $w-mv, Mv-w\in C_1$,

$$\begin{aligned} M^{2k}h(A^2(v))\Phi\left(\frac{DEC_R(v)}{A^2(v)}\right) &\geqslant h(A^2(w))\Phi\left(\frac{DEC_R(w)}{A^2(w)}\right) \\ &\geqslant m^{2k}h(A^2(v))\Phi\left(\frac{DEC_R(v)}{A^2(v)}\right). \end{aligned} \tag{3.1}$$

(iii) If functions $\frac{f^2}{\varphi(f,g)}, \frac{g^2}{\varphi(f,g)}$ are similarly ordered, then the above statements (i) and (ii) are valid with $DEC_R \to -DEC_R$.

(iv) The above statements (i) and (ii) are valid with $DEC_R \to DEC_L$.

Proof. We use a result from [11] which states that the functional $\eta(x)=h(v(x))\Phi\left(\frac{g(x)}{v(x)}\right)$ is superadditive on a convex cone if h is non-negative submultiplicative, g is superadditive and Φ is h-concave non-decreasing. If functions $\frac{f^2}{\varphi(f,g)}, \frac{g^2}{\varphi(f,g)}$ are oppositely ordered, we define v and g as $v(w)=A^2(w)$ and $g(w)=DEC_R(w)$. Since the function $x\mapsto x^2$ is superadditive, we have that $w\mapsto A^2(w)$ is also superadditive. By Theorem 3.1(i), g is also non-negative, superadditive and positive homogeneous of order 2. By Theorem 9 in [11] we have that η is superadditive, and by Corollary 10 in [11] for the functional η we obtain the inequality which can be transformed to (3.1). Other parts of Theorem 3.3 are proved in similar manner. $\square$

REMARK 3.4. Let us consider a very simple situation if $\Phi(x)=x$, i.e. $h(t)=t$, $k=1$. If functions $\frac{f^2}{\varphi(f,g)}, \frac{g^2}{\varphi(f,g)}$ are oppositely ordered, then for $w,v\in C_1$ and $M\geqslant m>0$ such that $w-mv, Mv-w\in C_1$,

$$M^2DEC_R(v)\geqslant DEC_R(w)\geqslant m^2DEC_R(v).$$

In particular, if $M=1$, then we obtain a monotonicity property of DEC_R:

$$DEC_R(v) \geqslant DEC_R(w) \text{ if } v-w \in C_1.$$

If functions $\frac{f^2}{\varphi(f,g)}$, $\frac{g^2}{\varphi(f,g)}$ are similarly ordered, then the above inequalities are true in the opposite direction.

EXAMPLE 3.5. These results with an isotonic linear functional A are allowing us to work with different examples of functionals. The simplest examples are summation and integration. But there are existing some exotic examples too. For instance, in [13] a refinement of the Cauchy inequality is given with substitution: integral $\longrightarrow$ the Jackson q-integral which is defined as:

$$\int_0^1 f(t)d_qt = (1-q)\sum_{k=0}^{\infty} f(q^k)q^k, \quad 0<q<1.$$

Furthermore, recently a theory of time scale measure spaces is developed ([3]) and integral over time scale set i.e. $\int_E f(t)d\mu_\Delta$ is also an isotonic linear functional. So, all results from two previous sections are valid in such particular cases.

In last few decades we are witnesses of a great development of fractional calculus theory, [7]. A lot of operators are investigated and some of them are isotonic linear. Let us mention here the Riemann-Liouville fractional integral operator defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\sigma)^{\alpha-1} f(\sigma)\,d\sigma, \quad \alpha>0,$$

which is an isotonic linear functional for fixed $t>0$. Or, let us mention even more general operator, the so-called fractional hypergeometric operator, ([1]), defined as:

$$I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} = \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^t \sigma^\mu(1-\sigma)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta+\mu,-\nu,\alpha;1-\frac{\sigma}{t}\right) f(\sigma)\,d\sigma$$

where the function ${}_2F_1(a,b,c,t)$ is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1)\dots(a+n-1)$, $(a)_0=1$, $t>0$, $\alpha > \max\{0,-\beta-\mu\}$, $\mu>-1$, $\beta-1<\eta<0$.

4. Discrete case

The begining point of this paper is the DEC inequality given in the discrete form. So, it is instructive to write results from the previous sections in discrete form. By $S_+(n)$ we denote the convex cone of such non-negative $\overline{p}=(p_1,p_2,\dots,p_n)$. For a function $a:\{1,2,\dots,n\}\to\mathbf{R}$, $i\mapsto a_i$, an isotonic linear functional A is defined as $A(a)=\sum_{i=1}^n a_i$. Let $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ be two positive n-tuples and φ be a positive function of two variables.

Now, functionals DEC_R and DEC_L have the following formulae:

$$DEC_R(\overline{p}) = \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i \varphi(a_i,b_i) \sum_{i=1}^n p_i \frac{a_i^2 b_i^2}{\varphi(a_i,b_i)},$$

$$DEC_L(\overline{p}) = \sum_{i=1}^n p_i \varphi(a_i,b_i) \sum_{i=1}^n p_i \frac{a_i^2 b_i^2}{\varphi(a_i,b_i)} - \left(\sum_{i=1}^n p_i a_i b_i\right)^2.$$

As a consequence of Theorem 2.1 and Theorem 3.1 a functional DEC_L is non-negative and superadditive. If n-tuples $(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered n-tuples, then DEC_R is a non-negative superadditive functional, while if the above n-tuples are similarly ordered, then $-DEC_R$ is non-negative superadditive.

Even those results are consequences of the more general situation, here we show one another proof of superadditivity of DEC_R.

Let us suppose that $(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered n-tuples. The difference $DEC(\overline{p}+\overline{q}) - DEC(\overline{p}) - DEC(\overline{q})$ is equal to

$$\sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n q_i b_i^2 + \sum_{i=1}^n q_i a_i^2 \sum_{i=1}^n p_i b_i^2 - \sum_{i=1}^n p_i \varphi_i \sum_{i=1}^n q_i \frac{a_i^2 b_i^2}{\varphi_i} - \sum_{i=1}^n q_i \varphi_i \sum_{i=1}^n p_i \frac{a_i^2 b_i^2}{\varphi_i} = L_n,$$

where $\varphi_i = \varphi(a_i,b_i)$. Transform the difference $L_n - L_{n-1}$ such that in the first bracket we put terms with factor p_n and in the second we put terms with the factor q_n. We get

$$\begin{aligned}
L_n - L_{n-1} &= p_n \left(a_n^2 \sum_{i=1}^{n-1} q_i b_i^2 + b_n \sum_{i=1}^{n-1} q_i a_i^2 - \varphi_n \sum_{i=1}^{n-1} q_i \frac{a_i^2 b_i^2}{\varphi_i} - \frac{a_n^2 b_n^2}{\varphi_n} \sum_{i=1}^{n-1} q_i \varphi_i \right) \\
&\quad + q_n \left(b_n^2 \sum_{i=1}^{n-1} p_i a_i^2 + a_n^2 \sum_{i=1}^{n-1} p_i b_i^2 - \frac{a_n^2 b_n^2}{\varphi_n} \sum_{i=1}^{n-1} p_i \varphi_i - \varphi_n \sum_{i=1}^{n-1} p_i \frac{a_i^2 b_i^2}{\varphi_i} \right) \\
&= \frac{p_n}{\varphi_n} \sum_{i=1}^{n-1} \frac{q_i}{\varphi_i} (a_n^2 b_i^2 \varphi_n \varphi_i + a_i^2 b_n^2 \varphi_n \varphi_i - a_i^2 b_i^2 \varphi_n^2 - a_n^2 b_n^2 \varphi_i^2) \\
&\quad + \frac{q_n}{\varphi_n} \sum_{i=1}^{n-1} \frac{p_i}{\varphi_i} (a_n^2 b_i^2 \varphi_n \varphi_i + a_i^2 b_n^2 \varphi_n \varphi_i - a_i^2 b_i^2 \varphi_n^2 - a_n^2 b_n^2 \varphi_i^2) \\
&= \frac{p_n}{\varphi_n} \sum_{i=1}^{n-1} \frac{q_i}{\varphi_i} (b_i^2 \varphi_n - b_n^2 \varphi_i)(a_n^2 \varphi_i - a_i^2 \varphi_n) \\
&\quad + \frac{q_n}{\varphi_n} \sum_{i=1}^{n-1} \frac{p_i}{\varphi_i} (b_i^2 \varphi_n - b_n^2 \varphi_i)(a_n^2 \varphi_i - a_i^2 \varphi_n) \geqslant 0
\end{aligned}$$

where the last inequality holds since $(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered n-tuples, namely the following holds

$$\left(\frac{a_i^2}{\varphi_i} - \frac{a_n^2}{\varphi_n}\right)\left(\frac{b_i^2}{\varphi_i} - \frac{b_n^2}{\varphi_n}\right) \leqslant 0.$$

After multiplying with $-\varphi_i^2\varphi_n^2$ we get

$$(a_n^2\varphi_i - a_i^2\varphi_n)(b_i^2\varphi_n - b_n^2\varphi_i) \geqslant 0$$

what is used in the last inequality. So, we have $L_n \geqslant L_{n-1} \geqslant \ldots \geqslant L_1 = 0$ and $DEC(\overline{p}+\overline{q}) - DEC(\overline{p}) - DEC(\overline{q}) \geqslant 0$.

REMARK 4.1. We have different cases of fulfilment of condition "$(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered n-tuples".

(i) Suppose that (D1) and (D2) are fulfilled. We put $a = \frac{a_i}{b_i}$, $b = \frac{a_j}{b_j}$. Having in mind that $\varphi_i = \varphi(a_i,b_i) = b_i^2\varphi(\frac{a_i}{b_i},1) = b_i^2\varphi(a,1)$ and $\varphi_j = \varphi(a_i,b_i) = b_j^2\varphi(b,1)$ we get

$$\frac{b}{a}\frac{\varphi(a,1)}{\varphi(b,1)} = \frac{a_jb_jb_i^2\varphi(a,1)}{a_ib_ib_j^2\varphi(b,1)} = \frac{a_jb_j}{a_ib_i}\frac{\varphi_i}{\varphi_j}.$$

Since $\frac{a_ib_j}{a_jb_i} = \frac{a}{b}$ we get that (D2) get the form

$$\frac{a_jb_j}{a_ib_i}\frac{\varphi_i}{\varphi_j} + \frac{a_ib_i}{a_jb_j}\frac{\varphi_j}{\varphi_i} \leqslant \frac{a_ib_j}{a_jb_i} + \frac{a_jb_i}{a_ib_j}.$$

This can be written like

$$\frac{a_i^2b_i^2}{\varphi_i^2} + \frac{a_j^2b_j^2}{\varphi_j^2} \leqslant \frac{a_i^2b_j^2}{\varphi_i\varphi_j} + \frac{a_j^2b_i^2}{\varphi_j\varphi_i},$$

namely the following holds

$$(\frac{a_i^2}{\varphi_i} - \frac{a_j^2}{\varphi_j})(\frac{b_i^2}{\varphi_i} - \frac{b_j^2}{\varphi_j}) \leqslant 0.$$

This means that $(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered n-tuples.

(i') If $\varphi(x,y)$ is a positive function satisfying (D1) and (D3) then $(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered for any positive n-tuples $(a_1,\ldots,a_n)$ and $(b_1,\ldots,b_n)$. This follows from Lemma 1.1 and (i), but the direct proof is shorter than the above proof from (i).

(ii) Suppose that $\varphi(a,b) = 1$. Then property (D1) is not valid. Then $\varphi_i = \varphi_j = 1$ and the "$(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered" corresponds to: "$(a_i^2)_i, (b_i^2)_i$ are oppositely ordered" which corresponds to a form of the Chebyshev inequality.

So, in these two cases we have that the functional DEC_R is non-negative superadditive.

As a consequence of Theorem 3.3 we have that if h and Φ satisfy assumptions of Theorem 3.3, and if $(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered, then the functional

$$\eta(\overline{p}) = h(P_n^2)\Phi\left(\frac{DEC_R(\overline{p})}{P_n^2}\right)$$

is superadditive, where $P_n=\sum_{i=1}^n p_i>0$, and if h is positive homogeneous of order k, $\overline{p},\overline{q}\in S_+(n)$, $P_n,Q_n>0$, and $M\geqslant m>0$ are such that $M\overline{p}\geqslant\overline{q}\geqslant m\overline{p}$, then

$$M^{2k}h(P_n^2)\Phi\left(\frac{DEC(\overline{p})}{P_n^2}\right)\geqslant h(Q_n^2)\Phi\left(\frac{DEC(\overline{q})}{Q_n^2}\right)\geqslant m^{2k}h(P_n^2)\Phi\left(\frac{DEC(\overline{p})}{P_n^2}\right). \tag{4.1}$$

The following theorem is an refinement of the second inequality in (1.3).

THEOREM 4.2. *Let $a_i,b_i>0$, $\varphi(a_i,b_i)>0$, $(i=1,2,\dots,n)$. Let $\overline{p}\in S_+(n)$ and $p_0=\min\{p_1,\dots,p_n\}$. If $(\frac{a_i^2}{\varphi(a_i,b_i)})_i$ and $(\frac{b_i^2}{\varphi(a_i,b_i)})_i$ are oppositely ordered, then*

$$\begin{aligned}\sum_{i=1}^n p_ia_i^2\sum_{i=1}^n p_ib_i^2 &\geqslant \sum_{i=1}^n p_i\varphi(a_i,b_i)\sum_{i=1}^n p_i\frac{a_i^2b_i^2}{\varphi(a_i,b_i)}\\ &\quad+p_0\left(\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2-\sum_{i=1}^n\varphi(a_i,b_i)\sum_{i=1}^n\frac{a_i^2b_i^2}{\varphi(a_i,b_i)}\right)\\ &\geqslant \sum_{i=1}^n p_i\varphi(a_i,b_i)\sum_{i=1}^n p_i\frac{a_i^2b_i^2}{\varphi(a_i,b_i)}.\end{aligned}$$

Proof. If $\Phi(x)=x$ and $M=1$, then inequality (4.1) gives

$$DEC_R(\overline{p})\geqslant DEC_R(\overline{q})$$

for $\overline{p}\geqslant\overline{q}$. After substitution $q_i=p_0$ for any $i=1,2,\dots,n$ and simple transformation we get the first inequality.

The second inequality is valid because

$$DEC_R(\overline{q})=p_0\left(\sum_{i=1}^n a_i^2\sum_{i=1}^n b_i^2-\sum_{i=1}^n\varphi(a_i,b_i)\sum_{i=1}^n\frac{a_i^2b_i^2}{\varphi(a_i,b_i)}\right)$$

is non-negative. □

5. The DEC functional for index sets

Let $I\subseteq\mathbf{N}$. Let φ be a positive function of two variables and let $p_i,a_i,b_i\geqslant 0$, $\varphi(a_i,b_i)>0$, $(i\in I)$. Let us define the index set function $DEC_i(I)$ by

$$DEC_i(I)=\sum_{i\in I}p_ia_i^2\sum_{i\in I}p_ib_i^2-\sum_{i\in I}p_i\varphi(a_i,b_i)\sum_{i\in I}p_i\frac{a_i^2b_i^2}{\varphi(a_i,b_i)}$$

if all above sums exist.

THEOREM 5.1. *Let $I,J\subset\mathbf{N}$ be sets such that $I\cap J=\emptyset$. Let φ be a positive function of two variables and let $p_i\geqslant 0,\ a_i,b_i,\varphi_i=\varphi(a_i,b_i)>0,\ (i\in I\cup J)$. If $\left(\frac{a_i^2}{\varphi(a_i,b_i)}\right)_{i\in I\cup J}$ and $\left(\frac{b_i^2}{\varphi(a_i,b_i)}\right)_{i\in I\cup J}$ are oppositely ordered, then*

$$DEC_i(I\cup J)\geqslant DEC_i(I)+DEC_i(J). \tag{5.1}$$

Proof. We use a discrete functional A defined as: $A(f)=\sum_{i\in I\cup J}f_i$, where $f(i)=f_i$ and adjointed functional DEC_R. Let $\overline{p}=(p_i)_{i\in I\cup J}$. Let us define two sequences $\overline{r}$ and $\overline{q}$ as following

$$r_i=\begin{cases}p_i & \text{if } i\in I\\ 0 & \text{if } i\in J\end{cases}\quad\text{and}\quad q_i=\begin{cases}0 & \text{if } i\in I\\ p_i & \text{if } i\in J.\end{cases}$$

Since $r_i+q_i=p_i$ for $i\in I\cup J$, $DEC_R(\overline{r})=DEC_i(I)$, $DEC_R(\overline{q})=DEC_i(J)$ and $DEC_R(\overline{p})=DEC_i(I\cup J)$, inequality (5.1) holds by Theorem 3.1. $\square$

A simple consequence of the above theorem is the following chain of refinements.

COROLLARY 5.2. *Suppose that $(a_i),(b_i),(p_i)$ and φ satisfy assumptions of Theorem 5.1. Let $I_k=\{1,2,\ldots,k\},\ k\in\mathbf{N}$. Then*

$$DEC_i(I_k)\geqslant DEC_i(I_{k-1})\geqslant\ldots\geqslant DEC_i(I_2)\geqslant 0.$$

Especially,

$$DEC_i(I_k)\geqslant\max\{DEC_i(J):\ \text{where } J\subseteq I, card(J)=2\}.$$

6. Means again

Let us consider a function

$$F(r)=\left(\frac{\int_a^b K^r(x)\,dx}{\int_a^b G^r(x)\,dx}\right)^{1/r},\ r\neq 0,\ F(0)=\exp\left(\frac{\int_a^b\log(K(x)/G(x))\,dx}{b-a}\right),$$

where K and G are positive functions. By Theorem 7.30 from [12], if G and K/G are oppositely ordered then $F(r)$ is decreasing on $\mathbf{R}$.

Let f,g and φ be such that $\frac{f}{\varphi(f,g)}$ and $\frac{g}{\varphi(f,g)}$ are oppositely ordered and define functions K and G as:

$$G(x)=\frac{g(x)}{\varphi(f(x),g(x))},\quad K(x)=\frac{f(x)g(x)}{\varphi^2(f(x),g(x))}.$$

Then a function F is decreasing on $\mathbf{R}$ where

$$F(r)=\begin{cases}\left(\dfrac{\int_a^b\frac{f^r(x)g^r(x)}{\varphi^{2r}(f(x),g(x))}\,dx}{\int_a^b\frac{g^r(x)}{\varphi^r(f(x),g(x))}\,dx}\right)^{1/r}, & r\neq 0,\\ \exp\left(\dfrac{\int_a^b\log\frac{f(x)}{\varphi(f(x),g(x))}\,dx}{b-a}\right), & r=0.\end{cases}$$

In the similar manner we get that $F_1(r)$ is decreasing on $\mathbf{R}$ where

$$F_1(r) = \begin{cases} \left(\dfrac{\int_a^b \frac{f^r(x)g^r(x)}{\varphi^{2r}(f(x),g(x))}\,dx}{\int_a^b \frac{f^r(x)}{\varphi^r(f(x),g(x))}\,dx} \right)^{1/r}, & r \neq 0, \\ \exp\left(\dfrac{\int_a^b \log \frac{g(x)}{\varphi(f(x),g(x))}\,dx}{b-a} \right), & r = 0. \end{cases}$$

From the inequality $F(1)F_1(1) \leqslant F(0)F_1(0)$ we get

$$\left(\int_a^b \frac{f(x)g(x)}{\varphi^2(f(x),g(x))} dx \right)^2 \leqslant \int_a^b \frac{f(x)}{\varphi(f(x),g(x))} dx \int_a^b \frac{g(x)}{\varphi(f(x),g(x))} dx$$
$$\times \exp\left(\frac{\int_a^b \log \frac{f(x)g(x)}{\varphi^2(f(x),g(x))} dx}{b-a} \right). \tag{6.1}$$

Consider the particular case of a nonsymmetric mean $\varphi(f,g) = f^\alpha g^{1-\alpha}$, $0 \leqslant \alpha \leqslant 1$, $\alpha \neq \frac{1}{2}$.

Then $\frac{fg}{\varphi^2(f,g)} = \left(\frac{f}{g}\right)^{1-2\alpha}$, $\frac{f}{\varphi(f,g)} = \left(\frac{f}{g}\right)^{1-\alpha}$, $\frac{g}{\varphi(f,g)} = \left(\frac{f}{g}\right)^{-\alpha}$.

Putting $s = \left(\frac{f}{g}\right)^{1-2\alpha}$ we get $\left(\frac{f}{g}\right)^{1-\alpha} = s^{\frac{1-\alpha}{1-2\alpha}}$, $\left(\frac{f}{g}\right)^{-\alpha} = s^{\frac{-\alpha}{1-2\alpha}}$. Denote $p = \frac{1-2\alpha}{1-\alpha}$. Then $1/p + 1/q = 1$ for $q = \frac{1-2\alpha}{-\alpha}$ and inequality (6.1) becomes

$$\left(\int_a^b s(x)\,dx \right)^2 \leqslant \int_a^b s^{\frac{1}{p}}(x)\,dx \int_a^b s^{\frac{1}{q}}(x)\,dx \cdot \exp\left(\frac{\int_a^b \log s(x)\,dx}{b-a} \right).$$

We have $\alpha = (p-1)/(p-2)$ and note that

$$\alpha \in \left(0, \frac{1}{2}\right) \quad \text{iff} \quad 0 < p < 1,\ q < 0 \quad \text{and} \quad \alpha \in \left(\frac{1}{2}, 1\right) \quad \text{iff} \quad p < 0,\ 0 < q < 1.$$

The above inequality is true when $0 < p < 1$, $q < 0$ or $p < 0$, $0 < q < 1$.

If $\varphi(x,y) = \sqrt{xy}$ we get by replacing $s(x) = \sqrt{\frac{f(x)}{g(x)}}$ the Hölder type inequality

$$\left(\int_a^b dx \right)^2 \leqslant \int_a^b s(x)\,dx \int_a^b s^{-1}(x)\,dx.$$

Another interesting inequality we get for $\varphi(x,y) = (x^2+y^2)^{\frac{1}{2}}$:

$$\left(\int_a^b \frac{f(x)g(x)dx}{f^2(x)+g^2(x)} \right)^2 \leqslant \int_a^b \frac{f(x)dx}{\sqrt{f^2(x)+g^2(x)}} \int_a^b \frac{g(x)dx}{\sqrt{f^2(x)+g^2(x)}}$$
$$\times \exp\left(\frac{\int_a^b \log\left(\frac{f(x)g(x)}{f^2(x)+g^2(x)} \right) dx}{b-a} \right).$$

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