Journal of
Mathematical
Inequalities
Volume 9, Number 4 (2015), 1259–1269 doi:10.7153/jmi-09-96

NEW RESULTS ABOUT HARDY–TYPE INEQUALITY

KRISTINA KRULIĆ HIMMELREICH, JOSIP PEČARIĆ,
DORA POKAZ AND MARJAN PRALJAK

(*Communicated by S. Varošanec*)

Abstract. We give a Levinson type generalization of Hardy's inequality with convex functions replaced by 3-convex functions at a point. Several results and examples are provided, both one-dimensional and multidimensional.

1. Introduction

The well known Hardy's inequality (see [6, 7, 8]) states

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \quad p>1, \tag{1}$$

where f is a non-negative function such that $f \in L^p(\mathbb{R}_+)$. If $\mathbb{R}_+$ is replaced by a finite interval $(0,b)$, then the following inequality holds

$$\int_0^b \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{(p-1)/p}\right] f^p(x)\,dx, \quad p>1,$$

for a non-negative $f \in L^p(0,b)$ (see [3, 4]). Rewriting (1) with the function f replaced with $f^{1/p}$ and then letting $p \to \infty$ we obtain the limiting case of Hardy's inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) dx < e\int_0^\infty f(x)\,dx, \tag{2}$$

which holds for all positive functions $f \in L^1(\mathbb{R}_+)$. This inequality is known as Pólya–Knopp's inequality (see [9]). Again, if we work on a finite interval $(0,b)$ the following inequality holds

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) dx < e\int_0^b \left(1-\frac{x}{b}\right) f(x)\,dx$$

Mathematics subject classification (2010): Primary 26D10; Secondary 26D15.
Keywords and phrases: Hardy's inequality, convex function, 3-convex function.
Acknowledgement. This work has been fully supported by the Croatian Science Foundation under the project 5435.

for functions $f \in L^1(0,b)$ (see [4]). If $p>1$ and f is a non-negative function such that $f \in L^p(\mathbb{R}_+)$, then

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy \leqslant \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y)\, dy,$$

and if, in addition, $g \in L^q(\mathbb{R}_+)$ where $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leqslant \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_0^\infty f^p(x)\, dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)\, dy \right)^{\frac{1}{q}}. \tag{3}$$

Inequality (3) is called Hilbert's or Hardy-Hilbert's inequality. In monograph [11] one can find generalizations, refinements, and variants of the famous Hardy's inequality.

On the other hand, Godunova [5] proved that the inequality

$$\int_{\mathbb{R}_+^n} \Phi\left(\frac{1}{x_1 \cdots x_n} \int_{\mathbb{R}_+^n} l\left(\frac{y_1}{x_1}, \dots, \frac{y_n}{x_n} \right) f(y_1, \dots, y_n) d\mathbf{y} \right) \frac{d\mathbf{x}}{x_1 \cdots x_n} \leqslant \int_{\mathbb{R}_+^n} \frac{\Phi(f(\mathbf{x}))}{x_1 \cdots x_n} d\mathbf{x}, \tag{4}$$

holds for a non-negative function $l : \mathbb{R}_+^n \to \mathbb{R}_+$, such that

$$\int_{\mathbb{R}_+^n} l(\mathbf{x}) d\mathbf{x} = 1,$$

a convex function $\Phi : [0,\infty) \to [0,\infty)$, and a non-negative function f on $\mathbb{R}_+^n$ such that the function $x \mapsto \Phi(f(\mathbf{x}))/(x_1 \cdots x_n)$ is integrable on $\mathbb{R}_+^n$.

By using the result given in (4) Godunova obtained many general inequalities which include Hardy's (1), Pólya–Knopp's (2) and Hardy–Hilbert's inequality (3). For more details see [12].

We also note that Hardy's inequality (1) shows that the Hardy operator H, defined by

$$Hf(x) = \frac{1}{x} \int_0^x f(t)\, dt,$$

maps L^p into itself with operator norm $p/(p-1)$. The operator can be generalized by adding a kernel

$$A_k f(x) = \frac{1}{K(x)} \int_0^\infty f(t) k(x,t)\, dt, \tag{5}$$

where

$$K(x) = \int_0^\infty k(x,t)\, dt < \infty.$$

Here $k(x,y)$ is a general measurable and non-negative function.

Further generalization include measure spaces $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$, so A_k from (5) can be generalized as follows:

$$A_kf(x) := \frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y), \tag{6}$$

where $f:\Omega_2\to\mathbb{R}$ is a measurable function, $k:\Omega_1\times\Omega_2\to\mathbb{R}$ is a measurable and non-negative kernel and

$$K(x) := \int_{\Omega_2} k(x,y)\,d\mu_2(y) < \infty,\ x\in\Omega_1. \tag{7}$$

In this setting K. Krulić et al. [10] proved the following Hardy-type inequality.

THEOREM 1. *Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1\times\Omega_2$, and K be defined on Ω_1 by (7).*

Suppose that $K(x)>0$ for all $x\in\Omega_1$, that the function $x\mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y\in\Omega_2$, and that v is defined on Ω_2 by

$$v(y) := \int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}\,d\mu_1(x) < \infty.$$

If Φ is a convex function on an interval $I\subseteq\mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x)\Phi(A_kf(x))\,d\mu_1(x) \leqslant \int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y) \tag{8}$$

holds for all measurable functions $f:\Omega_2\to I$, where A_k is defined by (6).

Notice that in the case of a concave function Φ, the sign of inequality (8) would be reversed.

In this paper we will make a further generalization of inequality (8) and obtain a Hardy inequality of Levinson type. Instead of convex functions we will work with the class of 3-convex functions at a point, following the idea of I. A. Baloch et al. [2] and J. Pečarić et al. [13]. We will point out the dual class of functions and the corresponding dual inequality. Particular applications of the main result will give us several examples of one-dimensional and multidimensional inequalities of Levinson type.

CONVENTIONS. An interval I in $\mathbb{R}$ is any convex subset of $\mathbb{R}$, while $\mathrm{Int}I$ denotes its interior. By $\mathbb{R}_+$ we denote the set of all positive real numbers, i.e. $\mathbb{R}_+=(0,\infty)$. A kth order divided difference of a function $f:I\to\mathbb{R}$, where I is an interval in $\mathbb{R}$, at distinct points $x_0,\ldots,x_k\in I$ is defined recursively by

$$[x_i]f = f(x_i), \qquad \text{for} \quad i=0,\ldots,k$$

and

$$[x_0,\ldots,x_k]f = \frac{[x_1,\ldots,x_k]f-[x_0,\ldots,x_{k-1}]f}{x_k-x_0}.$$

A function $f:I\to\mathbb{R}$ is called k-convex if $[x_0,\ldots,x_k]f\geqslant 0$ for all choices of $k+1$ distinct points $x_0,\ldots,x_k\in I$. If the kth derivative $f^{(k)}$ of a k-convex function exists, then $f^{(k)} \geqslant 0$, but $f^{(k)}$ may not exist (for properties of divided differences and k-convex functions see [12]). For $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^n$, $\mathbf{u} = (u_1, u_2, \dots, u_n)$, $\mathbf{v} = (v_1, v_2, \dots, v_n)$, let

$$\frac{\mathbf{u}}{\mathbf{v}} = \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \dots, \frac{u_n}{v_n}\right) \text{ and } \mathbf{u}^{\mathbf{v}} = u_1^{v_1} u_2^{v_2} \cdots u_n^{v_n}.$$

In particular, $\mathbf{u}^{\mathbf{1}} = \prod_{i=1}^n u_i$, $\mathbf{u}^{\mathbf{2}} = (\prod_{i=1}^n u_i)^2$, and $\mathbf{u}^{-\mathbf{1}} = (\prod_{i=1}^n u_i)^{-1}$, where $\mathbf{n} = (n, n, \dots, n)$. We write $\mathbf{u} < \mathbf{v}$ if componentwise $u_i < v_i$, $i = 1, \dots, n$. Relations $\leqslant$, $>$, and $\geqslant$ are defined analogously. Finally, we denote $(\mathbf{0}, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{0} < \mathbf{x} < \mathbf{b}\}$ and $(\mathbf{b}, \infty) = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{b} < \mathbf{x} < \infty\}$.

2. Main results with applications

In the statements of our results we will replace convex functions with the following class of functions (see [2], [13]).

DEFINITION 1. Let $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a function and $c \in \operatorname{Int} I$. We say that $f \in \mathscr{K}_1^c(I)$ (resp. $f \in \mathscr{K}_2^c(a,b)$) if there exists a constant α such that the function $F(x) = f(x) - \frac{\alpha}{2}x^2$ is concave (resp. convex) on $I \cap (-\infty, c]$ and convex (resp. concave) on $I \cap [c, \infty)$.

The next two remarks give some properties of functions in $\mathscr{K}_i^c(I)$, $i = 1, 2$.

REMARK 1. If $f \in \mathscr{K}_i^c(I)$, $i = 1, 2$, and $f''(c)$ exists, then $f''(c) = \alpha$. Indeed, let $f \in \mathscr{K}_1^c(I)$. Due to the concavity and convexity of F, for every distinct points $x_j \in I \cap (-\infty, c]$ and $y_j \in I \cap [c, \infty)$, $j = 1, 2, 3$, we have

$$[x_1, x_2, x_3]F = [x_1, x_2, x_3]f - \frac{\alpha}{2} \leqslant 0 \leqslant [y_1, y_2, y_3]f - \frac{\alpha}{2} = [y_1, y_2, y_3]F.$$

Therefore, if $f''_-(c)$ and $f''_+(c)$ exist, letting $x_j \nearrow c$ and $y_j \searrow c$, we get

$$f''_-(c) \leqslant \alpha \leqslant f''_+(c).$$

Similarly, for $f \in \mathscr{K}_2^c(I)$, we have $f''_+(c) \leqslant \alpha \leqslant f''_-(c)$.

REMARK 2. Function $f : I \to \mathbb{R}$ is 3-convex (resp. 3-concave) if and only if $f \in \mathscr{K}_1^c(I)$ (resp. $f \in \mathscr{K}_2^c(I)$) for every $c \in \operatorname{Int} I$. In other words, a function is 3-convex on an interval if and only if it is 3-convex at every point of its interior, so the property from the definition of $\mathscr{K}_1^c(I)$ can be described as “3-convexity at point c”.

We will often work with two sets of measure spaces and functions that satisfy the assumptions of Theorem 1. Therefore, let us denote the operator

$$\hat{A}_k g(x) := \frac{1}{\hat{K}(x)} \int_{\hat{\Omega}_2} \hat{k}(x,y) g(y) \, d\hat{\mu}_2(y), \tag{9}$$

where $(\hat{\Omega}_1, \hat{\Sigma}_1, \hat{\mu}_1)$ and $(\hat{\Omega}_2, \hat{\Sigma}_2, \hat{\mu}_2)$ are measure spaces, $g : \hat{\Omega}_2 \to \mathbb{R}$ is a measurable function, $\hat{k} : \hat{\Omega}_1 \times \hat{\Omega}_2 \to \mathbb{R}$ is a measurable and non-negative kernel and

$$\hat{K}(x) := \int_{\hat{\Omega}_2} \hat{k}(x,y) \, d\hat{\mu}_2(y) < \infty, x \in \hat{\Omega}_1. \tag{10}$$

Furthermore, for a weight function $\hat{u}$ on $\hat{\Omega}_1$ let

$$\hat{v}(y) := \int_{\hat{\Omega}_1} \hat{u}(x) \frac{\hat{k}(x,y)}{\hat{K}(x)} \, d\hat{\mu}_1(x) < \infty.$$

Throughout the rest of the paper we assume that all the integrals are well defined and finite. The following theorem is our main result.

THEOREM 2. *Let spaces* $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$ *and functions* u,k,K,v,A_k *be as in Theorem 1. Furthermore, let* $(\hat{\Omega}_1,\hat{\Sigma}_1,\hat{\mu}_1)$, $(\hat{\Omega}_2,\hat{\Sigma}_2,\hat{\mu}_2)$, $\hat{u},\hat{k},\hat{K},\hat{v},\hat{A}_k$ *be another set of spaces and functions as in Theorem 1. If* $f : \Omega_2 \to I \cap (-\infty, c]$ *and* $g \colon \hat{\Omega}_2 \to I \cap [c,\infty)$ *are measurable functions satisfying*

$$\begin{aligned} &\int_{\Omega_1} u(x)(A_k f(x))^2 \, d\mu_1(x) - \int_{\Omega_2} v(y) f^2(y) \, d\mu_2(y) \\ &= \int_{\hat{\Omega}_1} \hat{u}(x)(\hat{A}_k g(x))^2 \, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y) g^2(y) \, d\hat{\mu}_2(y), \end{aligned} \tag{11}$$

then for every $\Phi \in \mathscr{K}_1^c(I)$ *the following inequality holds*

$$\begin{aligned} &\int_{\hat{\Omega}_1} \hat{u}(x)\Phi(\hat{A}_k g(x)) \, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)\Phi(g(y)) d\hat{\mu}_2(y) \\ &\leqslant \int_{\Omega_1} u(x)\Phi(A_k f(x)) \, d\mu_1(x) - \int_{\Omega_2} v(y)\Phi(f(y)) \, d\mu_2(y). \end{aligned} \tag{12}$$

If $\Phi \in \mathscr{K}_2^c(I)$ *in the above setting, then (12) holds with the sign reversed.*

Proof. From Definition 1 there exists a constant α such that $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$ is concave on $I \cap (-\infty, c]$ so we can apply Theorem 1 on the function F and get

$$\int_{\Omega_1} u(x)F(A_k f(x)) \, d\mu_1(x) - \int_{\Omega_2} v(y)F(f(y)) \, d\mu_2(y) \geqslant 0.$$

By the definition of the function F we have

$$\int_{\Omega_1} u(x) \left[\Phi(A_k f(x)) - \frac{\alpha}{2}(A_k f(x))^2\right] d\mu_1(x) - \int_{\Omega_2} v(y) \left[\Phi(f(y)) - \frac{\alpha}{2} f^2(y)\right] d\mu_2(y) \geqslant 0.$$

Since integral is a linear functional we can write

$$\begin{aligned} &\int_{\Omega_1} u(x)\Phi(A_k f(x)) \, d\mu_1(x) - \int_{\Omega_2} v(y)\Phi(f(y)) \, d\mu_2(y) \\ &\geqslant \frac{\alpha}{2} \left\{ \int_{\Omega_1} u(x)(A_k f(x))^2 \, d\mu_1(x) - \int_{\Omega_2} v(y) f^2(y) \, d\mu_2(y) \right\}. \end{aligned} \tag{13}$$

For the same constant α the second part of Definition 1 gives us a convex function $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$ on $I \cap [c,\infty)$. Now, from Theorem 1 we have

$$\int_{\hat{\Omega}_1} \hat{u}(x)F(\hat{A}_k g(x)) \, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)F(g(y)) \, d\hat{\mu}_2(y) \leqslant 0.$$

Similarly as in the first part of the proof, we obtain

$$\int_{\hat{\Omega}_1} \hat{u}(x) \left[\Phi(\hat{A}_k g(x)) - \frac{\alpha}{2}(\hat{A}_k g(x))^2\right] d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y) \left[\Phi(g(y)) - \frac{\alpha}{2} g^2(y)\right] d\hat{\mu}_2(y) \leqslant 0$$

and also

$$\begin{aligned}&\int_{\hat{\Omega}_1} \hat{u}(x)\Phi(\hat{A}_k g(x))\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)\Phi(g(y))\, d\hat{\mu}_2(y)\\ &\leqslant \frac{\alpha}{2}\left\{\int_{\hat{\Omega}_1} \hat{u}(x)(\hat{A}_k g(x))^2\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)g^2(y)\, d\hat{\mu}_2(y)\right\}. \qquad (14)\end{aligned}$$

Due to assumption (11) the right hand sides of inequalities (13) and (14) are equal. Hence, we obtain (12). In the case $\Phi \in \mathscr{K}_2^c(I)$, the function $F(x) = \Phi(x) - \frac{\alpha}{2}x^2$ is convex on $I \cap (-\infty, c]$ and concave on $I \cap [c, \infty)$. Following the idea of the first part of the proof we get our statement. □

REMARK 3. The assumption of equality (11) in Theorem 1 can be weakened. More concretely, if

(a) $\alpha \geqslant 0$ and

$$\begin{aligned}&\int_{\Omega_1} u(x)(A_k f(x))^2\, d\mu_1(x) - \int_{\Omega_2} v(y)f^2(y)\, d\mu_2(y)\\ &\geqslant \int_{\hat{\Omega}_1} \hat{u}(x)(\hat{A}_k g(x))^2\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)g^2(y)\, d\hat{\mu}_2(y), \qquad (15)\end{aligned}$$

or

(b) $\alpha \leqslant 0$ and

$$\begin{aligned}&\int_{\Omega_1} u(x)(A_k f(x))^2\, d\mu_1(x) - \int_{\Omega_2} v(y)f^2(y)\, d\mu_2(y)\\ &\leqslant \int_{\hat{\Omega}_1} \hat{u}(x)(\hat{A}_k g(x))^2\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)g^2(y)\, d\hat{\mu}_2(y), \qquad (16)\end{aligned}$$

then (12) holds. Indeed, if we multiply (15) with $\frac{\alpha}{2} \geqslant 0$ we get

$$\begin{aligned}&\frac{\alpha}{2}\left\{\int_{\Omega_1} u(x)(A_k f(x))^2\, d\mu_1(x) - \int_{\Omega_2} v(y)f^2(y)\, d\mu_2(y)\right\}\\ &\geqslant \frac{\alpha}{2}\left\{\int_{\hat{\Omega}_1} \hat{u}(x)(\hat{A}_k g(x))^2\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)g^2(y)\, d\hat{\mu}_2(y)\right\} \qquad (17)\end{aligned}$$

so we can chain inequalities (13) and (14) to get (12). In the case when we multiply (16) with $\frac{\alpha}{2} \leqslant 0$ we again get (17) and the conclusion is the same. □

COROLLARY 1. *Let spaces* $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$, $(\hat{\Omega}_1, \hat{\Sigma}_1, \hat{\mu}_1)$, $(\hat{\Omega}_2, \hat{\Sigma}_2, \hat{\mu}_2)$ *and functions* $u, k, K, v, A_k, \hat{u}, \hat{k}, \hat{K}, \hat{v}, \hat{A}_k, f, g$ *be as in Theorem 2 and assume that (11) holds. If* Φ *is* 3*-convex on the interval* I*, then (12) holds. If* Φ *is* 3*-concave, then (12) holds with the sign reversed.*

Proof. If Φ is 3-convex, then by Remark 2 it is also from $\mathscr{K}_1^c(I)$ for every $c \in \operatorname{Int} I$, so we can apply Theorem 2. If Φ is 3-concave, then $\Phi \in \mathscr{K}_2^c(I)$ for every $c \in \operatorname{Int} I$, so we can again apply Theorem 2. □

EXAMPLE 1. Consider Theorem 2 with $\Omega_1 = \Omega_2 = (0,\infty)$, $I = (0,b)$ and $k(x,y) = 1$, $0 \leqslant y \leqslant x$, $k(x,y) = 0$, $y > x$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$ and $u(x) = \frac{1}{x}$. Then $v(y) = \frac{1}{y}$ and condition (11) becomes

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^2 \frac{dx}{x} - \int_0^\infty f^2(y)\frac{dy}{y}$$
$$= \int_{\hat{\Omega}_1} \hat{u}(x)(\hat{A}_k g(x))^2\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y) g^2(y)\, d\hat{\mu}_2(y).$$

Since $\Phi(u) = u^p$ for $p > 2$ or $p \in (0,1)$ is 3-convex, inequality (12) becomes

$$\int_{\hat{\Omega}_1} \hat{u}(x)\Phi(\hat{A}_k g(x))\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)\Phi(g(y)) d\hat{\mu}_2(y)$$
$$\leqslant \int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p \frac{dx}{x} - \int_0^\infty f^p(y)\frac{dy}{y} \tag{18}$$

so we get our type of the original Hardy inequality. If, furthermore, $\hat{\Omega}_1 = \hat{\Omega}_2 = (0,b)$ and $\hat{k}(x,y) = 1$, $0 \leqslant y \leqslant x$, $\hat{k}(x,y) = 0$, $y > x$, $d\hat{\mu}_1(x) = dx$, $d\hat{\mu}_2(y) = dy$ and $\hat{u}(x) = \frac{1}{x}$, then $\hat{v}(y) = \frac{1}{y} - \frac{1}{b}$ and condition (11) becomes

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^2 \frac{dx}{x} - \int_0^\infty f^2(y)\frac{dy}{y}$$
$$= \int_0^b \left(\frac{1}{x}\int_0^x g(y)dy\right)^2 \frac{dx}{x} - \int_0^b \left(\frac{1}{y} - \frac{1}{b}\right) g^2(y)dy,$$

while inequality (19) becomes

$$\int_0^b \left(\frac{1}{x}\int_0^x g(y)dy\right)^p \frac{dx}{x} - \int_0^b \left(\frac{1}{y} - \frac{1}{b}\right) g^p(y)dy$$
$$\leqslant \int_0^\infty \left(\frac{1}{x}\int_0^x f(y)dy\right)^p \frac{dx}{x} - \int_0^\infty f^p(y)\frac{dy}{y}.$$

If $\Phi(u) = u^p$, $p \in (1,2)$ or $p < 0$, then Φ is 3-concave and (18) holds with the sign reversed. □

EXAMPLE 2. Let $\Omega_1 = \Omega_2 = (0,\infty)$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures dx and dy, respectively, let $k(x,y) = \frac{(\frac{y}{x})^{-1/p}}{x+y}$, $p \in \mathbb{R}\setminus\{1\}$ and $u(x) = \frac{1}{x}$. Then $K(x) = K = \frac{\pi}{\sin(\pi/p)}$ and $v(y) = \frac{1}{y}$. If condition (11) holds, i. e. if

$$\frac{1}{K^2}\int_0^\infty \left(\int_0^\infty \left(\frac{y}{x}\right)^{-1/p} \frac{f(y)}{x+y} dy\right)^2 \frac{dx}{x} - \int_0^\infty f^2(y)\frac{dy}{y}$$
$$= \int_{\hat{\Omega}_1} \hat{u}(x)(\hat{A}_k g(x))^2\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y) g^2(y)\, d\hat{\mu}_2(y),$$

then applying inequality (12) for $\Phi(u) = u^p$ with $p \in \mathbb{R}_+ \setminus [1,2]$ yields

$$\int_{\hat{\Omega}_1} \hat{u}(x)\Phi(\hat{A}_k g(x))\, d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)\Phi(g(y))d\hat{\mu}_2(y)$$

$$\leqslant K^{-p} \int_0^\infty \left(\int_0^\infty y^{-1/p} \frac{f(y)}{x+y} dy \right)^p dx - \int_0^\infty f^p(y) \frac{dy}{y}, \tag{19}$$

while for $p \in (-\infty, 0) \cup (1,2)$ inequality (19) holds with the sign reversed. Replace $f(t)t^{-1/p}$ with $f(t)$ and we get our type of inequality for Hilbert's inequality. $\square$

In the previous examples we derived only inequalities over some subsets of $\mathbb{R}_+$. However, Theorem 1 and Theorem 2 cover much more general situations. We can apply that result to n-dimensional cells in $\mathbb{R}^n_+$ and thus, in particular, obtain a generalization of Godunova's inequality.

Applying Theorem 1 with $\Omega_1 = \Omega_2 = \mathbb{R}^n_+$, Lebesgue measures $d\mu_1(\mathbf{x}) = d\mathbf{x}$ and $d\mu_2(\mathbf{y}) = d\mathbf{y}$, and a kernel $k : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ of the form $k(\mathbf{x},\mathbf{y}) = l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$, where $l : \mathbb{R}^n_+ \to \mathbb{R}$ is a non-negative measurable function, K. Krulić et al [10] obtained the following generalization of Godunova's inequality.

COROLLARY 2. *Let l and u be non-negative measurable functions on $\mathbb{R}^n_+$ such that $0 < L(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}^n_+} l(\mathbf{y})\, d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}^n_+$, and that the function $\mathbf{x} \mapsto u(\mathbf{x}) \frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}$ is integrable on $\mathbb{R}^n_+$ for each fixed $\mathbf{y} \in \mathbb{R}^n_+$. Let the function v be defined on $\mathbb{R}^n_+$ by*

$$v(\mathbf{y}) = \int_{\mathbb{R}^n_+} u(\mathbf{x}) \frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})} d\mathbf{x}.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\mathbb{R}^n_+} u(\mathbf{x}) \Phi\left(\frac{1}{L(\mathbf{x})} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y}) d\mathbf{y} \right) d\mathbf{x} \leqslant \int_{\mathbb{R}^n_+} v(\mathbf{y}) \Phi(f(\mathbf{y})) d\mathbf{y}$$

holds for all measurable functions $f : \mathbb{R}^n_+ \to I$.

For $\int_{\mathbb{R}^n_+} l(\mathbf{t})\, d\mathbf{t} = 1$ and $u(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}}$, Corollary 2 from [10] reduces to Godunova's inequality (4). This shows that Corollary 2 is a genuine generalization of Godunova's inequality (4). Let $\hat{l}, \hat{u}, \hat{L}, \hat{v}$ be another set of functions as in Corollary 2, i. e. let $\hat{l}$ be a non-negative measurable function on $\mathbb{R}^m_+$ such that $0 < \hat{L}(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}^m_+} \hat{l}(\mathbf{y})\, d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}^m_+$ and for a weight function $\hat{u}$ on $\mathbb{R}^m_+$ let

$$\hat{v}(\mathbf{y}) = \int_{\mathbb{R}^m_+} \hat{u}(\mathbf{x}) \frac{\hat{l}\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{\hat{L}(\mathbf{x})} d\mathbf{x}.$$

THEOREM 3. *Let functions l, u, L, v be as in Corollary 2 and let $\hat{l}, \hat{u}, \hat{L}, \hat{v}$ be another set of functions as in Corollary 2. If $f \colon \mathbb{R}^n_+ \to I \cap (-\infty, c]$ and $g \colon \mathbb{R}^m_+ \to I \cap [c, \infty)$ are measurable functions satisfying*

$$\int_{\mathbb{R}^m_+} \hat{u}(\mathbf{x}) \left(\frac{1}{\hat{L}(\mathbf{x})} \int_{\mathbb{R}^m_+} \hat{l}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) g(\mathbf{y}) d\mathbf{y} \right)^2 d\mathbf{x} - \int_{\mathbb{R}^m_+} \hat{v}(\mathbf{y}) (g(\mathbf{y}))^2 d\mathbf{y}$$

$$= \int_{\mathbb{R}^n_+} u(\mathbf{x}) \left(\frac{1}{L(\mathbf{x})} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y}) d\mathbf{y} \right)^2 d\mathbf{x} - \int_{\mathbb{R}^n_+} v(\mathbf{y}) (f(\mathbf{y}))^2 d\mathbf{y} \tag{20}$$

then for every $\Phi \in \mathscr{K}_1^c(I)$ *the following inequality holds*

$$\int_{\mathbb{R}_+^m} \hat{u}(\mathbf{x})\Phi\left(\frac{1}{\hat{L}(\mathbf{x})}\int_{\mathbb{R}_+^m} \hat{l}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) g(\mathbf{y})d\mathbf{y}\right)d\mathbf{x} - \int_{\mathbb{R}_+^m} \hat{v}(\mathbf{y})\Phi(g(\mathbf{y}))d\mathbf{y}$$
$$\leqslant \int_{\mathbb{R}_+^n} u(\mathbf{x})\Phi\left(\frac{1}{L(\mathbf{x})}\int_{\mathbb{R}_+^n} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})d\mathbf{y}\right)d\mathbf{x} - \int_{\mathbb{R}_+^n} v(\mathbf{y})\Phi(f(\mathbf{y}))d\mathbf{y}. \tag{21}$$

If $\Phi \in \mathscr{K}_2^c(I)$ *in the above setting, then (12) holds with the sign reversed.*

COROLLARY 3. *Let* l, u, L, v, $\hat{l}$, $\hat{u}$, $\hat{L}$, $\hat{v}$, f, g *be as in Theorem 3 and assume that (20) holds. If* Φ *is a* 3*-convex function on the interval* I*, then (21) holds. If* Φ *is a* 3*-concave function, then (21) holds with the sign reversed.*

EXAMPLE 3. Let $n = 1$, $I = (0,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$,

$$u(x) = x^{-2\alpha} \text{ and } k(x,y) = \frac{\log y - \log x}{y - x}\left(\frac{y}{x}\right)^{-\alpha}, \quad \alpha \in (0,1).$$

For all $\alpha \in (0,1)$ we have

$$K(x) = \int_0^\infty \frac{\log y - \log x}{y - x}\left(\frac{y}{x}\right)^{-\alpha} dy = \int_0^\infty \frac{\log u}{u-1} u^{-\alpha} du$$
$$= \int_{-\infty}^{\infty} \frac{te^{(1-\alpha)t}}{e^t - 1} dt = \Psi'(\alpha) + \Psi'(1-\alpha) = \frac{\pi^2}{\sin^2 \pi\alpha},$$

where $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, x > 0,$ is the Digamma function and we used the identity $\Psi(1-x) = \Psi(x) + \pi \cot \pi x, x \in (0,1)$ (for details on Ψ see [1]). Now we have

$$v(y) = \frac{\sin^2 \pi\alpha}{\pi^2} \int_0^\infty \frac{\log x - \log y}{x - y}\left(\frac{y}{x}\right)^{\alpha} y^{-2\alpha} dx$$
$$= \frac{\sin^2 \pi\alpha}{\pi^2} y^{-2\alpha} \int_0^\infty \frac{\log u}{u-1} u^{-\alpha} du = y^{-2\alpha}, \quad (x = yu),$$

so condition (20) becomes

$$\frac{\sin^4 \pi\alpha}{\pi^4} \int_0^\infty \left(x^{-\alpha} \int_0^\infty \frac{\log y - \log x}{y - x}\left(\frac{y}{x}\right)^{-\alpha} f(y)dy\right)^2 dx - \int_0^\infty \left(\frac{f(y)}{y^\alpha}\right)^2 dy$$
$$= \int_{\mathbb{R}_+^m} \hat{u}(\mathbf{x})\left(\frac{1}{\hat{L}(\mathbf{x})}\int_{\mathbb{R}_+^m} \hat{l}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) g(\mathbf{y})d\mathbf{y}\right)^2 d\mathbf{x} - \int_{\mathbb{R}_+^m} \hat{v}(\mathbf{y})(g(\mathbf{y}))^2 d\mathbf{y}.$$

If this condition is fulfilled, then for $\Phi \in \mathscr{K}_1^c(I)$ we can apply inequality (12) and obtain

$$\int_{\mathbb{R}_+^m} \hat{u}(\mathbf{x})\Phi\left(\frac{1}{\hat{L}(\mathbf{x})}\int_{\mathbb{R}_+^m} \hat{l}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) g(\mathbf{y})d\mathbf{y}\right)d\mathbf{x} - \int_{\mathbb{R}_+^m} \hat{v}(\mathbf{y})\Phi(g(\mathbf{y}))d\mathbf{y}$$
$$\leqslant \int_0^\infty \Phi\left(\frac{\sin^2 \pi\alpha}{\pi^2} \int_0^\infty \frac{\log y - \log x}{y - x}\left(\frac{y}{x}\right)^{-\alpha} f(y)dy\right) x^{-2\alpha} dx - \int_0^\infty y^{-2\alpha}\Phi(f(y))dy. \quad \square$$

Further, by applying our result to the 3-convex function $\Phi(x) = e^x$ and making some suitable variable transformations we can obtain Pólya–Knopp type inequalities. We give the following example:

EXAMPLE 4. Let the assumptions in Theorem 2 be satisfied with functions f and g replaced by $\log f^p$ and $\log g^p$, respectively, i. e. the functions f and g satisfy $\log f^p \leqslant c \leqslant \log g^p$ and

$$\int_{\Omega_1} u(x)\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\log f^p(y)\,d\mu_2(y)\right)^2 d\mu_1(x) - \int_{\Omega_2} v(y)\log^2 f^p(y)\,d\mu_2(y)$$
$$= \int_{\hat{\Omega}_1} \hat{u}(x)\left(\frac{1}{\hat{K}(x)}\int_{\hat{\Omega}_2} \hat{k}(x,y)\log g^p(y)\,d\hat{\mu}_2(y)\right)^2 d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y)\log^2 g^p(y)\,d\hat{\mu}_2(y).$$

Since the function $\Phi(x) = e^x$ is 3-convex on $\mathbb{R}$, for every interval $I \subset \mathbb{R}$ and $c \in \operatorname{Int} I$ we have $\Phi \in \mathscr{K}_1^c(I)$. Therefore, inequality (12) applied for Φ yields

$$\int_{\hat{\Omega}_1} \hat{u}(x)\left[\exp\left(\frac{1}{\hat{K}(x)}\int_{\hat{\Omega}_2} \hat{k}(x,y)\log g(y)\,d\hat{\mu}_2(y)\right)\right]^p d\hat{\mu}_1(x) - \int_{\hat{\Omega}_2} \hat{v}(y) g^p(y)\,d\hat{\mu}_2(y)$$
$$\leqslant \int_{\Omega_1} u(x)\left[\exp\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\log f(y)\,d\mu_2(y)\right)\right]^p d\mu_1(x) - \int_{\Omega_2} v(y) f^p(y)\,d\mu_2(y). \tag{22}$$

In particular, if $p = 1, \Omega_1 = \Omega_2 = (0,\infty), \hat{\Omega}_1 = \hat{\Omega}_2 = (0,b), \hat{k}(x,y) = k(x,y) = 1, 0 < y < x, \hat{k}(x,y) = k(x,y) = 0, y \geqslant x$. (so that $\hat{K}(x) = K(x) = x$), $d\hat{\mu}_1(x) = d\mu_1(x) = dx$, $d\hat{\mu}_2(y) = d\mu_2(y) = dy$, $\hat{u}(x) = u(x) = \frac{1}{x}$ (so that $v(x) = \frac{1}{x}$, $\hat{v}(x) = \frac{1}{x} - \frac{1}{b}$) replacing $f(x)/x$ by $f(x)$ and $g(x)/x$ by $g(x)$ after a simple calculation we find that (22) is equal to

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x \log g(y)\,dy\right) dx - e\int_0^b \left(\frac{1}{y} - \frac{1}{b}\right) g(y)\,dy$$
$$\leqslant \int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(y)\,dy\right) dx - e\int_0^\infty f(y)\,dy$$

which is an inequality related to the classical Pólya–Knopp's inequality. □

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(Received October 29, 2014)

Kristina Krulić Himmelreich
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a, 10 000 Zagreb, Croatia
e-mail: krulic@ttf.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz baruna Filipovića 28a, 10 000 Zagreb, Croatia
e-mail: pecaric@element.hr

Dora Pokaz
Faculty of Civil Engineering
University of Zagreb
Kačićeva 26, 10 000 Zagreb, Croatia
e-mail: dora@grad.hr

Marjan Praljak
Faculty of Food Technology and Biotechnology
University Of Zagreb, Pierottijeva 6, 10 000 Zagreb, Croatia
e-mail: mpraljak@pbf.hr