Journal of Mathematical Inequalities
Volume 9, Number 4 (2015), 1245–1257 doi:10.7153/jmi-09-95

CONVEX ORDERING PROPERTIES AND APPLICATIONS

AURELIA FLOREA, EUGEN PĂLTĂNEA AND DUMITRU BĂLĂ

(*Communicated by M. Matić*)

Abstract. A relevant application of the stochastic convex order is the well-known weighted Hermite-Hadamard inequality, where the weight is provided by a given probability distribution. Our goal is to show that, starting from such a fixed weight, we can fill the whole space between the Hermite-Hadamard bounds by highlighting some parametric families of probability distributions. Thus, we propose two alternative constructions based on the convex ordering properties.

1. Introduction

The paper refers to the following result of Hermite (1883) and Hadamard (1893).

THEOREM 1. *If $f:[a,b]\to\mathbb{R}$ is a convex function, then*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}. \tag{1}$$

A rich literature has been stimulated by this result. We mention here a brief selection of relevant extensions. Fejér [4] highlights in 1906 the first weighted version of (1).

THEOREM 2. *If $f:[a,b]\to\mathbb{R}$ is convex and $g:[a,b]\to[0,\infty)$ is integrable and symmetric about $(a+b)/2$, i.e. $g(a+b-x)=g(x)$, $\forall\, x\in(a,b)$, with $\int_a^b g(x)\,dx>0$, then*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}\leqslant\frac{f(a)+f(b)}{2}. \tag{2}$$

Remark that for $g(x)=1$, $\forall\, x\in[a,b]$, we obtain the Hermite-Hadamard inequality. Another interesting generalization is due to Brenner and Alzer [2].

THEOREM 3. *If $f:[a,b]\to\mathbb{R}$ is convex and $g:[a,b]\to[0,\infty)$ is integrable and symmetric about $c\in(a,b)$, such that $\int_{c-t}^{c+t} g(x)\,dx>0$, $\forall\, 0\leqslant t\leqslant\min\{c-a,b-c\}$, then*

$$f(c)\leqslant\frac{\int_{c-t}^{c+t} f(x)g(x)\,dx}{\int_{c-t}^{c+t} g(x)\,dx}\leqslant\frac{b-c}{b-a}f(a)+\frac{c-a}{b-a}f(b). \tag{3}$$

Mathematics subject classification (2010): 26A51, 26B25, 26D10.
Keywords and phrases: Convex functions, Hermite-Hadamard inequality, convex order.

© ELEMENT, Zagreb
Paper JMI-09-95

Mention that Bakula, Pečarić and Perić have recently provided an extension of Theorem 3 for positive linear functionals (see [1]).

Fink proves in [5] a general weighted version of the Hermite-Hadamard inequality.

THEOREM 4. *Let* μ *be a real Borel measure on* $[a,b]$*, with* $\mu([a,b]) > 0$*. If* $f : [a,b] \to \mathbb{R}$ *is convex, then (under some restrictions of positivity on* μ*),*

$$f(x_\mu) \leqslant \frac{1}{\mu([a,b])} \int_a^b f(x)\, d\mu(x) \leqslant \frac{b-x_\mu}{b-a} f(a) + \frac{x_\mu - a}{b-a} f(b), \tag{4}$$

where $x_\mu = \frac{\int_a^b x\, d\mu(x)}{\mu([a,b])}$ *is the barycenter of* μ*.*

Florea and Niculescu find in [6] a complete characterization of the measures μ satisfying the right inequality of (4). A comprehensive treatment of convex functions can be found, for example, in [8].

In this paper, we look at the weighted version (4) of the classical inequality of Hermite and Hadamard from the perspective provided by the stochastic convex order. This approach is mainly due to Cal and Cárcamo. Thus, in the paper [3] the weight μ is regarded as a probability measure on $[a,b]$ and the inequalities are interpreted in terms of the convex ordering between random variables. Recently, also in [10–13, 15, 16] are studied the Hermite-Hadamard inequalities based on the convex ordering properties. Rajba [12] was the first who used the Ohlin's lemma [9] on convex stochastic ordering, to get a simple proof of some known Hermite-Hadamard type inequalities as well as to obtaining new Hermite-Hadamard type inequalities. In [12] are given some measures μ satisfying the inequality (4), as well as some generalizations of the inequality (3).

In this context, we will prove that the majorant and the minorant in inequalities of type (4) can be respectively obtained by a continuous deformation of given probability measure μ. We think this preoccupation is instructive, helping to better understand this inequality.

Let us recall some basic notions and results on the stochastic convex order (see, for example, [14]).

DEFINITION 1. Let ξ and η be two random variables. We say that

1. ξ is smaller than η in the *convex order* (denoted by $\xi \leqslant_{cx} \eta$) if

$$\mathbb{E}[f(\xi)] \leqslant \mathbb{E}[f(\eta)]$$

for all real convex functions f.

2. ξ is smaller than η in the increasing convex order (denoted by $\xi \leqslant_{icx} \eta$) if

$$\mathbb{E}[f(\xi)] \leqslant \mathbb{E}[f(\eta)]$$

for all increasing real convex functions f.

The following theorem (see Theorem 3.A.1 in [14]) establishes an useful criterion for the convex order and the increasing convex order.

THEOREM 5. *Let $\overline{F}$ and $\overline{G}$ be the survival functions of the random variables ξ and η, respectively, that is $\overline{F}(t) = \mathbb{P}\{\xi > t\}$ and $\overline{G}(t) = \mathbb{P}\{\eta > t\}$, for $t \in \mathbb{R}$. If $\mathbb{E}[\xi] = \mathbb{E}[\eta]$ then*

$$\xi \leqslant_{cx} \eta \;\Leftrightarrow\; \int_x^\infty \overline{F}(u)du \leqslant \int_x^\infty \overline{G}(u)du,\ \forall\, x \in \mathbb{R} \;\Leftrightarrow\; \xi \leqslant_{icx} \eta.$$

For a random variable ξ, with values in $[a,b]$, let us denote:

- $F(x) = \mathbb{P}\{\xi \leqslant x\}$, $x \in \mathbb{R}$, – *the distribution function* of ξ;
- $p : [a,b] \to [0,\infty)$ – *the density function* of ξ (if exists!);
- $\mathbb{E}[f(\xi)] = \int_{\mathbb{R}} f(x)\,dF(x) = \int_a^b f(x)p(x)\,dx$ – *the mean* (or *the expectation*) of the random variable $f(\xi)$ (where $f : [a,b] \to \mathbb{R}$ is integrable).

In [3] the inequality (1) is written as

$$f(\mathbb{E}[\xi]) \leqslant \mathbb{E}[f(\xi)] \leqslant \mathbb{E}[f(\xi^*)], \tag{5}$$

where ξ is a random variable with uniform distribution on $[a,b]$, ξ^* a random variable with uniform distribution on $\{a,b\}$ and $f : [a,b] \to \mathbb{R}$ is a convex function. In fact, the right inequality of (5) say

$$\xi \leqslant_{cx} \xi^*.$$

The same authors provide the following general multi-dimensional extension.

THEOREM 6. *Let $K \subset \mathbb{R}^n$ be a compact convex set. Denote K^* the set extreme points of K. For a given K-valued random vector ξ, there is a K^*-valued random vector ξ^* such that the multi-dimensional Hermite-Hadamard type inequality*

$$\mathbb{E}[f(\xi)] \leqslant \mathbb{E}[f(\xi^*)] \tag{6}$$

holds for all convex functions $f : K \to \mathbb{R}$.

In this paper, we will develop the treatment of the weighted Hermite-Hadamard inequalities in the frame of convex order.

2. Main results

In the following, we will consider a real interval $[a,b]$ and a fixed point $m \in (a,b)$. Let us define the discrete random variables ξ_0 and ξ_1 with values in $[a,b]$, having the distributions

$$\xi_0 : \begin{pmatrix} m \\ 1 \end{pmatrix} \quad \text{and} \quad \xi_1 : \begin{pmatrix} a & b \\ \frac{b-m}{b-a} & \frac{m-a}{b-a} \end{pmatrix}. \tag{7}$$

The corresponding survival functions of ξ_0 and ξ_1 are

$$\overline{F}_0(x) = \begin{cases} 1, & x < m \\ 0, & x \geqslant m \end{cases} \text{ and } \overline{F}_1(x) = \begin{cases} 1, & x < a \\ \frac{m-a}{b-a}, & a \leqslant x < b \\ 0, & x \geqslant b \end{cases}, \tag{8}$$

respectively. For an integrable function $f:[a,b]\to\mathbb{R}$, we have

$$\mathbb{E}\left[f\left(\xi_0\right)\right]=f(m) \text{ and } \mathbb{E}\left[f\left(\xi_1\right)\right]=\frac{b-m}{b-a}f\left(a\right)+\frac{m-a}{b-a}f\left(b\right). \tag{9}$$

Also, note that the two random variables have the same mean m and

$$m=a+\int_a^b \overline{F}_0(x)\,dx=a+\int_a^b \overline{F}_1(x)\,dx.$$

In fact, we have a specific formula for the mean of $[a,b]$-valued random variables.

LEMMA 1. *Let ξ be a random variable taking the values in the interval $[a,b]$. If $\overline{F}$ is the survival function of ξ, then*

$$\mathbb{E}[\xi]=a+\int_a^b \overline{F}(x)dx.$$

Proof. Let $F=1-\overline{F}$, $F(x)=\mathbb{P}\{\xi\leqslant x\}$, $x\in\mathbb{R}$, be the distribution function of ξ. Recall that F is right-continuous. Since $F(x)=0$ for $x<a$ and $F(x)=1$ for $x\geqslant b$, we obtain $\int_{-\infty}^a xdF(x)=a(F(a)-0)=aF(a)$ and $\int_b^\infty xdF(x)=0$. Hence

$$\mathbb{E}[\xi]=\int_{-\infty}^\infty xdF(x)=\int_{-\infty}^a xdF(x)+\int_a^b xdF(x)+\int_b^\infty xdF(x)=aF(a)-\int_a^b xd\overline{F}(x).$$

Integrating by parts, we find

$$\int_a^b xd\overline{F}(x)=b\overline{F}(b)-a\overline{F}(a)-\int_a^b \overline{F}(x)dx=-\left[a\overline{F}(a)+\int_a^b \overline{F}(x)dx\right].$$

Since $F(a)+\overline{F}(a)=1$, we obtain the conclusion. □

Now, we can adapt Theorem 5 to the case of random variables with values in a given interval.

LEMMA 2. *Let ξ and η two random variables taking values in $[a,b]$, with the survival functions $\overline{F}$ and $\overline{G}$, respectively. Assume $\int_a^b\overline{F}(x)dx=\int_a^b\overline{G}(x)dx$. Then $\xi\leqslant_{cx}\eta$ if and only if $\int_t^b\overline{F}(x)dx\leqslant\int_t^b\overline{G}(x)dx$, for all $t\in(a,b)$.*

Proof. According to our hypothesis and Lemma 1, we find $\mathbb{E}[\xi]=\mathbb{E}[\eta]$. On the other hand, we have

$$\int_t^\infty\overline{F}(x)dx=a-t+\int_a^b\overline{F}(x)dx=a-t+\int_a^b\overline{G}(x)dx=\int_t^\infty\overline{G}(x)dx, \text{ for } t<a;$$

$$\int_t^\infty\overline{F}(x)dx=\int_t^b\overline{F}(x)dx\leqslant\int_t^b\overline{G}(x)dx=\int_t^\infty\overline{G}(x)dx, \quad \text{for } a\leqslant t<b$$

$$\int_t^\infty\overline{F}(x)dx=0=\int_t^\infty\overline{G}(x)dx, \quad \text{for } t\geqslant b$$

Then, applying Theorem 5, we obtain the conclusion. □

A significant consequence of the above lemmas is the following probabilistic version of Theorem 4.

THEOREM 7. *For a random variable ξ taking values in $[a,b]$, with the mean $\mathbb{E}[\xi] = m$, we have*

$$\xi_0 \leqslant_{cx} \xi \leqslant_{cx} \xi_1.$$

In particular, if ξ has a density function $p : [a,b] \to [0,\infty)$, then

$$f(m) \leqslant \int_a^b f(x)\, p(x)\, dx \leqslant \frac{b-m}{b-a} f(a) + \frac{m-a}{b-a} f(b), \tag{10}$$

for any convex function $f : [a,b] \to \mathbb{R}$.

Proof. Based on Lemma 1 and Lemma 2, it suffices to show

$$\int_t^b \overline{F_0}(x)\, dx \leqslant \int_t^b \overline{F}(x)\, dx \leqslant \int_t^b \overline{F_1}(x)\, dx,\ \forall\, t \in [a,b].$$

For $t \in [m,b]$, we have $\int_t^b \overline{F}(x)\, dx \geqslant 0 = \int_t^b \overline{F}_0\, dx$. Assume that there is $t \in [a,m]$ such that $\int_t^b \overline{F}(x) < \int_t^b \overline{F_0}(x)\, dx = m - t$. In this case, $\int_a^b \overline{F}(x)\, dx < \int_a^t \overline{F}(x)\, dx + m - t \leqslant t - a + m - t = m - a$, in contradiction with the assumption $\mathbb{E}[\xi] = m$. So $\int_t^b \overline{F_0}(x)\, dx \leqslant \int_t^b \overline{F}(x)\, dx,\ \forall\, t \in [a,b]$. Assume now that there is $t \in (a,b)$ such that $\int_t^b \overline{F}(x)\, dx > \int_t^b \overline{F_1}(x)\, dx = \frac{(m-a)(b-t)}{b-a}$. Since the function $\overline{F}$ is nonincreasing, we have $\int_t^b \overline{F}(x)\, dx \leqslant (b-t)\overline{F}(t)$. Thus, we obtain $\overline{F}(t) > \frac{m-a}{b-a}$. Therefore, $\int_a^b \overline{F}(x)\, dx = \int_a^t \overline{F}(x)\, dx + \int_t^b \overline{F}(x)\, dx \geqslant (t-a)\overline{F}(t) + \int_t^b \overline{F}(x)\, dx > \frac{(m-a)(t-a)}{b-a} + \frac{(m-a)(b-t)}{b-a} = m - a$, in contradiction with the assumption $\mathbb{E}[\xi] = m$. So $\int_t^b \overline{F}(x)\, dx \leqslant \int_t^b \overline{F_1}(x)\, dx,\ \forall\, t \in [a,b]$. Therefore, $\xi_0 \leqslant_{cx} \xi \leqslant_{cx} \xi_1$. That is $\mathbb{E}[f(\xi_0)] \leqslant \mathbb{E}[f(\xi)] \leqslant \mathbb{E}[f(\xi_1)]$, for a convex function $f : [a,b] \to \mathbb{R}$. But, if ξ has the density p then $\mathbb{E}[f(\xi)] = \int_a^b f(x)\, p(x)\, dx$ and we obtain (10). This completes the proof. $\square$

REMARK 1. Using the Ohlin's lemma [9], Rajba [11] gave an alternative very simple proof of Theorem 7. The Ohlin's lemma is also used to study inequalities of the Hermite-Hadamard type in the papers [10,13,15,16]. In the papers [10,15,16], furthermore, to examine the Hermite-Hadamard type inequalities is used the Levin-Stečkin's theorem [7] (see also [8]), as well as Lemma 2 [10], which is some modification of the Levin-Stečkin's theorem. The Levin-Stečkin's theorem [7] as well as Lemma 2 in [10] give necessary and sufficient conditions for the stochastic convex ordering. Moreover, for random variables with values in some closed interval, from the Levin-Stečkin's theorem [7] it follows Theorem 5.

Let us consider now a (fixed) continuous random variable ξ taking values in $[a,b]$. Assume that ξ has a density function $p : [a,b] \to [0,\infty)$ (i.e. p is integrable on $[a,b]$, with $\int_a^b p(x) dx = 1$) and the mean $\mathbb{E}[\xi] = \int_a^b x p(x) dx = m$. Our goal is to show that the interval between the Hermite-Hadamard bounds

$$\Bigg[\overbrace{\mathbb{E}[f(\xi_0)]}^{\text{H-H minorant}} \underbrace{\quad\cdots\quad}_{\text{left spacing}} \mathbb{E}[f(\xi)] \underbrace{\quad\cdots\quad}_{\text{right spacing}} \overbrace{\mathbb{E}[f(\xi_1)]}^{\text{H-H majorant}}\Bigg].$$

can be "filled" by considering an associated family $(\xi_\lambda)_{\lambda\in[0,1]}$ of random variables which is totally ordered with respect to the stochastic convex order.

Let $F(x)=\mathbb{P}\{\xi\leqslant x\}$, $x\in\mathbb{R}$ be the distribution function of ξ. For $x\in[a,b]$, we have $F(x)=\int_a^x p(t)dt$. Since F is a continuous (nondecreasing) function, with $F(a)=0$ and $F(b)=1$, there is $c\in(a,b)$ such that

$$F(c)=\int_a^c p(t)dt=\frac{b-m}{b-a}\in(0,1). \tag{11}$$

DEFINITION 2. Let $p:[a,b]\to[0,\infty)$ be a density function with the mean $m\in(a,b)$ and $c\in(a,b)$ such that the relation (11) holds. The parametric family of functions $(p_\lambda)_{\lambda\in(0,1)}$, where $p_\lambda:[a,b]\to[0,\infty)$, induced by the density p is defined as follows:

$$p_\lambda(x)=\begin{cases}\frac{1}{2(1-\lambda)}p\left(\frac{x-a(2\lambda-1)}{2(1-\lambda)}\right), & x\in[a,u_\lambda]\\ 0, & x\in(u_\lambda,v_\lambda)\\ \frac{1}{2(1-\lambda)}p\left(\frac{x-b(2\lambda-1)}{2(1-\lambda)}\right), & x\in[v_\lambda,b]\end{cases},\ \text{for } \lambda\in[1/2,1), \tag{12}$$

where

$$\begin{cases}u_\lambda:=(2\lambda-1)a+2(1-\lambda)c\\ v_\lambda:=(2\lambda-1)b+2(1-\lambda)c\end{cases},$$

and

$$p_\lambda(x)=\begin{cases}\frac{1}{2\lambda}p\left(\frac{x+m(2\lambda-1)}{2\lambda}\right), & x\in[s_\lambda,t_\lambda]\\ 0, & x\in[a,s_\lambda)\cup(t_\lambda,b]\end{cases},\ \text{for } \lambda\in(0,1/2), \tag{13}$$

where

$$\begin{cases}s_\lambda:=2\lambda a+(1-2\lambda)m\\ t_\lambda:=2\lambda b+(1-2\lambda)m\end{cases}.$$

Observe that $p_{1/2}=p$. Moreover, the definition (13) can be even applied for $\lambda=1/2$. The significance of the family of functions $(p_\lambda)_{\lambda\in(0,1)}$ introduced in Definition 2 is highlighted by the following lemma.

LEMMA 3. *$p_\lambda:[a,b]\to[0,\infty)$ is a density function on $[a,b]$ with the mean m, for all $\lambda\in(0,1)$.*

Proof. Clearly, p_λ is integrable on $[a,b]$ for $\lambda\in(0,1)$.
Assume $\lambda\in(1/2,1)$. We have

$$\int_a^{u_\lambda}p_\lambda(x)dx=\frac{1}{2(1-\lambda)}\int_a^{(2\lambda-1)a+2(1-\lambda)c}p\left(\frac{x-a(2\lambda-1)}{2(1-\lambda)}\right)dx=\int_a^c p(t)dt$$

and

$$\int_{v_\lambda}^b p_\lambda(x)dx=\frac{1}{2(1-\lambda)}\int_{(2\lambda-1)b+2(1-\lambda)c}^b p\left(\frac{x-b(2\lambda-1)}{2(1-\lambda)}\right)dx=\int_c^b p(t)dt.$$

Then

$$\int_a^b p_\lambda(x)dx = \int_a^{u_\lambda} p_\lambda(x)dx + \int_{v_\lambda}^b p_\lambda(x)dx = \int_a^c p(t)dt + \int_c^b p(t)dt = \int_a^b p(t)dt = 1.$$

In addition, from (11), we obtain

$$\int_a^{u_\lambda} p_\lambda(x)dx = \frac{b-m}{b-a} \text{ and } \int_{v_\lambda}^b p_\lambda(x)dx = \frac{m-a}{b-a}. \tag{14}$$

The mean of p_λ is given by

$$\begin{aligned}\int_a^b xp_\lambda(x)dx &= \frac{1}{2(1-\lambda)}\int_a^{u_\lambda} xp\left(\frac{x-a(2\lambda-1)}{2(1-\lambda)}\right)dx + \frac{1}{2(1-\lambda)}\int_{v_\lambda}^b xp\left(\frac{x-b(2\lambda-1)}{2(1-\lambda)}\right)dx \\ &= \int_a^c p(t)[2(1-\lambda)t + a(2\lambda-1)]dt + \int_c^b p(t)[2(1-\lambda)t + b(2\lambda-1)]dt \\ &= 2(1-\lambda)\int_a^b tp(t)dt + (2\lambda-1)\left(a\int_a^c p(t)dt + b\int_c^b p(t)dt\right) \\ &= 2(1-\lambda)m + (2\lambda-1)\left[\frac{a(b-m)}{b-a} + \frac{b(m-a)}{b-a}\right] = m.\end{aligned}$$

For $\lambda \in (0,1/2)$, we find

$$\int_a^b p_\lambda(x)dx = \frac{1}{2\lambda}\int_{2\lambda a+(1-2\lambda)m}^{2\lambda b+(1-2\lambda)m} p\left(\frac{x+m(2\lambda-1)}{2\lambda}\right)dx = \int_a^b p(t)dt = 1$$

and

$$\int_a^b xp_\lambda(x)dx = \int_a^b p(t)[2\lambda t + m(1-2\lambda)]dt = 2\lambda\int_a^b tp(t)dt + m(1-2\lambda)\int_a^b p(t)dt = m.$$

Therefore the $p_\lambda : [a,b] \to [0,\infty)$ is a density function on $[a,b]$ with the mean m, for all $\lambda \in (0,1)$. □

Based on the above construction of the family of density functions $(p_\lambda)_{\lambda\in(0,1)}$ (Definition 2), we will formulate the main result of this paper.

THEOREM 8. *Let ξ_λ be a continuous random variable with the density function p_λ, for $\lambda \in (0,1)$. Let ξ_0 and ξ_1 be two random variable with the distribution defined by (7). The family $(\xi_\lambda)_{\lambda\in[0,1]}$ of $[a,b]$-valued random variables has the following properties:*

1. $\xi_\lambda \overset{(d)}{\to} \xi_1$, *for* $\lambda \uparrow 1$;
2. $\xi_\lambda \overset{(d)}{\to} \xi_0$, *for* $\lambda \downarrow 0$;
3. $\xi_\lambda \overset{(d)}{\to} \xi_{\lambda_0}$, *for* $\lambda \to \lambda_0$;

4. if $\lambda,\mu \in [0,1]$*, such that* $\lambda < \mu$*, then* $\xi_\lambda \leqslant_{cx} \xi_\mu$*;*

5. for every convex function $f : [a,b] \to \mathbb{R}$*,*

$$I_f : [0,1] \to \mathbb{R},\ I_f(\lambda) := \mathbb{E}\left[f(\xi_\lambda)\right] = \int_a^b f(x) p_\lambda(x) dx,$$

is a continuous nondecreasing function on $[0,1]$*, with the image*

$$I_f([0,1]) = \left[f(m), \frac{(b-m)f(a) + (m-a)f(b)}{b-a} \right].$$

Here, " $\overset{(d)}{\to}$ *" denotes the convergence in distribution.*

Proof. For $\lambda \in [0,1]$, let us denote by F_λ and $\overline{F}_\lambda$ the distribution function and the survival function of ξ_λ, respectively.

1. We have to show $\lim\limits_{\lambda \to 1-} F_\lambda(x) = F_1(x)$ at the continuity points $x \in [a,b]$ of F_1. Assume $x \in (a,b)$. Since $\lim\limits_{\lambda \to 1-} u_\lambda = a$ and $\lim\limits_{\lambda \to 1-} v_\lambda = b$ there is $\lambda_x \in (1/2,1)$ such that $u_\lambda < x < v_\lambda$, $\forall \lambda \in (\lambda_x, 1)$. Then, from (14), $F_\lambda(x) = \int_a^{u_\lambda} p_\lambda(t)dt = \frac{b-m}{b-a} = F_1(x)$, $\forall \lambda \in (\lambda_x,1)$. So, $\xi_\lambda \overset{(d)}{\to} \xi_1$, for $\lambda \uparrow 1$.

2. For $x \in [a,m)$, there is $\lambda_x \in (0,1/2)$ such that $s_\lambda > x$, $\forall\, \lambda \in (0,\lambda_x)$. Then $F_\lambda(x) = 0 = F_0(x)$, $\forall\, \lambda \in (0,\lambda_x)$. Similarly, for $x \in (m,b]$, there is $\lambda_x \in (0,1/2)$ such that $t_\lambda < x$, $\forall\, \lambda \in (0,\lambda_x)$. Thus, $F_\lambda(x) = \int_{s_\lambda}^{t_\lambda} p_\lambda(t)dt = 1 = F_0(x)$, $\forall\, \lambda \in (0,\lambda_x)$. As follows, $\xi_\lambda \overset{(d)}{\to} \xi_0$, for $\lambda \downarrow 0$.

3. It is sufficient to note that the function $F_\lambda(x) = \int_a^x p_\lambda(t)dt$ is continuous in $\lambda \in (0,1)$, for all $x \in [a,b]$.

4. Theorem 7 ensures $\xi_0 \leqslant_{cx} \xi_\lambda \leqslant_{cx} \xi_1$, for all $\lambda \in (0,1)$. Then, we have only to establish the convex ordering between ξ_λ and ξ_μ for $0 < \lambda < \mu \leqslant 1/2$ and for $1/2 \leqslant \lambda < \mu < 1$. Let $\varphi : [a,b] \to \mathbb{R}$ be the function defined by

$$\varphi(x) = \int_x^b \overline{F}_\mu(t)dt - \int_x^b \overline{F}_\lambda(t)dt,\ x \in [a,b].$$

We have $\varphi'(x) = \overline{F}_\lambda(x) - \overline{F}_\mu(x) = F_\mu(x) - F_\lambda(x)$, for $x \in [a,b]$.

In the first case, $0 < \lambda < \mu \leqslant 1/2$, we obtain

$$F_\lambda(x) = \int_a^x p_\lambda(t)dt = \begin{cases} 0, & x \in [a, s_\lambda] \\ \int_a^{m+\frac{x-m}{2\lambda}} p(z)dz, & x \in (s_\lambda, t_\lambda) \\ 1, & x \in [t_\lambda, b] \end{cases}.$$

Thus, after some calculations, we find

$$\varphi'(x) = \begin{cases} 0, & x \in \left[a, s_\mu\right] \\ \int_a^{m+\frac{x-m}{2\mu}} p(z)dz, & x \in \left(s_\mu, s_\lambda\right) \\ \int_{m+\frac{x-m}{2\lambda}}^{m+\frac{x-m}{2\mu}} p(z)dz, & x \in [s_\lambda, m] \\ -\int_{m+\frac{x-m}{2\mu}}^{m+\frac{x-m}{2\lambda}} p(z)dz, & x \in (m, t_\lambda) \\ -\int_{m+\frac{x-m}{2\mu}}^{b} p(z)dz, & x \in \left[t_\lambda, t_\mu\right] \\ 0, & x \in \left(t_\mu, b\right] \end{cases}.$$

Therefore, according to the sign of the derivative, φ is nondecreasing on the interval $[0,m]$ and nonincreasing on the interval $[m,b]$. Since $\varphi(a) = (m-a)-(m-a) = 0$ and $\varphi(b) = 0-0 = 0$, we conclude that $\varphi(x) \geqslant 0,\ \forall\, x \in [a,b]$.

In the second case, $1/2 \leqslant \lambda < \mu < 1$, we have

$$F_\lambda(x) = \int_a^x p_\lambda(t)dt = \begin{cases} \int_a^{a+\frac{x-a}{2(1-\lambda)}} p(z)dz, & x \in [a, u_\lambda] \\ \frac{b-m}{b-a}, & x \in (u_\lambda, v_\lambda) \\ 1 - \int_{b-\frac{b-x}{2(1-\lambda)}}^{b} p(z)dz, & x \in [v_\lambda, b] \end{cases}$$

and

$$\varphi'(x) = \begin{cases} \int_{a+\frac{x-a}{2(1-\lambda)}}^{a+\frac{x-a}{2(1-\mu)}} p(z)dz, & x \in \left[a, u_\mu\right] \\ \int_{a+\frac{x-a}{2(1-\lambda)}}^{c} p(z)dz, & x \in \left(u_\mu, u_\lambda\right) \\ 0, & x \in [u_\lambda, v_\lambda] \\ -\int_c^{b-\frac{b-x}{2(1-\lambda)}} p(z)dz, & x \in \left(v_\lambda, v_\mu\right) \\ -\int_{b-\frac{b-x}{2(1-\mu)}}^{b-\frac{b-x}{2(1-\lambda)}} p(z)dz, & x \in \left[v_\mu, b\right] \end{cases}.$$

Hence, φ is nondecreasing on the interval $[0, v_\lambda]$ and nonincreasing on the interval $[v_\lambda, b]$. From $\varphi(a) = \varphi(b) = 0$, we conclude that $\varphi(x) \geqslant 0,\ \forall\, x \in [a,b]$.

As follows, for $0 < \lambda < \mu < 1$, we have $\int_x^b \overline{F}_\lambda(t)dt \leqslant \int_x^b \overline{F}_\mu(t)dt,\ \forall\, x \in [a,b]$, with $\int_a^b \overline{F}_\lambda(t)dt = \int_a^b \overline{F}_\mu(t)dt$. Thus, from Lemma 2, we obtain $\xi_\lambda \leqslant_{cx} \xi_\mu$, for $0 < \lambda < \mu < 1$.

5. Let $f : [a,b] \to \mathbb{R}$ be a convex function. From the definition of the convex order and the previous result 4, the function I_f is nondecreasing on $[0,1]$. We have

$$I_f(\lambda) = \int_{s_\lambda}^{t_\lambda} f(x)p_\lambda(x)dx = \int_a^b f(m+2\lambda(t-m))p(t)dt, \text{ for } \lambda \in (0,1/2) \qquad (15)$$

and

$$\begin{aligned} I_f(\lambda) &= \int_a^{u_\lambda} f(x)p_\lambda(x)dx + \int_{v_\lambda}^b f(x)p_\lambda(x)dx \\ &= \int_a^c f(a+2(1-\lambda)(t-a))p(t)dt \\ &\quad + \int_c^b f(b+2(1-\lambda)(t-b))p(t)dt, \text{ for } \lambda \in [1/2,1). \end{aligned} \tag{16}$$

The function f is then bounded on $[a,b]$ and continuous on (a,b). Clearly, we have $I_f(\lambda) = I_{\overline{f}}(\lambda)$ where $\overline{f}$ is the continuous function on $[a,b]$ defined by $\overline{f}(x) = f(x)$, for $x \in (a,b)$, $\overline{f}(a) = \lim_{x\to a+} f(x)$ and $\overline{f}(b) = \lim_{x\to b-} f(x)$. Then, based on the properties of the integrals involving parameters, the function I_f is continuous on $(0,1)$. Also, we can directly verify $\lim_{\lambda\to 0+} I_f(\lambda) = I_f(0)$ and $\lim_{\lambda\to 1-} I_f(\lambda) = I_f(1)$. Hence I_f is continuous on $[0,1]$, with the image $\left[f(m), \frac{(b-m)f(a)+(m-a)f(b)}{b-a}\right]$. □

REMARK 2. We have $I_f(1/2) = \int_a^b f(x)p(x)dx$. $I_f(0) = f(m)$ is the *Hermite-Hadamard minorant* and $I_f(1) = \frac{b-m}{b-a} f(a) + \frac{m-a}{b-a} f(b)$ is the *Hermite-Hadamard majorant*. Note that the continuity of I_f at 0 and 1 is explained by the proved convergences $\xi_\lambda \stackrel{(d)}{\to} \xi_0$, for $\lambda \downarrow 0$, and $\xi_\lambda \stackrel{(d)}{\to} \xi_1$, for $\lambda \uparrow 1$.

Further, we intend to complete our probabilistic approach by indicating a general method to construct a random variable with the values in the interval $[a,b]$ and the expectation $m \in (a,b)$. The construction starts from an arbitrary $[0,1]$-valued random variable. Moreover, starting from a family of $[0,1]$-valued random variables, which is totally ordered with respect to the increasing convex order, we will get a family of $[a,b]$-valued random variables which is totally ordered with respect to the convex order.

DEFINITION 3. Assume a real interval $[a,b]$ and $m \in (a,b)$. Let ξ be a random variable taking the values in the interval $[0,1]$. Suppose that the distribution function F of ξ has at most a finite number of points of discontinuity. We define the random variables $\xi_- = m + (a-m)\xi$, with the distribution function F_-, and $\xi_+ = m + (b-m)\xi$, with the distribution function F_+. A random variable $\widetilde{\xi}$ having the distribution function $F_* = \frac{b-m}{b-a} F_{\xi_-} + \frac{m-a}{b-a} F_{\xi_+}$, will be called an $([a,b];m)$-mixture of ξ.

We easily observe that $\widetilde{\xi}$ takes the values in the interval $[a,b]$. In particular, $\widetilde{0} = \xi_0$ and $\widetilde{1} = \xi_1$ (see Definition 7). The following theorem proves some properties of the $([a,b];m)$-mixtures.

THEOREM 9. *Let ξ and η be two random variables satisfying Definition 3, such that $\xi \leqslant_{icx} \eta$. Let us consider their corresponding $([a,b];m)$-mixtures $\widetilde{\xi}$ and $\widetilde{\eta}$, respectively. Then $\mathbb{E}\left[\widetilde{\xi}\right] = m$ and $\widetilde{\xi} \leqslant_{cx} \widetilde{\eta}$.*

Proof. Let F and G be the distribution functions of ξ and η, respectively. From Lemma 1 and considering Definition 3 we obtain

$$\begin{aligned}\mathbb{E}\left[\widetilde{\xi}\right] &= a+\int_a^b \overline{F}_*(x)dx = \frac{b-m}{b-a}\left[a+\int_a^b \overline{F}_-(x)dx\right]+\frac{m-a}{b-a}\left[a+\int_a^b \overline{F}_+(x)dx\right]\\ &= \frac{b-m}{b-a}\mathbb{E}[\xi_-]+\frac{m-a}{b-a}\mathbb{E}[\xi_+]\\ &= \frac{b-m}{b-a}\{m+(a-m)\mathbb{E}[\xi]\}+\frac{m-a}{b-a}\{m+(b-m)\mathbb{E}[\xi]\}=m.\end{aligned}$$

We also have $\mathbb{E}[\widetilde{\eta}]=m$. As follows,

$$\int_a^b \overline{F}_*(t)dt=\int_a^b \overline{G}_*(t)dt. \tag{17}$$

The inequality $\xi \leqslant_{icx} \eta$ implies

$$\int_x^1 \overline{F}(t)dt \leqslant \int_x^1 \overline{G}(t)dt,\ \forall\, x\in[0,1]. \tag{18}$$

From Definition 3 F, we obtain

$$F_-(x)=\mathbb{P}\{\xi_- \leqslant x\}=\mathbb{P}\left\{\xi \geqslant \frac{x-m}{a-m}\right\} \text{ and } F_+(x)=\mathbb{P}\left\{\xi \leqslant \frac{x-m}{b-m}\right\},\ \text{for } x\in[a,b].$$

Then

$$\begin{aligned}\overline{F}_*(x) &= 1-\left[\frac{b-m}{b-a}\mathbb{P}\left\{\xi \geqslant \frac{x-m}{a-m}\right\}+\frac{m-a}{b-a}\mathbb{P}\left\{\xi \leqslant \frac{x-m}{b-m}\right\}\right]\\ &= \begin{cases} 1-\frac{b-m}{b-a}\mathbb{P}\left\{\xi \geqslant \frac{x-m}{a-m}\right\}, & \text{for } x\in[a,m)\\ \frac{m-a}{b-a}\overline{F}\left(\frac{x-m}{b-m}\right), & \text{for } x\in(m,b]\end{cases}\end{aligned}$$

Since F has at most a finite number of discontinuity points, we obtain after some calculations

$$\int_x^b \overline{F}_*(t)dt=\frac{(b-m)(m-a)}{b-a}\int_{\frac{x-m}{b-m}}^1 \overline{F}(z)dz,\ \text{for } x\in[m,b],$$

and

$$\int_x^b \overline{F}_*(t)dt=\frac{(b-m)(m-a)}{b-a}\int_{\frac{x-m}{a-m}}^1 \overline{F}(z)dz+m-x,\ \text{for } x\in[a,m).$$

But we have similar results for the expression of $\int_x^b \overline{G}_*(t)dt$. Therefore, using the relation (18), we find

$$\int_x^b \overline{F}_*(t)dt \leqslant \int_x^b \overline{G}_*(t)dt,\ \forall\, x\in[a,b]. \tag{19}$$

From (17), (19) and Lemma 2 we deduce $\widetilde{\xi} \leqslant_{cx} \widetilde{\eta}$. □

Finally, we apply our last construction to obtain new results of Hermite-Hadamard type.

DEFINITION 4. A stochastic process $(X_\lambda)_{\lambda\in[0,1]}$ with values in $[0,1]$ is said to be continuous increasing convex ordered if

1. $X_0 = 0$ and $X_\lambda \overset{(d)}{\to} X_0$, for $\lambda \downarrow 0$;
2. $X_1 = 1$ and $X_\lambda \overset{(d)}{\to} X_1$, for $\lambda \uparrow 1$;
3. $X_\lambda \overset{(d)}{\to} X_{\lambda_0}$, for $\lambda \to \lambda_0 \in (0,1)$;
4. $0 \leqslant \lambda_1 < \lambda_2 \leqslant 1 \ \Rightarrow\ X_{\lambda_1} \leqslant_{icx} X_{\lambda_2}$.

EXAMPLE 1. Let X_λ be a Beta random variable with the distribution function

$$F_\lambda(x) = \frac{\sin(\lambda\pi)}{\pi}\int_0^x t^{\lambda-1}(1-t)^{-\lambda}dt,\ x\in[0,1],$$

for all $\lambda \in (0,1)$. Assume $X_0 = 0$ and $X_1 = 1$. Then $(X_\lambda)_{\lambda\in[0,1]}$ is a continuous increasing convex ordered stochastic process.

We obtain an alternative of Theorem 8.

THEOREM 10. *Let $(X_\lambda)_{\lambda\in[0,1]}$ be a continuous increasing convex ordered stochastic process. For $\lambda\in[0,1]$, denote $\widetilde{X}_\lambda$ the $([a,b];m)$-mixture of X_λ. Then*

$$0 \leqslant \lambda_1 < \lambda_2 \leqslant 1 \ \Rightarrow\ \mathbb{E}\left[f\left(\widetilde{X}_{\lambda_1}\right)\right] \leqslant \mathbb{E}\left[f\left(\widetilde{X}_{\lambda_2}\right)\right],$$

for all convex functions $f:[a,b]\to\mathbb{R}$, i.e. $(\widetilde{X}_\lambda)_{\lambda\in[0,1]}$ is a totally ordered convex family of random variables taking values in $[a,b]$, with the common mean m.

The proof is based on Theorem 9. The details are ommited.

Acknowledgements. We express our many thanks to the editor and to the referee for careful reading the manuscript and for valuable comments and suggestions, helping to improve the presentation of this paper.

This work was partially supported by the Grant number 3C/2014, awarded in the internal Grant competition of the University of Craiova.

REFERENCES

[1] M. K. BAKULA, J. PEČARIĆ AND J. PERIĆ, *Extensions of the Hermite-Hadamard inequality with applications*, Math. Inequal. Appl. **4** (2012), 899–921.

[2] J. L. BRENNER, H. ALZER, *Integral inequalities for concave functions with applications to special functions*, Proc. Roy. Soc. Edinburgh Sect. A **118** (1991), 173–192.

[3] J. DE LA CAL AND J. CÁRCAMO, *Multidimensional Hermite-Hadamard inequalities and the convex order*, J. Math. Anal. Appl. **324** (2006), 248–261.

[4] L. FEJÉR, *Über die Fourierreihen, II*, Naturwiss. Anz. Ungar. Akad. Wiss. **24** (1906), 369–390.

[5] A. M. FINK, *A best possible Hadamard Inequality*, Math. Inequal. Appl. **1** (1998), 223–230.

[6] A. FLOREA AND C. P. NICULESCU, *A Hermite-Hadamard inequality for convex-concave symmetric functions*, Bull. Soc. Sci. Math. Roum. **50** (2007), 149–156.

[7] V. I. LEVIN AND S. B. STEČKIN, *Inequalities*, Amer. Math. Soc. Transl. **14** (1960), 1–29.

[8] C. P. NICULESCU AND L.-E. PERSSON, *Convex Functions and their Applications: A Contemporary Approach* (CMS Books in Mathematics), Springer-Verlag New York Inc., New York, 2006.

[9] J. OHLIN, *On a class of measures of dispersion with application to optimal reinsurance*, ASTIN Bulletin **5** (1969), 249–266.

[10] A. OLBRYŚ AND T. SZOSTOK, *Inequalities of the Hermite-Hadamard type involving numerical differentiation formulas*, arXiv preprint, arXiv: 1411.7859v1 [math.CA] (2014).

[11] T. RAJBA, *On probabilistic characterizations of convexity and delta-convexity*, Conference on Inequalities and Applications **14**, September 7–13, 2014, Hajdúszoboszló (Hungary).

[12] T. RAJBA, *On the Ohlin lemma for Hermite-Hadamard-Fejér type inequalities*, Math. Inequal. Appl. **17** (2014), 557–571.

[13] T. RAJBA, *On strong delta-convexity and Hermite-Hadamard type inequalities for delta-convex functions of higher order*, Math. Inequal. Appl. **18** (2015), 267–293.

[14] M. SHAKED AND J. G. SHANTHIKUMAR, *Stochastic Orders*, Springer Series in Statistics, New York, 2007.

[15] T. SZOSTOK, *Levin Stečkin theorem and inequalities of the Hermite-Hadamard type*, arXiv preprint, *arXiv:1411.7708v1 [math.CA]* (2014).

[16] T. SZOSTOK, *Ohlin's lemma and some inequalities of the Hermite-Hadamard type*, Aequat. Math. (2014), DOI 10.1007/s00010-014-0286-2.

(Received October 26, 2014)

Aurelia Florea
University of Craiova
Romania
e-mail: aurelia_florea@yahoo.com

Eugen Păltănea
Transilvania University of Braşov
Romania
e-mail: epaltanea@unitbv.ro

Dumitru Bălă
University of Craiova
Romania
e-mail: dumitru_bala@yahoo.com