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LEVINSON'S INEQUALITY FOR HILBERT SPACE OPERATORS

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Abstract. Levinson's operator inequality is given for unital fields of positive linear mappings and a large class of functions. Order among quasi-arithmetic means is considered under similar conditions.

1. Introduction

Let $\mathscr{B}(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H.We denote by $\mathscr{B}_h(H)$ the real subspace of all self-adjoint operators on H. Bounds of $X \in \mathscr{B}_h(H)$ are defined by $m_X := \inf\{\langle X\xi,\xi\rangle : \xi \in H, \|\xi\| = 1\}$ and $M_X := \sup\{\langle X\xi,\xi\rangle : \xi \in H, \|\xi\| = 1\}$

A continuous real valued function f defined on an interval I is said to be operator convex if $f(\lambda X + (1-\lambda)Y) \leqslant \lambda f(X) + (1-\lambda)f(Y)$ for all self-adjoint operators X,Y with spectra contained in I and all $\lambda \in [0,1]$. If the function f is operator convex, then so-called Jensen's operator inequality $f(\Phi(X)) \leqslant \Phi(f(X))$ holds for any unital positive linear mapping Φ on $\mathscr{B}(H)$ and any $X \in \mathscr{B}_h(H)$ with spectrum contained in I. Many other versions of Jensen's operator inequality can be found in [4, 3].

Assume furthermore that $(\Phi_1,\dots,\Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$. If in addition $\sum_{i=1}^n \Phi_i(1_H) = 1_K$, we say that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$ is *unital.*

Now we give the definiton of classes of functions for which we observe Levinson's operator inequality:

DEFINITION 1. Let $f \in \mathscr{C}(I)$ be a real valued function on an arbitrary interval I in $\mathbb{R}$ and $c \in I^\circ$, where I° is the interior of I.

We say that $f \in \mathscr{K}_1^c(I)$ (resp. $f \in \mathscr{K}_2^c(I)$) if there exists a constant α such that the function $F(t) = f(t) - \frac{\alpha}{2}t^2$ is concave (resp. convex) on $I\bigcap(-\infty,c]$ and convex (resp. concave) on $I\bigcap[c,\infty)$.

Moreover, we say that $f \in \dot{\mathscr{K}}_1^c(I)$ (resp. $f \in \dot{\mathscr{K}}_2^c(I)$) if there exists a constant α such that the function $F(t) = f(t) - \frac{\alpha}{2}t^2$ is operator concave (resp. operator convex) on $I\bigcap(-\infty,c]$ and operator convex (resp. operator concave) on $I\bigcap[c,\infty)$.

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$\mathcal{K}_1^c(I)$ can be interpreted as the class of "3-convex functions at point c" and generalizes 3-convex functions in the following sense: a function is 3-convex on I if and only if it is at every $c \in I^\circ$.

Next, we will review the history of research of Levison's inequality.

Levinson [5] considered an inequality as follows:

If $f:(0,2c) \to \mathbb{R}$ satisfies $f''' \geqslant 0$ and p_i, x_i, y_i, $i = 1,2,\ldots,n$, are such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $0 \leqslant x_i \leqslant c$ and

$$x_1 + y_1 = x_2 + y_2 = \ldots = x_n + y_n = 2c, \tag{1}$$

then the inequality

$$\sum_{i=1}^n p_i f(x_i) - f(\bar{x}) \leqslant \sum_{i=1}^n p_i f(y_i) - f(\bar{y}) \tag{2}$$

holds, where $\bar{x} = \sum_{i=1}^n p_i x_i$ and $\bar{y} = \sum_{i=1}^n p_i y_i$ denote the weighted arithmetic means.

Popoviciu [11] showed that the assumptions on the differentiability of f can be weakened and for (2) to hold it is enough to assume that f is 3-convex. Bullen [2] gave another proof of Popoviciu's result rescaled to a general interval $[a,b]$ as follows:

If $f:[a,b] \to \mathbb{R}$ is 3-convex and p_i, x_i, y_i, $i = 1,2,\ldots,n$, are such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $a \leqslant x_i, y_i \leqslant b$, (1) holds (for some $c \in [a,b]$) and

$$\max(x_1,\ldots,x_n) \leqslant \min(y_1,\ldots,y_n), \tag{3}$$

then (2) holds.

Mercer [6] made a significant improvement by replacing (1) with the weaker condition that the variances of the two sequences are equal:

$$\sum_{i=1}^n p_i (x_i - \bar{x})^2 = \sum_{i=1}^n p_i (y_i - \bar{y})^2, \tag{4}$$

Witkowski [12, 13] extended this result in several ways. Firstly, he showed that Levinson's inequality can be stated in a more general setting with random variables. Furthermore, he showed that it is enough to assume that f is 3-convex and that the assumption of equality of the variances can be weakened to inequality in a certain direction.

Baloch, Pečarić and Praljak [1] built on and extend the methods of Witkowski [12]. They introduced a new class of functions $\mathcal{K}_1^c((a,b))$ as in Definition 1. Their main result is that this class is the largest class of functions for which Levinson's inequality holds under Mercer's assumptions as follows:

(i) Let $a < x_i \leqslant c \leqslant y_i < b$, $p_i > 0$ for $i = 1,2,\ldots,n$, $\sum_{i=1}^n p_i = 1$ and (4) holds. If $f \in \mathcal{K}_1^c((a,b))$, then inequality (2) holds and if $f \in \mathcal{K}_1^c((a,b))$, then (2) holds with reverse sign of inequality.

(ii) Let $f:(a,b) \to \mathbb{R}$ be continuous and $c \in (a,b)$. If inequality (2) (resp. the reverse of (2)) holds for every $n \in \mathbb{N}$ and sequences p_i, x_i, y_i, $i = 1,\ldots,n$, such that $p_i > 0$, $\sum_{i=1}^n p_i = 1$, $a < x_i \leqslant c \leqslant y_i < b$ and (4) holds, then $f \in \mathcal{K}_1^c((a,b))$ (resp. $f \in \mathcal{K}_2^c((a,b))$).

In this paper, we give a general formulation of Levinson's operator inequality. In Section 2 we will prove the operator version of inequality (2) for $f \in \dot{\mathscr{K}}_1^c(I)$, which will generalize the above result (i) by Baloch, Pečarić and Praljak. In Section 3 by adding spectra conditions we will prove the same version of (2) for $f \in \mathscr{K}_1^c(I)$. In Section 4 we will give order among quasi-arithmetic means under similar conditions. As an application we give order among power means, which will generalize Bullen's results [2, Corollary].

2. Levinson's inequality with operator convexity and concavity

In this section we give version of Levinson's operator inequality for $f \in \dot{\mathscr{K}}_i^c(I)$, $i = 1,2$.

THEOREM 1. *Let $(X_1,\dots,X_n)$ be an n-tuple and $(Y_1,\dots,Y_k)$ be a k-tuple of self-adjoint operators $X_i, Y_j \in \mathscr{B}_h(H)$ with spectra contained in $[m_x, M_x]$ and $[m_y, M_y]$, respectively, such that $a < m_x \leqslant M_x \leqslant c \leqslant m_y \leqslant M_y < b$ for some $a,b,c \in \mathbb{R}$. Let $(\Phi_1,\dots,\Phi_n)$ be a unital n-tuple and $(\Psi_1,\dots,\Psi_k)$ be a unital k-tuple of positive linear mappings $\Phi_i, \Psi_j : \mathscr{B}(H) \to \mathscr{B}(K)$.*

If $f \in \dot{\mathscr{K}}_1^c((a,b))$ and $\alpha X \leqslant \alpha Y$, then

$$\sum_{i=1}^{n} \Phi_i\big(f(X_i)\big) - f\Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big) \leqslant \tfrac{\alpha}{2} X \leqslant \tfrac{\alpha}{2} Y \leqslant \sum_{j=1}^{k} \Psi_j\big(f(Y_j)\big) - f\Big(\sum_{j=1}^{k} \Psi_j(Y_j)\Big) \tag{5}$$

holds, where

$$X := \sum_{i=1}^{n} \Phi_i\big(X_i^2\big) - \Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big)^2, \quad Y := \sum_{j=1}^{k} \Psi_j\big(Y_j^2\big) - \Big(\sum_{j=1}^{k} \Psi_j(Y_j)\Big)^2.$$

If $f \in \dot{\mathscr{K}}_2^c((a,b))$ and $\alpha X \geqslant \alpha Y$, then the reverse inequalities are valid in (5).

Proof. We will give the proof for $f \in \dot{\mathscr{K}}_1^c((a,b))$. So, there is a constant α such that $F(t) = f(t) - \frac{\alpha}{2} t^2$ is operator concave on $[m_x, c] \subset (a,c]$. Then the reverse of Jensen's inequality for an operator concave function gives

$$\begin{aligned} 0 &\leqslant F\Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big) - \sum_{i=1}^{n} \Phi_i\big(F(X_i)\big) \\ &= f\Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big) - \tfrac{\alpha}{2}\Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big)^2 - \sum_{i=1}^{n} \Phi_i\big(f(X_i)\big) + \tfrac{\alpha}{2} \sum_{i=1}^{n} \Phi_i\big(X_i^2\big). \end{aligned}$$

It follows

$$\sum_{i=1}^{n} \Phi_i\big(f(X_i)\big) - f\Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big) \leqslant \frac{\alpha}{2} X. \tag{6}$$

Also, since F is operator convex on $[c,M_y] \subset [c,b)$, Jensen's operator inequality gives

$$\begin{aligned}
0 &\leqslant \sum_{j=1}^{k} \Psi_j\big(F(Y_j)\big) - F\Big(\sum_{j=1}^{k} \Psi_j(Y_j)\Big) \\
&= \sum_{j=1}^{k} \Psi_j\big(f(Y_j)\big) - \tfrac{\alpha}{2} \sum_{j=1}^{k} \Psi_j\big(Y_j^2\big) - f\Big(\sum_{j=1}^{k} \Psi_j(Y_j)\Big) + \tfrac{\alpha}{2}\Big(\sum_{j=1}^{k} \Psi_j(Y_j)\Big)^2.
\end{aligned}$$

It follows

$$\frac{\alpha}{2} Y \leqslant \sum_{j=1}^{k} \Psi_j\big(f(Y_j)\big) - f\Big(\sum_{j=1}^{k} \Psi_j(Y_j)\Big). \tag{7}$$

Combining inequalities (6) and (7) and taking into account that $\alpha X \leqslant \alpha Y$, we obtain the desired inequality (5). $\square$

Now, we give an example to clear the situation in Theorem 1.

EXAMPLE 1. Let $c > 1$, $d = \frac{\sqrt{c^5}+c^3/2}{\sqrt{c^3}-1}$, $f(t) = \sqrt{t^3} - t^2/2$ on $(0,c]$ and $f(t) = d\sqrt{t} - d/t - t^2$ on $[c,\infty)$. Then $f \in \dot{\mathscr{K}}_2^c((0,\infty))$ for $\alpha = -2$.

Let $\Phi, \Psi : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ be defined by $\Phi((m_{ij})_{1\leqslant i,j\leqslant 3}) = \frac{1}{2}(m_{ij})_{1\leqslant i,j\leqslant 2}$ and $\Psi(M) = \frac{1}{3}\mathrm{Tr}(M) I_2$ for every $M = (m_{ij})_{1\leqslant i,j\leqslant 3} \in M_3(\mathbb{C})$.

If

$$X_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 3 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 15 & 3 & 2 \\ 3 & 18 & 1 \\ 2 & 1 & 9 \end{pmatrix},$$

then $0.1 I_3 < X_1, X_2 < 5.3 I_3 < 8.3 I_3 < Y_1 < 20.2$ and we can choose any c from $[5.3, 8.3]$. Next, we have

$$\begin{aligned}
X = \Phi\big(X_1^2\big) + \Phi\big(X_2^2\big) - \Big(\Phi(X_1) + \Phi(X_2)\Big)^2 &= \begin{pmatrix} 1.5 & 0.25 \\ 0.25 & 1.75 \end{pmatrix} \\
< Y = \Psi\big(Y_1^2\big) - \Big(\Psi(Y_1)\Big)^2 &= \begin{pmatrix} 70/3 & 0 \\ 0 & 70/3 \end{pmatrix}.
\end{aligned}$$

Then, according to Theorem 1, we obtain (rounded to six decimal places)

$$F_1 = \begin{pmatrix} 0.286126 & 0.172952 \\ 0.172952 & 0.484473 \end{pmatrix} < X < Y < F_2 = (70/3 + 0.0667891\, d) I_2,$$

where

$$F_1 = \sqrt{\Big(\Sigma_{i=1}^2 \Phi\big(X_i\big)\Big)^3} - \tfrac{1}{2}\Big(\Sigma_{i=1}^2 \Phi\big(X_i\big)\Big)^2 - \Sigma_{i=1}^2 \Big(\Phi\big(\sqrt{X_i^3}\big) - \tfrac{1}{2}\Phi\big(X_i^2\big)\Big),$$

$$F_2 = d\sqrt{\Psi\big(Y_1\big)} - d\Big(\Psi\big(Y_1\big)\Big)^{-1} - \Big(\Psi\big(Y_1\big)\Big)^2 - d\Psi\big(\sqrt{Y_1}\big) + d\Psi\big(Y_1^{-1}\big) + \Psi\big(Y_1^2\big).$$

This shows that inequality (5) can be strict.

Next, we give the following obvious corollary to Theorem 1 with convex combinations of operators X_i, $i=1,\dots,n$ and Y_j, $j=1,\dots,k$. This is a generalization of [1, Theorem 2.6].

COROLLARY 2. *Let $(X_1,\dots,X_n)$ be an n-tuple and $(Y_1,\dots,Y_k)$ be a k-tuple of self-adjoint operators $X_i,Y_j\in\mathscr{B}_h(H)$ with spectra contained in $[m_x,M_x]$ and $[m_y,M_y]$, respectively, such that $a<m_x\leqslant M_x\leqslant c\leqslant m_y\leqslant M_y<b$ for some $a,b,c\in\mathbb{R}$. Let $(p_1,\dots,p_n)$ be an n-tuple and $(q_1,\dots,q_k)$ be a k-tuple of positive scalars such that $\sum_{i=1}^n p_i=1$ and $\sum_{j=1}^k q_j=1$.*

If $f\in\dot{\mathscr{K}}_1^c((a,b))$ and $\alpha P\leqslant\alpha Q$, then

$$\sum_{i=1}^n p_i f(X_i)-f\left(\bar{X}\right)\leqslant\frac{\alpha}{2}P\leqslant\frac{\alpha}{2}Q\leqslant\sum_{j=1}^k q_j f(Y_j)-f\left(\bar{Y}\right) \tag{8}$$

holds, where

$$P:=\sum_{i=1}^n p_i\left(X_i-\bar{X}\right)^2,\quad Q:=\sum_{j=1}^k q_j\left(Y_j-\bar{Y}\right)^2,$$

and $\bar{X}:=\sum_{i=1}^n p_iX_i$, $\bar{Y}:=\sum_{j=1}^k q_jY_j$ denote the weighted arithmetic means of operators.

If $f\in\dot{\mathscr{K}}_2^c([m_x,M_y])$ and $\alpha P\geqslant\alpha Q$, then reverse inequalities are valid in (8).

Proof. We apply Theorem 1 for positive linear mappings $\Phi_i,\Psi_j:\mathscr{B}(H)\to\mathscr{B}(K)$ determined by $\Phi_i:B\mapsto p_iB$, $i=1,\dots,n$ and $\Psi_j:B\mapsto q_jB$, $j=1,\dots,k$. $\square$

REMARK 1. 1) If f is convex (resp. concave), then $f''_-(c)\leqslant\alpha\leqslant f''_+(c)$ (resp. $f''_+(c)\leqslant\alpha\leqslant f''_-(c)$), see [1]. So, condition $\alpha X\leqslant\alpha Y$ (resp. $\alpha X\geqslant\alpha Y$) in Theorem 1 can be weakened to $X\leqslant Y$ (resp. $Y\leqslant X$).

2) Setting $n=k$, $p_i=q_i$ and $A:=\sum_{i=1}^n p_i\left(X_i-\bar{X}\right)^2=\sum_{i=1}^n p_i\left(Y_i-\bar{Y}\right)^2$ (compare with (4)) in Corollary 2, we get (compare with (2))

$$\sum_{i=1}^n p_i f(X_i)-f\left(\bar{X}\right)\leqslant\frac{\alpha}{2}A\leqslant\sum_{i=1}^n p_i f(Y_i)-f\left(\bar{Y}\right) \tag{9}$$

if $f\in\dot{\mathscr{K}}_1^c((a,b))$. Or, we get reverse inequalities in (9) if $f\in\dot{\mathscr{K}}_2^c((a,b))$. This is a generalization of Mercer's result [6] on operators.

3. Levinson's inequality without operator concavity and convexity

In this section we give Levinson's operator inequality for $f\in\mathscr{K}_i^c(I)$, $i=1,2$.

Operator convexity plays an essential role in (5). In fact, this inequality will be false if we replace an operator convex function by a general convex function. We give a simple counterexample:

COUNTEREXAMPLE 1. Let $c > 0$, $d = \sqrt{c^7} + \sqrt{c^3}/2$, $f(t) = t^4 - t^2/2$ on $(0,c]$ and $f(t) = d\sqrt{t} - t^2$ on $[c,\infty)$. Then $f \in \mathscr{K}_2^c((0,\infty))$ for $\alpha = -2$, but $f \notin \dot{\mathscr{K}}_2^c((0,\infty))$ because the function $t \mapsto t^4$ is not operator convex on $(0,\infty)$. Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be mapping defined by $\Phi(M) = M/2$ for every $M \in M_2(\mathbb{C})$. If

$$X_1 = \begin{pmatrix} 50 & -30 \\ -30 & 20 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 9 & -1 \\ -1 & 10 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 100 & 1 \\ 1 & 200 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 200 & 2 \\ 2 & 300 \end{pmatrix},$$

then $I_2 < X_1, X_2 < 69I_2 < 99I_2 < Y_1, Y_2 < 301I_2$ and we can choose any c from $[69,99]$. Next, we have (by setting $\Phi_1 = \Phi_2 = \Psi_1 = \Psi_2$ in Theorem 1)

$$X = \tfrac{1}{2}(X_1 - X_2)^2 = \begin{pmatrix} 420.25 & 410.25 \\ 410.25 & 25. \end{pmatrix} < Y = \tfrac{1}{2}(Y_1 - Y_2)^2 = \begin{pmatrix} 2500 & 0.25 \\ 0.25 & 2500 \end{pmatrix}.$$

Then, we obtain (rounded to six decimal places)

$$F_1 \lessgtr\!\!\!\!/ \; -X > -Y > F_2,$$

where

$$F_1 = \sum_{i=1}^{2}\left(\frac{X_i^4}{2} - \frac{X_i^2}{4}\right) - \left(\sum_{i=1}^{2}\frac{X_i}{2}\right)^4 + \frac{1}{2}\left(\sum_{i=1}^{2}\frac{X_i}{2}\right)^2 = \begin{pmatrix} 2.37074\cdot 10^6 & 347175. \\ 347175. & 34362.5 \end{pmatrix},$$

$$\begin{aligned} F_2 &= \sum_{i=1}^{2}\left(d\frac{Y_i^{1/2}}{2} - \frac{Y_i^2}{2}\right) - d\left(\sum_{i=1}^{2}\frac{Y_i}{2}\right)^{1/2} + \left(\sum_{i=1}^{2}\frac{Y_i}{2}\right)^2 \\ &= -\begin{pmatrix} 2500 & 0.25 \\ 0.25 & 2500 \end{pmatrix} - d\begin{pmatrix} 0.176379 & 0.000965 \\ 0.000965 & 0.0801 \end{pmatrix}. \end{aligned}$$

So, the reverse inequality is false in (5) under the operator order.

In the following theorem we give a general result when Levinson's operator inequality (5) holds for $f \in \mathscr{K}_1^c([m_x, M_y])$ with conditions on the spectra of operators. There have been many interesting works devoted to obtain operator inequalities under spectra conditions. The reader is referred to [9, 10] and references therein.

THEOREM 3. *Let $(X_1,\ldots,X_n)$ be an n-tuple and $(Y_1,\ldots,Y_k)$ be a k-tuple of self-adjoint operators $X_i, Y_j \in \mathscr{B}_h(H)$. Let m_{X_i}, M_{X_i} be bounds of X_i and m_{Y_j}, M_{Y_j} be bounds of Y_j, such that $a < m_{X_i} \leqslant M_{X_i} \leqslant c \leqslant m_{Y_j} \leqslant M_{Y_j} < b$ for some $a,b,c \in \mathbb{R}$ and every $i = 1,\ldots,n$, $j = 1,\ldots,k$. Let $(\Phi_1,\ldots,\Phi_n)$ be a unital n-tuple and $(\Psi_1,\ldots,\Psi_k)$ be a unital k-tuple of positive linear mappings $\Phi_i, \Psi_j : \mathscr{B}(H) \to \mathscr{B}(K)$. Let m_X, M_X and m_Y, M_Y be bounds of $\bar{X} = \sum_{i=1}^n \Phi_i(X_i)$ and $\bar{Y} = \sum_{j=1}^k \Psi_j(Y_j)$, respectively, such that*

$$\begin{aligned} (m_X, M_X) \cap [m_{X_i}, M_{X_i}] &= \varnothing, \quad i = 1,\ldots,n, \\ (m_Y, M_Y) \cap [m_{Y_j}, M_{Y_j}] &= \varnothing, \quad j = 1,\ldots,k. \end{aligned} \tag{10}$$

If $f \in \mathscr{K}_1^c((a,b))$ and $\alpha X \leqslant \alpha Y$ hold, then

$$\sum_{i=1}^{n}\Phi_i\big(f(X_i)\big) - f\big(\bar{X}\big) \leqslant \frac{\alpha}{2}X \leqslant \frac{\alpha}{2}Y \leqslant \sum_{j=1}^{k}\Psi_j\big(f(Y_j)\big) - f\big(\bar{Y}\big) \tag{$*$}$$

is valid (i.e. (5) *is valid), where*

$$X := \sum_{i=1}^{n} \Phi_i\left(X_i^2\right) - \bar{X}^2, \quad Y := \sum_{j=1}^{k} \Psi_j\left(Y_j^2\right) - \bar{Y}^2.$$

If $f \in \mathscr{K}_2^c((a,b))$ *and* $\alpha X \geqslant \alpha Y$ *hold, then reverse inequalities are valid in* $(*)$.

Proof. The proof is similar to the one for Theorem 1. We apply Jensen's operator inequality without operator convexity and concavity (see [9, Theorem 1.]). We omit the details of the proof. □

As an application of Theorem 3, we obtain many interesting inequalities. For example, we obtain the following inequalities for some power functions.

EXAMPLE 2. The function $f(t) = t^p$, $p = 3,4,\ldots$ is an element of $\mathscr{K}_1^c((-c,\infty))$ for $\alpha = p(p-1)c^{p-2}$ and $c \in \mathbb{R}^+$. Let mappings Φ_i, Ψ_j and operators X_i, Y_j, X, Y, $\bar{X}$, $\bar{Y}$ be as in Theorem 3. If (10) and $X \leqslant Y$ hold, then

$$\sum_{i=1}^{n} \Phi_i\left(X_i^p\right) - \bar{X}^p \leqslant \binom{n}{2} c^{p-2} X \leqslant \binom{n}{2} c^{p-2} Y \leqslant \sum_{j=1}^{k} \Psi_j\left(Y_j^p\right) - \bar{Y}^p.$$

REMARK 2.

1) In Counterexample 1 bounds of X_1 are $m_{X_1} = 1.45898$, $M_{X_1} = 68.54102$, and of $X = X_1/2 + X_2/2$ are $m_X = 5.13824$, $M_X = 39.3618$. Then $(m_X, M_X) \cap [m_{X_1}, M_{X_1}] \neq \varnothing$, so spectra conditions don't hold and we cannot apply Theorem 3.

2) If we add spectra conditions (10) in Corollary 2, we can put weaker conditions on the function f: it is sufficient that f is in class $\mathscr{K}_1^c((a,b))$ instead of in $\dot{\mathscr{K}}_1^c((a,b))$.

3) Similarly as in Definition 1, we can observe a class of functions $\dot{\mathscr{K}}_3^c(I)$, when F is operator concave on $I \cap (-\infty, c]$ and real convex function on $I \cap [c, \infty)$. So, we can obtain the following result:

Let mappings Φ_i, Ψ_j *and operators* X_i, Y_j, X, Y, $\bar{X}$, $\bar{Y}$ *be as in Theorem* 3. *If* $f \in \mathscr{K}_3^c((a,b))$, $\alpha X \leqslant \alpha Y$ *and* $(m_Y, M_Y) \cap [m_{Y_j}, M_{Y_j}] = \varnothing$, *(* $j = 1,\ldots,k$ *) hold, then* (5) *is valid*

In the same way, we can consider other combinations.

Next, we give a version of Levison's operator inequality with the scalar product.

THEOREM 4. *Let* $(X_1,\ldots,X_n)$ *be an* n*-tuple and* $(Y_1,\ldots,Y_k)$ *be a* k*-tuple of self-adjoint operators* $X_i, Y_j \in \mathscr{B}_h(H)$ *with spectra contained in* $[m_x, M_x]$ *and* $[m_y, M_y]$, *respectively, such that* $a < m_x \leqslant M_x \leqslant c \leqslant m_y \leqslant M_y < b$ *for some* $a,b,c \in \mathbb{R}$. *Let* $(z_1,\ldots,z_n)$ *be an* n*-tuple and* $(w_1,\ldots,w_k)$ *be a* k*-tuple of vectors* $z_i, w_j \in H$, *such that* $\sum_{i=1}^{n} \| z_i \|^2 = 1$ *and* $\sum_{i=1}^{k} \| w_i \|^2 = 1$.

If $f \in \mathscr{K}_1^c((a,b))$ *and* $\alpha \mathrm{x} \leqslant \alpha \mathrm{y}$, *then*

$$\sum_{i=1}^{n} \left\langle f(X_i) z_i, z_i \right\rangle - f(\bar{\mathrm{x}}) \leqslant \frac{\alpha}{2} \mathrm{x} \leqslant \frac{\alpha}{2} \mathrm{y} \leqslant \sum_{j=1}^{k} \left\langle f(Y_j) w_j, w_j \right\rangle - f(\bar{\mathrm{y}}), \tag{11}$$

holds, where

$$\mathrm{x} := \sum_{i=1}^{n}\left\langle\left(X_i - \bar{\mathrm{x}}1_H\right)^2 z_i, z_i\right\rangle, \qquad \bar{\mathrm{x}} := \sum_{i=1}^{n}\left\langle X_i z_i, z_i\right\rangle,$$
$$\mathrm{y} := \sum_{j=1}^{k}\left\langle\left(Y_j - \bar{\mathrm{y}}1_H\right)^2 w_j, w_j\right\rangle, \qquad \bar{\mathrm{y}} := \sum_{j=1}^{k}\left\langle Y_j w_j, w_j\right\rangle.$$

If $f \in \mathscr{K}_2^c((a,b))$ *and* $\alpha\mathrm{x} \geqslant \alpha\mathrm{y}$ *hold, then reverse inequalities are valid in* (11).

Proof. We use the same technique as in the proof of Theorem 1. For the sake of completeness, we give the proof.

Let $f \in \mathscr{K}_1^c((a,b))$. So, here is a constant α such that $F(t) = f(t) - \frac{\alpha}{2}t^2$ is concave on $[m_x, c] \subset (a,c]$. Then the converse of Jensen's inequality for a concave function implies

$$0 \leqslant F(\bar{\mathrm{x}}) - \sum_{i=1}^{n}\left\langle F(X_i)z_i, z_i\right\rangle = f(\bar{\mathrm{x}}) - \frac{\alpha}{2}\bar{\mathrm{x}}^2 - \sum_{i=1}^{n}\left\langle f(X_i)z_i, z_i\right\rangle + \frac{\alpha}{2}\sum_{i=1}^{n}\left\langle X_i^2 z_i, z_i\right\rangle.$$

So

$$\sum_{i=1}^{n}\left\langle f(X_i)z_i, z_i\right\rangle - f(\bar{\mathrm{x}}) \leqslant \frac{\alpha}{2}\sum_{i=1}^{n}\left\langle X_i^2 z_i, z_i\right\rangle - \frac{\alpha}{2}\bar{\mathrm{x}}^2 = \frac{\alpha}{2}\mathrm{x}. \tag{12}$$

Similarly, since F is convex on $[c, M_y] \subset [c,b)$, Jensen's inequality implies

$$\frac{\alpha}{2}\mathrm{y} = \frac{\alpha}{2}\sum_{j=1}^{k}\left\langle Y_j^2 w_j, w_j\right\rangle - \frac{\alpha}{2}\bar{\mathrm{y}}^2 \leqslant \sum_{j=1}^{k}\left\langle f(Y_j)w_j, w_j\right\rangle - f(\bar{\mathrm{y}}). \tag{13}$$

Combining inequalities (12) and (13) and taking into account that $\alpha\mathrm{x} \leqslant \alpha\mathrm{y}$, we obtain (11). □

4. Quasi-arithmetic means

We define the quasi-arithmetic mean of operators:

$$\mathfrak{M}_\varphi(\mathbf{X}, \mathbf{\Phi}, n) := \varphi^{-1}\left(\sum_{i=1}^{n}\Phi_i\left(\varphi(X_i)\right)\right), \tag{14}$$

where $(X_1, \ldots, X_n)$ is an n-tuple of self-adjoint operators in $\mathscr{B}_h(H)$ with spectra in I, $(\Phi_1, \ldots, \Phi_n)$ is a unital n-tuple of positive linear mappings $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$ and $\varphi : I \to \mathbb{R}$ is a strictly monotone function. There have been many works devoted to observing the order among these means, see e.g. [7, 8, 4, 3].

The power mean is a special case of the quasi-arithmetic mean:

$$\mathfrak{M}_r(\mathbf{X}, \mathbf{\Phi}, n) := \begin{cases} \left(\sum_{i=1}^{n}\Phi_i\left(X_i^r\right)\right)^{1/r}, & r \in \mathbb{R}\backslash\{0\}, \\ \exp\left(\sum_{i=1}^{n}\Phi_i\left(\ln(X_i)\right)\right), & r = 0, \end{cases} \tag{15}$$

where $X_1, \ldots, X_n$ are positive operators.

4.1. Results with operator convexity and concavity

In this subsection we give order among quasi-arithmetic means under similar conditions as in Section 2.

First, as a generalization of [2, Corollary] on operators and quasi-arithmetic means, we obtain the following results.

THEOREM 5. *Let $(X_1,\dots,X_n)$ be an n-tuple and $(Y_1,\dots,Y_k)$ be a k-tuple of self-adjoint operators $X_i, Y_j \in \mathscr{B}_h(H)$ with spectra contained in $[m_x, M_x]$ and $[m_y, M_y]$, respectively, such that $a < m_x \leqslant M_x \leqslant c_1 \leqslant m_y \leqslant M_y < b$ for some $a, b, c_1 \in \mathbb{R}$. Let $(\Phi_1,\dots,\Phi_n)$ be a unital n-tuple and $(\Psi_1,\dots,\Psi_k)$ be a unital k-tuple of positive linear mappings $\Phi_i, \Psi_j : \mathscr{B}(H) \to \mathscr{B}(K)$. Let $\psi, \varphi : (a,b) \to \mathbb{R}$ be strictly monotone functions, $c = \varphi(c_1)$ and I is the open interval between $\varphi(a)$ and $\varphi(b)$.*

If $\psi \circ \varphi^{-1} \in \dot{\mathscr{K}}_1^c(I)$ and $\alpha X_\varphi \leqslant \alpha Y_\varphi$, then

$$\begin{aligned} &\psi\big(\mathfrak{M}_\psi(\mathbf{X},\mathbf{\Phi},n)\big) - \psi\big(\mathfrak{M}_\varphi(\mathbf{X},\mathbf{\Phi},n)\big) \\ &\leqslant \tfrac{\alpha}{2} X_\varphi \leqslant \tfrac{\alpha}{2} Y_\varphi \leqslant \psi\big(\mathfrak{M}_\psi(\mathbf{Y},\mathbf{\Psi},k)\big) - \psi\big(\mathfrak{M}_\varphi(\mathbf{Y},\mathbf{\Psi},k)\big), \end{aligned} \tag{16}$$

where

$$\begin{aligned} X_\varphi &:= \sum_{i=1}^n \Phi_i\big(\varphi(X_i)^2\big) - \Big(\sum_{i=1}^n \Phi_i\left(\varphi(X_i)\right)\Big)^2, \\ Y_\varphi &:= \sum_{j=1}^k \Psi_j\big(\varphi(Y_j)^2\big) - \Big(\sum_{j=1}^k \Psi_j\left(\varphi(Y_j)\right)\Big)^2. \end{aligned} \tag{17}$$

If $\psi \circ \varphi^{-1} \in \dot{\mathscr{K}}_2^c(I)$ and $\alpha X_\varphi \geqslant \alpha Y_\varphi$ hold, then reverse inequalities are valid in (16).

Proof. Suppose that φ is a strictly increasing function in (a,b).

Then $m_x 1_H \leqslant X_i \leqslant M_x 1_H$ and $m_y 1_H \leqslant Y_j \leqslant M_y 1_H$ implies $\varphi(m_x)1_H \leqslant \varphi(X_i) \leqslant \varphi(M_x)1_H$ and $\varphi(m_y)1_H \leqslant \varphi(Y_j) \leqslant \varphi(M_y)1_H$ for $i = 1,\dots,n$, $j = 1,\dots,k$. Furthermore, $a < m_x \leqslant M_x \leqslant c_1 \leqslant m_y \leqslant M_y < b$ implies $\varphi(a) < \varphi(m_x) \leqslant \varphi(M_x) \leqslant c \leqslant \varphi(m_y) \leqslant \varphi(M_y) < \varphi(b)$.

So, for a function $f \in \dot{\mathscr{K}}_1^c((\varphi(a), \varphi(b)))$ there is a constant α such that $F(t) = f(t) - \frac{\alpha}{2}t^2$ is operator concave on $[\varphi(m_x), c] \subset (\varphi(a), c]$. Then the converse of Jensen's inequality for operators $(\varphi(X_1),\dots,\varphi(X_n))$ gives

$$\sum_{i=1}^n \Phi_i\big(f(\varphi(X_i))\big) - f\Big(\sum_{i=1}^n \Phi_i(\varphi(X_i))\Big) \leqslant \frac{\alpha}{2}\left[\sum_{i=1}^n \Phi_i\big(\varphi(X_i)^2\big) - \Big(\sum_{i=1}^n \Phi_i(\varphi(X_i))\Big)^2\right].$$

Also, since F is operator convex on $[c, \varphi(M_y)] \subset [c, \varphi(b))$, Jensen's inequality for operators $(\varphi(Y_1),\dots,\varphi(Y_k))$ gives

$$\frac{\alpha}{2}\left[\sum_{j=1}^k \Psi_j\big(\varphi(Y_j)^2\big) - \Big(\sum_{j=1}^k \Psi_j(\varphi(Y_j))\Big)^2\right] \leqslant \sum_{j=1}^k \Psi_j\big(f(\varphi(Y_j))\big) - f\Big(\sum_{j=1}^k \Psi_j(\varphi(Y_j))\Big).$$

Setting $f=\psi\circ\varphi^{-1}$ in above two inequalities and taking into account that $\alpha X_\varphi \leqslant \alpha Y_\varphi$ holds, we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\Phi_i\big(\psi(X_i)\big)-\psi\circ\varphi^{-1}\Big(\sum_{i=1}^{n}\Phi_i(\varphi(X_i))\Big)\leqslant \tfrac{\alpha}{2}X_\varphi\\ &\leqslant \tfrac{\alpha}{2}Y_\varphi \leqslant \sum_{j=1}^{k}\Psi_j\big(\psi(Y_j)\big)-\psi\circ\varphi^{-1}\Big(\sum_{j=1}^{k}\Psi_j(\varphi(Y_j))\Big),\end{aligned}$$

which gives the desired inequality (16).

Analogously, we can prove (16) in the case when φ is a strictly decreasing function. □

Setting ψ equal to the identity function in Theorem 5, we can obtain inequality (18). We give this result with weakened assumptions.

THEOREM 6. *Let mappings Φ_i, Ψ_j, operators X_i, Y_j and a,b,c_1 be as in Theorem 5. Let $f:(a,b)\to\mathbb{R}$ such that $\varphi := f|_{(a,c_1]}$, $\psi := f|_{[c_1,b)}$ be strictly monotone functions, $c=\varphi(c_1)$ and I is the open interval between $f(a)$ and $f(b)$.*

If $f^{-1}\in \dot{\mathscr{K}}_1^c(I)$ and $\alpha Y_\psi \leqslant \alpha X_\varphi$, then

$$\mathfrak{M}_\psi(\mathbf{Y},\boldsymbol{\Psi},k)-\mathfrak{M}_1(\mathbf{Y},\boldsymbol{\Psi},k)\leqslant \frac{\alpha}{2}Y_\psi \leqslant \frac{\alpha}{2}X_\varphi \leqslant \mathfrak{M}_\varphi(\mathbf{X},\boldsymbol{\Phi},n)-\mathfrak{M}_1(\mathbf{X},\boldsymbol{\Phi},n), \tag{18}$$

where

$$\begin{aligned}Y_\psi &:= \Big(\sum_{j=1}^{k}\Psi_j\left(\psi(Y_j)\right)\Big)^2-\sum_{j=1}^{k}\Psi_j\big(\psi(Y_j)^2\big),\\ X_\varphi &:= \Big(\sum_{i=1}^{n}\Phi_i\left(\varphi(X_i)\right)\Big)^2-\sum_{i=1}^{n}\Phi_i\big(\varphi(X_i)^2\big),\end{aligned}$$

If $f^{-1}\in \dot{\mathscr{K}}_2^c(I)$ and $\alpha Y_\psi \geqslant \alpha X_\varphi$ holds, then reverse inequalities are valid in (18).

Proof. If both functions φ and ψ are strictly increasing or decreasing these results follow directly from Theorem 5. The remaining two cases can be proven by using the same technique as in the proof of Theorem 5. □

REMARK 3. Let Φ_i, Ψ_j be mappings, X_i, Y_j be positive operators as in Theorem 5 and $0<m_x\leqslant M_x\leqslant c\leqslant m_y\leqslant M_y<b$. Setting $f(t)=t^s$, $s\geqslant 1$ for $t\in(0,c]$, $f(t)=dt^r$, $r\leqslant -1$ or $\frac{1}{2}\leqslant r\leqslant 1$ for $t\in[c,\infty)$, where $d=c^{s/r}$ and $\alpha=0$ in Theorem 6, we obtain order among power means as follows

$$\mathfrak{M}_r(\mathbf{Y},\boldsymbol{\Psi},k)-\mathfrak{M}_1(\mathbf{Y},\boldsymbol{\Psi},k)\leqslant 0\leqslant \mathfrak{M}_s(\mathbf{X},\boldsymbol{\Phi},n)-\mathfrak{M}_1(\mathbf{X},\boldsymbol{\Phi},n). \tag{19}$$

We remark that (19) holds for all positive operators $X_1,\dots,X_n$, Y_1, $\dots,Y_k$ without condition $M_x\leqslant c\leqslant m_y$. Really, LHS (resp. RHS) of (19) holds since $t\mapsto t^s$ (resp. $t\mapsto t^r$) is operator concave (resp. operator convex) on $(0,\infty)$, see [8, 4].

Setting $\alpha\neq 0$ in Theorem 6, we can obtain a refinement of (19) for some r,s. In this way we obtain inequalities for instance as in the following corollary.

COROLLARY 7. *Let Φ_i, Ψ_j be mappings, X_i, Y_j be positive operators as in Theorem 5 and $0 < m_x \leqslant M_x \leqslant c \leqslant m_y \leqslant M_y < b$.*

(*i*) *If $s \geqslant 1$ and $C_{1/2} \leqslant C_s$, then*

$$\mathfrak{M}_{1/2}(\mathbf{Y},\boldsymbol{\Psi},k) - \mathfrak{M}_1(\mathbf{Y},\boldsymbol{\Psi},k) \leqslant \alpha C_{1/2} \leqslant \alpha C_s \leqslant \mathfrak{M}_s(\mathbf{X},\boldsymbol{\Phi},n) - \mathfrak{M}_1(\mathbf{X},\boldsymbol{\Phi},n) \quad (20)$$

for every $\alpha \in (0, 2c^{1-2s})$, where

$$C_{1/2} := \Big(\sum_{j=1}^{k} \Psi_i(\sqrt{Y_j})\Big)^2 - \sum_{j=1}^{k} \Psi_j\big(Y_j\big), \quad C_s := \Big(\sum_{i=1}^{n} \Phi_i(X_i^s)\Big)^2 - \sum_{i=1}^{n} \Phi_i\big(X_i^{2s}\big).$$

(*ii*) *If $C_{1/2} \leqslant C_{\exp}$, then*

$$\mathfrak{M}_{1/2}(\mathbf{Y},\boldsymbol{\Psi},k) - \mathfrak{M}_1(\mathbf{Y},\boldsymbol{\Psi},k) \leqslant \alpha C_{1/2} \leqslant \alpha C_{\exp} \leqslant \mathfrak{M}_{\exp}(\mathbf{X},\boldsymbol{\Phi},n) - \mathfrak{M}_1(\mathbf{X},\boldsymbol{\Phi},n) \quad (21)$$

for every $\alpha \in (0, 2c^{1-2s})$, where

$$C_{\exp} := \Big(\sum_{i=1}^{n} \Phi_i(\exp X_i)\Big)^2 - \sum_{i=1}^{n} \Phi_i\big((\exp X_i)^2\big).$$

Proof. (*i*) We set $f(t) = t^s$, $s \geqslant 1$ for $t \in (0,c]$ and $f(t) = d\sqrt{t}$ for $t \in [c,\infty)$, where $d = c^{s-1/2}$. Then f is strictly monotone on $(0,\infty)$ and $f^{-1} \in \dot{\mathscr{K}}_1^c((0,\infty))$ for every $\alpha \in (0, 2/a^2)$. So, $\alpha Y_\psi \leqslant \alpha X_\varphi$ and (18) give $\alpha C_{1/2} \leqslant \alpha C_s$ and (20), since the power means are positively homogeneous.

(*iii*) Setting $f(t) = \exp(t)$ for $t \in (0,c]$, $f(t) = d\sqrt{t}$ for $t \in [c,\infty)$, where $d = \exp(c)/\sqrt{c}$, and using the same technique as above, we obtain $C_{1/2} \leqslant C_{\exp}$ and (21). □

4.2. Results without operator convexity and concavity

For wider application it is interesting to consider inequalities involving quasi-arithmetic means under similar conditions as in Section 3. Thus, if spectra conditions hold, then (16) is valid for all strictly monotone functions $\varphi, \psi : (a,b) \to \mathbb{R}$ such that $\psi \circ \varphi^{-1} \in \mathscr{K}_1^c(I)$ and (18) is valid for every strictly monotone function $f : (a,b) \to \mathbb{R}$ such that $f^{-1} \in \mathscr{K}_1^c(I)$. Now we give these results.

THEOREM 8. *Let $(X_1,\ldots,X_n)$ be an n-tuple and $(Y_1,\ldots,Y_k)$ be a k-tuple of self-adjoint operators $X_i, Y_j \in \mathscr{B}_h(H)$. Let m_{X_i}, M_{X_i} be bounds of X_i and m_{Y_j}, M_{Y_j} be bounds of Y_j, such that $a < m_{X_i} \leqslant M_{X_i} \leqslant c_1 \leqslant m_{Y_j} \leqslant M_{Y_j} < b$ for some $a,b,c \in \mathbb{R}$ and every $i = 1,\ldots,n$, $j = 1,\ldots,k$. Let $(\Phi_1,\ldots,\Phi_n)$ be a unital n-tuple and $(\Psi_1,\ldots,\Psi_k)$ be a unital k-tuple of positive linear mappings $\Phi_i, \Psi_j : \mathscr{B}(H) \to \mathscr{B}(K)$. Let $m_{\varphi_X}, M_{\varphi_X}$ and $m_{\varphi_Y}, M_{\varphi_Y}$ be bounds of $\mathfrak{M}_\varphi(\mathbf{X},\boldsymbol{\Phi},n)$ and $\mathfrak{M}_\varphi(\mathbf{Y},\boldsymbol{\Psi},k)$, respectively, such that*

$$\begin{aligned} (m_{\varphi_X}, M_{\varphi_X}) \cap [m_{X_i}, M_{X_i}] &= \varnothing, \quad i = 1,\ldots,n, \\ (m_{\varphi_Y}, M_{\varphi_Y}) \cap [m_{Y_j}, M_{Y_j}] &= \varnothing, \quad j = 1,\ldots,k. \end{aligned} \quad (22)$$

Let $\psi, \varphi : (a,b) \to \mathbb{R}$ be strictly monotone functions, $c = \varphi(c_1)$ and I is the open interval between $\varphi(a)$ and $\varphi(b)$.

If $\psi \circ \varphi^{-1} \in \dot{\mathscr{K}}_1^c(I)$ *and* $\alpha X_\varphi \leqslant \alpha Y_\varphi$, *then*

$$\begin{aligned} &\psi\big(\mathfrak{M}_\psi(\mathbf{X},\mathbf{\Phi},n)\big) - \psi\big(\mathfrak{M}_\varphi(\mathbf{X},\mathbf{\Phi},n)\big) \\ &\leqslant \tfrac{\alpha}{2}X_\varphi \leqslant \tfrac{\alpha}{2}Y_\varphi \leqslant \psi\big(\mathfrak{M}_\psi(\mathbf{Y},\mathbf{\Psi},k)\big) - \psi\big(\mathfrak{M}_\varphi(\mathbf{Y},\mathbf{\Psi},k)\big), \end{aligned} \tag{$**$}$$

is valid (i.e. (16) *is valid), where* X_φ, Y_φ *are defined by* (17)*.*

If $\psi \circ \varphi^{-1} \in \dot{\mathscr{K}}_2^c(I)$ *and* $\alpha X_\varphi \geqslant \alpha Y_\varphi$ *hold, then reverse inequalities are valid in* $(**)$.

Proof. Suppose that φ is a strictly increasing function in (a,b).
Then $m_{X_i}1_H \leqslant X_i \leqslant M_{X_i}1_H$ and $m_{\varphi_X}1_K \leqslant \mathfrak{M}_\varphi(\mathbf{X},\mathbf{\Phi},n) \leqslant M_{\varphi_X}1_K$, implies

$$\begin{aligned} &\varphi(m_{X_i})1_H \leqslant \varphi(X_i) \leqslant \varphi(M_{X_i})1_H, \quad i=1,\dots,n, \\ \text{and} \qquad &\varphi(m_{\varphi_X})1_K \leqslant \varphi\big(\mathfrak{M}_\varphi(\mathbf{X},\mathbf{\Phi},n)\big) = \sum_{i=1}^n \Phi_i(\varphi(A_i)) \leqslant \varphi(M_{\varphi_X})1_K. \end{aligned}$$

It follows that

$$\begin{aligned} &\big(m_{\varphi_X}, M_{\varphi_X}\big) \cap [m_{X_i}, M_{X_i}] = \varnothing, \quad i=1,\dots,n, \\ \Rightarrow \quad &\big(\varphi(m_{\varphi_X}), \varphi(M_{\varphi_X})\big) \cap [\varphi(m_{X_i}), \varphi(M_{X_i})] = \varnothing, \quad i=1,\dots,n. \end{aligned} \tag{23}$$

Similarly, we can prove that

$$\begin{aligned} &\big(m_{\varphi_Y}, M_{\varphi_Y}\big) \cap [m_{Y_j}, M_{Y_j}] = \varnothing, \quad j=1,\dots,k, \\ \Rightarrow \quad &\big(\varphi(m_{\varphi_Y}), \varphi(M_{\varphi_Y})\big) \cap [\varphi(m_{Y_j}), \varphi(M_{Y_j})] = \varnothing, \quad j=1,\dots,k. \end{aligned} \tag{24}$$

Furthermore, $a < m_{X_i} \leqslant M_{X_i} \leqslant c_1 \leqslant m_{Y_j} \leqslant M_{Y_j} < b$ implies

$$\varphi(a) < \varphi(m_{X_i}) \leqslant \varphi(M_{X_i}) \leqslant c \leqslant \varphi(m_{Y_j}) \leqslant \varphi(M_{Y_j}) < \varphi(b)$$

for every $i=1,\dots,n,\ j=1,\dots,k$.

Now, we use a similar technique as in the proof of Theorem 5: Since $\psi \circ \varphi^{-1} \in \mathscr{K}_1^c(I)$, there is a constant α such that $F(t) = \psi \circ \varphi^{-1}(t) - \frac{\alpha}{2}t^2$ is concave on $(\varphi(a), c]$. Then the converse of Jensen's inequality for operators $(\varphi(X_1),\dots,\varphi(X_n))$ and with spectra condition (23) gives (see [9])

$$\sum_{i=1}^n \Phi_i\big(\psi(X_i)\big) - \psi \circ \varphi^{-1}\Big(\sum_{i=1}^n \Phi_i(\varphi(X_i))\Big) \leqslant X_\varphi. \tag{25}$$

Also, since F is convex on $[c, \varphi(b))$, Jensen's inequality for operators $(\varphi(Y_1),\dots,\varphi(Y_k))$ and spectra condition (24) gives

$$Y_\varphi \leqslant \sum_{j=1}^k \Psi_j\big(\psi(Y_j)\big) - \psi \circ \varphi^{-1}\Big(\sum_{j=1}^k \Psi_j(\varphi(Y_j))\Big), \tag{26}$$

Combining (25) and (26) and taking into account that $\alpha X_\varphi \leqslant \alpha Y_\varphi$, we obtain the desired inequality $(**)$.

Analogously, we can prove $(**)$ in the case when φ is a strictly decreasing function. $\square$

Applying Theorem 8, we obtain a generalization and refining of Bullen's result [2, Corollary] for power means.

COROLLARY 9. *Let Φ_i, Ψ_j be mappings, X_i, Y_j be positive operators as in Theorem 8 and $0 < m_{X_i} \leqslant M_{X_i} \leqslant c \leqslant m_{Y_j} \leqslant M_{Y_j} < b$ for some $a,b,c \in \mathbb{R}$ and every $i = 1,\dots,n$, $j = 1,\dots,k$. Let $m_{\varphi_X}, M_{\varphi_X}$ and $m_{\varphi_Y}, M_{\varphi_Y}$ be bounds of $\mathfrak{M}_\varphi(\mathbf{X},\boldsymbol{\Phi},n)$ and $\mathfrak{M}_\varphi(\mathbf{Y},\boldsymbol{\Psi},k)$, respectively. Let*

$$C_{s_X} := \begin{cases} \left(\sum\limits_{i=1}^{n} \Phi_i(X_i^s)\right)^2 - \sum\limits_{i=1}^{n} \Phi_i\left(X_i^{2s}\right), & s \neq 0 \\ \left(\sum\limits_{i=1}^{n} \Phi_i(\ln X_i)\right)^2 - \sum\limits_{i=1}^{n} \Phi_i\left(\ln^2(X_i)\right), & s = 0, \end{cases}$$

and C_{s_Y}, C_{0_Y} be analogous notations for operators $Y_1, \dots, Y_k$. Let $\alpha = \frac{r}{s}\left(\frac{r}{s} - 1\right) c^{\frac{r}{s}-2}$ if $rs \neq 0$, $\alpha = r^2 \exp(cr)$ if $s = 0$ and $\alpha = -\frac{1}{s} c^{-2}$ if $r = 0$.

(i) If $r < 0 < s$, $2s \leqslant r \leqslant s < 0$, $s < 0 < r$ or $0 < s \leqslant r \leqslant 2s$, $C_{s_X} \leqslant C_{s_Y}$ and spectra conditions (22) hold, then

$$\mathfrak{M}_s(\mathbf{X},\boldsymbol{\Phi},n)^r - \mathfrak{M}_r(\mathbf{X},\boldsymbol{\Phi},n)^r \leqslant \alpha C_{s_X} \leqslant \alpha C_{s_Y} \leqslant \mathfrak{M}_s(\mathbf{Y},\boldsymbol{\Psi},k)^r - \mathfrak{M}_r(\mathbf{Y},\boldsymbol{\Psi},k)^r. \quad (27)$$

(ii) If $r \leqslant 2s < 0$, $0 < 2s \leqslant r$, $C_{s_X} \geqslant C_{s_Y}$ or $s \leqslant r < 0$ or $0 < r \leqslant s$, $C_{s_X} \leqslant C_{s_Y}$ and (22) holds, then reverse inequalities are valid in (27).

(iii) If $r < 0$, $C_{0_X} \leqslant C_{0_Y}$ and (22) holds, then (27) is valid for $s = 0$.

But, if $r > 0$, $C_{0_X} \geqslant C_{0_Y}$ and (22) holds, then reverse inequalities are valid in (27) for $s = 0$.

(iv) If $s > 0$, $C_{s_X} \leqslant C_{s_Y}$ and (22) holds, then

$$\ln\left(\frac{\mathfrak{M}_0(\mathbf{X},\boldsymbol{\Phi},n)}{\mathfrak{M}_s(\mathbf{X},\boldsymbol{\Phi},n)}\right) \leqslant \alpha C_{s_X} \leqslant \alpha C_{s_Y} \leqslant \ln\left(\frac{\mathfrak{M}_0(\mathbf{Y},\boldsymbol{\Psi},k)}{\mathfrak{M}_s(\mathbf{Y},\boldsymbol{\Psi},k)}\right). \quad (28)$$

If $s < 0$, $C_{s_X} \geqslant C_{s_Y}$ and spectra conditions:

$$(m_{\varphi_Y}, M_{\varphi_Y}) \cap [m_{Y_i}, M_{Y_i}] = \varnothing, \quad i = 1,\dots,k, \quad (29)$$

hold, then reverse inequalities are valid in (28).

Proof. (i)–(ii): We set $\varphi(t) = t^s$, $\psi(t) = t^r$ and $f(t) = t^{\frac{r}{s}}$, $r,s \neq 0$. Let us consider a function $F(t) = t^{\frac{r}{s}} - \frac{\alpha}{2} t^2$ for $\alpha = \frac{r}{s}(\frac{r}{s} - 1) c^{\frac{r}{s}-2}$. Since $F''(t) = \frac{r}{s}(\frac{r}{s} - 1)(t^{\frac{r}{s}-2} - c^{\frac{r}{s}-2})$, then c is inflection point of F.

If $\frac{r}{s} < 0$ or $1 \leqslant \frac{r}{s} \leqslant 2$, then $f \in \mathscr{K}_2^c((0,\infty))$ and $\alpha > 0$. So, applying Theorem 8 we obtain (27) in the case (i).

If $\frac{r}{s} \geqslant 2$, then $\alpha > 0$ or if $0 < \frac{r}{s} \leqslant 1$, then $\alpha < 0$. Also, $f \in \mathscr{K}_1^c((0,\infty))$ and reverses of inequalities (27) hold in the case (ii).

(iii): If $s = 0$, we set $\varphi(t) = \ln t$, $\psi(t) = t^r$ and $f(t) = \exp(tr)$, $r \neq 0$. Let $F(t) = \exp(tr) - \frac{\alpha}{2}t^2$ for $\alpha = r^2 \exp(cr)$. Then $f \in \mathscr{K}_1^c((0,\infty))$ for $r > 0$, or $f \in \mathscr{K}_2^c((0,\infty))$ for $r < 0$. So, applying Theorem 8 we obtain (27) under conditions $r > 0$, $C_{0_X} \geqslant C_{0_Y}$ and (22). Or, we obtain (27) under conditions $r < 0$, $C_{0_X} \leqslant C_{0_Y}$ and (22).

(iv): If $r = 0$, we set $\varphi(t) = t^s$, $\psi(t) = \ln t$ and $f(t) = \frac{1}{s}\ln t$, $s \neq 0$. Let $F(t) = \frac{1}{s}\ln t - \frac{\alpha}{2}t^2$ for $\alpha = -\frac{1}{sc^2}$. Then $f \in \mathscr{K}_1^c((0,\infty))$ for $s > 0$, or $f \in \mathscr{K}_3^c((0,\infty))$ for $s < 0$. So, applying Theorem 8 we obtain (28) under conditions $s < 0$, $C_{s_X} \geqslant C_{s_Y}$ and (22). Or, we obtain (28) under conditions $s > 0$, $C_{s_X} \leqslant C_{s_Y}$ and (29). $\square$

Finally, we give version of Theorem 6 without operator convexity or concavity. The proof is similar to the one for Theorem 8 and we omit it.

THEOREM 10. *Let Φ_i, Ψ_j be mappings, X_i, Y_j be operators as in Theorem 8 and $a < m_{X_i} \leqslant M_{X_i} \leqslant c_1 \leqslant m_{Y_j} \leqslant M_{Y_j} < b$ for some $a,b,c \in \mathbb{R}$ and every $i = 1,\dots,n$, $j = 1,\dots,k$. Let $m_{\varphi_X}, M_{\varphi_X}$ and $m_{\varphi_Y}, M_{\varphi_Y}$ be bounds of $\mathfrak{M}_\varphi(\mathbf{X},\mathbf{\Phi},n)$ and $\mathfrak{M}_\varphi(\mathbf{Y},\mathbf{\Psi},k)$, respectively, such that spectra conditions (22) hold. Let $f : (a,b) \to \mathbb{R}$ such that $\varphi := f|_{(a,c_1]}$, $\psi := f|_{[c_1,b)}$ be strictly monotone functions, $c = \varphi(c_1)$ and I is the open interval between $f(a)$ and $f(b)$.*

If $f^{-1} \in \mathscr{K}_1^c(I)$ and $\alpha Y_\psi \leqslant \alpha X_\varphi$, then (18) is valid, where X_φ, Y_φ are defined by (17).

If $f^{-1} \in \mathscr{K}_2^c(I)$ and $\alpha Y_\psi \geqslant \alpha X_\varphi$ holds, then reverse inequalities are valid in (18).

REMARK 4. By setting $r = 1$ in Corollary 9, we obtain order between $\mathfrak{M}_s(\mathbf{Y},\mathbf{\Psi},k)$ and $\mathfrak{M}_s(\mathbf{X},\mathbf{\Phi},n)$. Applying Theorem 10, we can obtain another order among power means as follows. Setting $f(t) = t^s$, $s \geqslant 1$ for $t \in (0,c]$, $f(t) = dt^r$, $r \leqslant 1$ for $t \in [c,\infty)$, where $d = c^{s/r}$ and $\alpha = 0$ in this theorem, we obtain obvious inequality

$$\mathfrak{M}_r(\mathbf{Y},\mathbf{\Psi},k) - \mathfrak{M}_1(\mathbf{Y},\mathbf{\Psi},k) \leqslant 0 \leqslant \mathfrak{M}_s(\mathbf{X},\mathbf{\Phi},n) - \mathfrak{M}_1(\mathbf{X},\mathbf{\Phi},n)$$

under spectra conditions (22).

But, for some $\alpha > 0$, we can obtain refining of the above inequalities as follows:

Let C_{s_X}, C_{r_Y} and C_{0_Y} as in Corollary 6. If $s \geqslant 1$, $0 \leqslant r \leqslant 1$, $C_{r_Y} \leqslant C_{s_X}$ and spectra conditions (29) hold, then

$$\mathfrak{M}_r(\mathbf{Y},\mathbf{\Psi},k) - \mathfrak{M}_1(\mathbf{Y},\mathbf{\Psi},k) \leqslant \alpha C_{s_Y} \leqslant \alpha C_{s_X} \leqslant \mathfrak{M}_s(\mathbf{X},\mathbf{\Phi},n) - \mathfrak{M}_1(\mathbf{X},\mathbf{\Phi},n)$$

is valid for every $\alpha \in (0, c^{-2+(1-r)/r^2}(1-r)/r^2)$.

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