

## LEVINSON'S INEQUALITY FOR HILBERT SPACE OPERATORS

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(Communicated by N. Elezović)

*Abstract.* Levinson's operator inequality is given for unital fields of positive linear mappings and a large class of functions. Order among quasi-arithmetic means is considered under similar conditions.

### 1. Introduction

Let  $\mathcal{B}(H)$  be the algebra of all bounded linear operators on a complex Hilbert space  $H$ . We denote by  $\mathcal{B}_h(H)$  the real subspace of all self-adjoint operators on  $H$ . Bounds of  $X \in \mathcal{B}_h(H)$  are defined by  $m_X := \inf \{ \langle X\xi, \xi \rangle : \xi \in H, \|\xi\| = 1 \}$  and  $M_X := \sup \{ \langle X\xi, \xi \rangle : \xi \in H, \|\xi\| = 1 \}$

A continuous real valued function  $f$  defined on an interval  $I$  is said to be operator convex if  $f(\lambda X + (1 - \lambda)Y) \leq \lambda f(X) + (1 - \lambda)f(Y)$  for all self-adjoint operators  $X, Y$  with spectra contained in  $I$  and all  $\lambda \in [0, 1]$ . If the function  $f$  is operator convex, then so-called Jensen's operator inequality  $f(\Phi(X)) \leq \Phi(f(X))$  holds for any unital positive linear mapping  $\Phi$  on  $\mathcal{B}(H)$  and any  $X \in \mathcal{B}_h(H)$  with spectrum contained in  $I$ . Many other versions of Jensen's operator inequality can be found in [4, 3].

Assume furthermore that  $(\Phi_1, \dots, \Phi_n)$  is an  $n$ -tuple of positive linear mappings  $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ . If in addition  $\sum_{i=1}^n \Phi_i(1_H) = 1_K$ , we say that  $\sum_{i=1}^n \Phi_i(1_H) = 1_K$  is unital.

Now we give the definition of classes of functions for which we observe Levinson's operator inequality:

**DEFINITION 1.** Let  $f \in \mathcal{C}(I)$  be a real valued function on an arbitrary interval  $I$  in  $\mathbb{R}$  and  $c \in I^\circ$ , where  $I^\circ$  is the interior of  $I$ .

We say that  $f \in \mathcal{K}_1^c(I)$  (resp.  $f \in \mathcal{K}_2^c(I)$ ) if there exists a constant  $\alpha$  such that the function  $F(t) = f(t) - \frac{\alpha}{2}t^2$  is concave (resp. convex) on  $I \cap (-\infty, c]$  and convex (resp. concave) on  $I \cap [c, \infty)$ .

Moreover, we say that  $f \in \dot{\mathcal{K}}_1^c(I)$  (resp.  $f \in \dot{\mathcal{K}}_2^c(I)$ ) if there exists a constant  $\alpha$  such that the function  $F(t) = f(t) - \frac{\alpha}{2}t^2$  is operator concave (resp. operator convex) on  $I \cap (-\infty, c]$  and operator convex (resp. operator concave) on  $I \cap [c, \infty)$ .

*Mathematics subject classification* (2010): Primary 47A63; Secondary 47B15, 47A64.

*Keywords and phrases:* Self-adjoint operator, positive linear mapping, convex function, Levinson's inequality, quasi-arithmetic mean.

$\mathcal{K}_1^c(I)$  can be interpreted as the class of “3-convex functions at point  $c$ ” and generalizes 3-convex functions in the following sense: a function is 3-convex on  $I$  if and only if it is at every  $c \in I^\circ$ .

Next, we will review the history of research of Levinson’s inequality.

Levinson [5] considered an inequality as follows:

If  $f : (0, 2c) \rightarrow \mathbb{R}$  satisfies  $f'' \geq 0$  and  $p_i, x_i, y_i, i = 1, 2, \dots, n$ , are such that  $p_i > 0, \sum_{i=1}^n p_i = 1, 0 \leq x_i \leq c$  and

$$x_1 + y_1 = x_2 + y_2 = \dots = x_n + y_n = 2c, \tag{1}$$

then the inequality

$$\sum_{i=1}^n p_i f(x_i) - f(\bar{x}) \leq \sum_{i=1}^n p_i f(y_i) - f(\bar{y}) \tag{2}$$

holds, where  $\bar{x} = \sum_{i=1}^n p_i x_i$  and  $\bar{y} = \sum_{i=1}^n p_i y_i$  denote the weighted arithmetic means.

Popoviciu [11] showed that the assumptions on the differentiability of  $f$  can be weakened and for (2) to hold it is enough to assume that  $f$  is 3-convex. Bullen [2] gave another proof of Popoviciu’s result rescaled to a general interval  $[a, b]$  as follows:

If  $f : [a, b] \rightarrow \mathbb{R}$  is 3-convex and  $p_i, x_i, y_i, i = 1, 2, \dots, n$ , are such that  $p_i > 0, \sum_{i=1}^n p_i = 1, a \leq x_i, y_i \leq b, (1)$  holds (for some  $c \in [a, b]$ ) and

$$\max(x_1, \dots, x_n) \leq \min(y_1, \dots, y_n), \tag{3}$$

then (2) holds.

Mercer [6] made a significant improvement by replacing (1) with the weaker condition that the variances of the two sequences are equal:

$$\sum_{i=1}^n p_i (x_i - \bar{x})^2 = \sum_{i=1}^n p_i (y_i - \bar{y})^2, \tag{4}$$

Witkowski [12, 13] extended this result in several ways. Firstly, he showed that Levinson’s inequality can be stated in a more general setting with random variables. Furthermore, he showed that it is enough to assume that  $f$  is 3-convex and that the assumption of equality of the variances can be weakened to inequality in a certain direction.

Baloch, Pečarić and Praljak [1] built on and extend the methods of Witkowski [12]. They introduced a new class of functions  $\mathcal{K}_1^c((a, b))$  as in Definition 1. Their main result is that this class is the largest class of functions for which Levinson’s inequality holds under Mercer’s assumptions as follows:

(i) Let  $a < x_i \leq c \leq y_i < b, p_i > 0$  for  $i = 1, 2, \dots, n, \sum_{i=1}^n p_i = 1$  and (4) holds. If  $f \in \mathcal{K}_1^c((a, b))$ , then inequality (2) holds and if  $f \in \mathcal{K}_1^c((a, b))$ , then (2) holds with reverse sign of inequality.

(ii) Let  $f : (a, b) \rightarrow \mathbb{R}$  be continuous and  $c \in (a, b)$ . If inequality (2) (resp. the reverse of (2)) holds for every  $n \in \mathbb{N}$  and sequences  $p_i, x_i, y_i, i = 1, \dots, n$ , such that  $p_i > 0, \sum_{i=1}^n p_i = 1, a < x_i \leq c \leq y_i < b$  and (4) holds, then  $f \in \mathcal{K}_1^c((a, b))$  (resp.  $f \in \mathcal{K}_2^c((a, b))$ ).

In this paper, we give a general formulation of Levinson's operator inequality. In Section 2 we will prove the operator version of inequality (2) for  $f \in \mathcal{K}_1^{\bullet c}(I)$ , which will generalize the above result (i) by Baloch, Pečarić and Praljak. In Section 3 by adding spectra conditions we will prove the same version of (2) for  $f \in \mathcal{K}_1^c(I)$ . In Section 4 we will give order among quasi-arithmetic means under similar conditions. As an application we give order among power means, which will generalize Bullen's results [2, Corollary].

### 2. Levinson's inequality with operator convexity and concavity

In this section we give version of Levinson's operator inequality for  $f \in \mathcal{K}_i^{\bullet c}(I)$ ,  $i = 1, 2$ .

**THEOREM 1.** *Let  $(X_1, \dots, X_n)$  be an  $n$ -tuple and  $(Y_1, \dots, Y_k)$  be a  $k$ -tuple of self-adjoint operators  $X_i, Y_j \in \mathcal{B}_h(H)$  with spectra contained in  $[m_x, M_x]$  and  $[m_y, M_y]$ , respectively, such that  $a < m_x \leq M_x \leq c \leq m_y \leq M_y < b$  for some  $a, b, c \in \mathbb{R}$ . Let  $(\Phi_1, \dots, \Phi_n)$  be a unital  $n$ -tuple and  $(\Psi_1, \dots, \Psi_k)$  be a unital  $k$ -tuple of positive linear mappings  $\Phi_i, \Psi_j : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ .*

*If  $f \in \mathcal{K}_1^{\bullet c}((a, b))$  and  $\alpha X \leq \alpha Y$ , then*

$$\sum_{i=1}^n \Phi_i(f(X_i)) - f\left(\sum_{i=1}^n \Phi_i(X_i)\right) \leq \frac{\alpha}{2} X \leq \frac{\alpha}{2} Y \leq \sum_{j=1}^k \Psi_j(f(Y_j)) - f\left(\sum_{j=1}^k \Psi_j(Y_j)\right) \quad (5)$$

*holds, where*

$$X := \sum_{i=1}^n \Phi_i(X_i^2) - \left(\sum_{i=1}^n \Phi_i(X_i)\right)^2, \quad Y := \sum_{j=1}^k \Psi_j(Y_j^2) - \left(\sum_{j=1}^k \Psi_j(Y_j)\right)^2.$$

*If  $f \in \mathcal{K}_2^{\bullet c}((a, b))$  and  $\alpha X \geq \alpha Y$ , then the reverse inequalities are valid in (5).*

*Proof.* We will give the proof for  $f \in \mathcal{K}_1^{\bullet c}((a, b))$ . So, there is a constant  $\alpha$  such that  $F(t) = f(t) - \frac{\alpha}{2}t^2$  is operator concave on  $[m_x, c] \subset (a, c]$ . Then the reverse of Jensen's inequality for an operator concave function gives

$$\begin{aligned} 0 &\leq F\left(\sum_{i=1}^n \Phi_i(X_i)\right) - \sum_{i=1}^n \Phi_i(F(X_i)) \\ &= f\left(\sum_{i=1}^n \Phi_i(X_i)\right) - \frac{\alpha}{2}\left(\sum_{i=1}^n \Phi_i(X_i)\right)^2 - \sum_{i=1}^n \Phi_i(f(X_i)) + \frac{\alpha}{2}\sum_{i=1}^n \Phi_i(X_i^2). \end{aligned}$$

It follows

$$\sum_{i=1}^n \Phi_i(f(X_i)) - f\left(\sum_{i=1}^n \Phi_i(X_i)\right) \leq \frac{\alpha}{2} X. \quad (6)$$

Also, since  $F$  is operator convex on  $[c, M_y] \subset [c, b)$ , Jensen’s operator inequality gives

$$\begin{aligned} 0 &\leq \sum_{j=1}^k \Psi_j(F(Y_j)) - F\left(\sum_{j=1}^k \Psi_j(Y_j)\right) \\ &= \sum_{j=1}^k \Psi_j(f(Y_j)) - \frac{\alpha}{2} \sum_{j=1}^k \Psi_j(Y_j^2) - f\left(\sum_{j=1}^k \Psi_j(Y_j)\right) + \frac{\alpha}{2} \left(\sum_{j=1}^k \Psi_j(Y_j)\right)^2. \end{aligned}$$

It follows

$$\frac{\alpha}{2} Y \leq \sum_{j=1}^k \Psi_j(f(Y_j)) - f\left(\sum_{j=1}^k \Psi_j(Y_j)\right). \tag{7}$$

Combining inequalities (6) and (7) and taking into account that  $\alpha X \leq \alpha Y$ , we obtain the desired inequality (5).  $\square$

Now, we give an example to clear the situation in Theorem 1.

EXAMPLE 1. Let  $c > 1$ ,  $d = \frac{\sqrt{c^5+c^3}/2}{\sqrt{c^3-1}}$ ,  $f(t) = \sqrt{t^3} - t^2/2$  on  $(0, c]$  and  $f(t) = d\sqrt{t} - d/t - t^2$  on  $[c, \infty)$ . Then  $f \in \mathcal{K}_2^c((0, \infty))$  for  $\alpha = -2$ .

Let  $\Phi, \Psi : M_3(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  be defined by  $\Phi((m_{ij})_{1 \leq i, j \leq 3}) = \frac{1}{2}(m_{ij})_{1 \leq i, j \leq 2}$  and  $\Psi(M) = \frac{1}{3}\text{Tr}(M)I_2$  for every  $M = (m_{ij})_{1 \leq i, j \leq 3} \in M_3(\mathbb{C})$ .

If

$$X_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 3 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 15 & 3 & 2 \\ 3 & 18 & 1 \\ 2 & 1 & 9 \end{pmatrix},$$

then  $0.1I_3 < X_1, X_2 < 5.3I_3 < 8.3I_3 < Y_1 < 20.2$  and we can choose any  $c$  from  $[5.3, 8.3]$ . Next, we have

$$\begin{aligned} X &= \Phi(X_1^2) + \Phi(X_2^2) - \left(\Phi(X_1) + \Phi(X_2)\right)^2 = \begin{pmatrix} 1.5 & 0.25 \\ 0.25 & 1.75 \end{pmatrix} \\ &< Y = \Psi(Y_1^2) - \left(\Psi(Y_1)\right)^2 = \begin{pmatrix} 70/3 & 0 \\ 0 & 70/3 \end{pmatrix}. \end{aligned}$$

Then, according to Theorem 1, we obtain (rounded to six decimal places)

$$F_1 = \begin{pmatrix} 0.286126 & 0.172952 \\ 0.172952 & 0.484473 \end{pmatrix} < X < Y < F_2 = (70/3 + 0.0667891 d)I_2,$$

where

$$F_1 = \sqrt{\left(\sum_{i=1}^2 \Phi(X_i)\right)^3} - \frac{1}{2} \left(\sum_{i=1}^2 \Phi(X_i)\right)^2 - \sum_{i=1}^2 \left(\Phi(\sqrt{X_i^3}) - \frac{1}{2}\Phi(X_i^2)\right),$$

$$F_2 = d\sqrt{\Psi(Y_1)} - d\left(\Psi(Y_1)\right)^{-1} - \left(\Psi(Y_1)\right)^2 - d\Psi(\sqrt{Y_1}) + d\Psi(Y_1^{-1}) + \Psi(Y_1^2).$$

This shows that inequality (5) can be strict.

Next, we give the following obvious corollary to Theorem 1 with convex combinations of operators  $X_i, i = 1, \dots, n$  and  $Y_j, j = 1, \dots, k$ . This is a generalization of [1, Theorem 2.6].

**COROLLARY 2.** *Let  $(X_1, \dots, X_n)$  be an  $n$ -tuple and  $(Y_1, \dots, Y_k)$  be a  $k$ -tuple of self-adjoint operators  $X_i, Y_j \in \mathcal{B}_h(H)$  with spectra contained in  $[m_x, M_x]$  and  $[m_y, M_y]$ , respectively, such that  $a < m_x \leq M_x \leq c \leq m_y \leq M_y < b$  for some  $a, b, c \in \mathbb{R}$ . Let  $(p_1, \dots, p_n)$  be an  $n$ -tuple and  $(q_1, \dots, q_k)$  be a  $k$ -tuple of positive scalars such that  $\sum_{i=1}^n p_i = 1$  and  $\sum_{j=1}^k q_j = 1$ .*

*If  $f \in \mathcal{K}_1^c((a, b))$  and  $\alpha P \leq \alpha Q$ , then*

$$\sum_{i=1}^n p_i f(X_i) - f(\bar{X}) \leq \frac{\alpha}{2} P \leq \frac{\alpha}{2} Q \leq \sum_{j=1}^k q_j f(Y_j) - f(\bar{Y}) \tag{8}$$

holds, where

$$P := \sum_{i=1}^n p_i (X_i - \bar{X})^2, \quad Q := \sum_{j=1}^k q_j (Y_j - \bar{Y})^2,$$

and  $\bar{X} := \sum_{i=1}^n p_i X_i, \bar{Y} := \sum_{j=1}^k q_j Y_j$  denote the weighted arithmetic means of operators.

*If  $f \in \mathcal{K}_2^c([m_x, M_y])$  and  $\alpha P \geq \alpha Q$ , then reverse inequalities are valid in (8).*

*Proof.* We apply Theorem 1 for positive linear mappings  $\Phi_i, \Psi_j : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  determined by  $\Phi_i : B \mapsto p_i B, i = 1, \dots, n$  and  $\Psi_j : B \mapsto q_j B, j = 1, \dots, k$ .  $\square$

**REMARK 1.** 1) If  $f$  is convex (resp. concave), then  $f''_-(c) \leq \alpha \leq f''_+(c)$  (resp.  $f''_+(c) \leq \alpha \leq f''_-(c)$ ), see [1]. So, condition  $\alpha X \leq \alpha Y$  (resp.  $\alpha X \geq \alpha Y$ ) in Theorem 1 can be weakened to  $X \leq Y$  (resp.  $Y \leq X$ ).

2) Setting  $n = k, p_i = q_i$  and  $A := \sum_{i=1}^n p_i (X_i - \bar{X})^2 = \sum_{i=1}^n p_i (Y_i - \bar{Y})^2$  (compare with (4)) in Corollary 2, we get (compare with (2))

$$\sum_{i=1}^n p_i f(X_i) - f(\bar{X}) \leq \frac{\alpha}{2} A \leq \sum_{i=1}^n p_i f(Y_i) - f(\bar{Y}) \tag{9}$$

if  $f \in \mathcal{K}_1^c((a, b))$ . Or, we get reverse inequalities in (9) if  $f \in \mathcal{K}_2^c((a, b))$ . This is a generalization of Mercer's result [6] on operators.

### 3. Levinson's inequality without operator concavity and convexity

In this section we give Levinson's operator inequality for  $f \in \mathcal{K}_i^c(I), i = 1, 2$ .

Operator convexity plays an essential role in (5). In fact, this inequality will be false if we replace an operator convex function by a general convex function. We give a simple counterexample:

COUNTEREXAMPLE 1. Let  $c > 0$ ,  $d = \sqrt{c^7} + \sqrt{c^3}/2$ ,  $f(t) = t^4 - t^2/2$  on  $(0, c]$  and  $f(t) = d\sqrt{t} - t^2$  on  $[c, \infty)$ . Then  $f \in \mathcal{K}_2^c((0, \infty))$  for  $\alpha = -2$ , but  $f \notin \mathcal{K}_2^c((0, \infty))$  because the function  $t \mapsto t^4$  is not operator convex on  $(0, \infty)$ . Let  $\Phi : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$  be mapping defined by  $\Phi(M) = M/2$  for every  $M \in M_2(\mathbb{C})$ . If

$$X_1 = \begin{pmatrix} 50 & -30 \\ -30 & 20 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 9 & -1 \\ -1 & 10 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 100 & 1 \\ 1 & 200 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 200 & 2 \\ 2 & 300 \end{pmatrix},$$

then  $I_2 < X_1, X_2 < 69I_2 < 99I_2 < Y_1, Y_2 < 301I_2$  and we can choose any  $c$  from  $[69, 99]$ . Next, we have (by setting  $\Phi_1 = \Phi_2 = \Psi_1 = \Psi_2$  in Theorem 1)

$$X = \frac{1}{2}(X_1 - X_2)^2 = \begin{pmatrix} 420.25 & 410.25 \\ 410.25 & 25. \end{pmatrix} < Y = \frac{1}{2}(Y_1 - Y_2)^2 = \begin{pmatrix} 2500 & 0.25 \\ 0.25 & 2500 \end{pmatrix}.$$

Then, we obtain (rounded to six decimal places)

$$F_1 \not\leq -X > -Y > F_2,$$

where

$$\begin{aligned} F_1 &= \sum_{i=1}^2 \left( \frac{X_i^4}{2} - \frac{X_i^2}{4} \right) - \left( \sum_{i=1}^2 \frac{X_i}{2} \right)^4 + \frac{1}{2} \left( \sum_{i=1}^2 \frac{X_i}{2} \right)^2 = \begin{pmatrix} 2.37074 \cdot 10^6 & 347175. \\ & 34362.5 \end{pmatrix}, \\ F_2 &= \sum_{i=1}^2 \left( d \frac{Y_i^{1/2}}{2} - \frac{Y_i^2}{2} \right) - d \left( \sum_{i=1}^2 \frac{Y_i}{2} \right)^{1/2} + \left( \sum_{i=1}^2 \frac{Y_i}{2} \right)^2 \\ &= - \begin{pmatrix} 2500 & 0.25 \\ 0.25 & 2500 \end{pmatrix} - d \begin{pmatrix} 0.176379 & 0.000965 \\ 0.000965 & 0.0801 \end{pmatrix}. \end{aligned}$$

So, the reverse inequality is false in (5) under the operator order.

In the following theorem we give a general result when Levinson’s operator inequality (5) holds for  $f \in \mathcal{K}_1^c([m_x, M_y])$  with conditions on the spectra of operators. There have been many interesting works devoted to obtain operator inequalities under spectra conditions. The reader is referred to [9, 10] and references therein.

THEOREM 3. Let  $(X_1, \dots, X_n)$  be an  $n$ -tuple and  $(Y_1, \dots, Y_k)$  be a  $k$ -tuple of self-adjoint operators  $X_i, Y_j \in \mathcal{B}_h(H)$ . Let  $m_{X_i}, M_{X_i}$  be bounds of  $X_i$  and  $m_{Y_j}, M_{Y_j}$  be bounds of  $Y_j$ , such that  $a < m_{X_i} \leq M_{X_i} \leq c \leq m_{Y_j} \leq M_{Y_j} < b$  for some  $a, b, c \in \mathbb{R}$  and every  $i = 1, \dots, n, j = 1, \dots, k$ . Let  $(\Phi_1, \dots, \Phi_n)$  be a unital  $n$ -tuple and  $(\Psi_1, \dots, \Psi_k)$  be a unital  $k$ -tuple of positive linear mappings  $\Phi_i, \Psi_j : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ . Let  $m_X, M_X$  and  $m_Y, M_Y$  be bounds of  $\bar{X} = \sum_{i=1}^n \Phi_i(X_i)$  and  $\bar{Y} = \sum_{j=1}^k \Psi_j(Y_j)$ , respectively, such that

$$\begin{aligned} (m_X, M_X) \cap [m_{X_i}, M_{X_i}] &= \emptyset, \quad i = 1, \dots, n, \\ (m_Y, M_Y) \cap [m_{Y_j}, M_{Y_j}] &= \emptyset, \quad j = 1, \dots, k. \end{aligned} \tag{10}$$

If  $f \in \mathcal{K}_1^c((a, b))$  and  $\alpha X \leq \alpha Y$  hold, then

$$\sum_{i=1}^n \Phi_i(f(X_i)) - f(\bar{X}) \leq \frac{\alpha}{2} X \leq \frac{\alpha}{2} Y \leq \sum_{j=1}^k \Psi_j(f(Y_j)) - f(\bar{Y}) \tag{*}$$

is valid (i.e. (5) is valid), where

$$X := \sum_{i=1}^n \Phi_i(X_i^2) - \bar{X}^2, \quad Y := \sum_{j=1}^k \Psi_j(Y_j^2) - \bar{Y}^2.$$

If  $f \in \mathcal{K}_2^c((a,b))$  and  $\alpha X \geq \alpha Y$  hold, then reverse inequalities are valid in (\*).

*Proof.* The proof is similar to the one for Theorem 1. We apply Jensen's operator inequality without operator convexity and concavity (see [9, Theorem 1.]). We omit the details of the proof.  $\square$

As an application of Theorem 3, we obtain many interesting inequalities. For example, we obtain the following inequalities for some power functions.

EXAMPLE 2. The function  $f(t) = t^p$ ,  $p = 3, 4, \dots$  is an element of  $\mathcal{K}_1^c((-c, \infty))$  for  $\alpha = p(p-1)c^{p-2}$  and  $c \in \mathbb{R}^+$ . Let mappings  $\Phi_i, \Psi_j$  and operators  $X_i, Y_j, X, Y, \bar{X}, \bar{Y}$  be as in Theorem 3. If (10) and  $X \leq Y$  hold, then

$$\sum_{i=1}^n \Phi_i(X_i^p) - \bar{X}^p \leq \binom{n}{2} c^{p-2} X \leq \binom{n}{2} c^{p-2} Y \leq \sum_{j=1}^k \Psi_j(Y_j^p) - \bar{Y}^p.$$

REMARK 2.

1) In Counterexample 1 bounds of  $X_1$  are  $m_{X_1} = 1.45898, M_{X_1} = 68.54102$ , and of  $X = X_1/2 + X_2/2$  are  $m_X = 5.13824, M_X = 39.3618$ . Then  $(m_X, M_X) \cap [m_{X_1}, M_{X_1}] \neq \emptyset$ , so spectra conditions don't hold and we cannot apply Theorem 3.

2) If we add spectra conditions (10) in Corollary 2, we can put weaker conditions on the function  $f$ : it is sufficient that  $f$  is in class  $\mathcal{K}_1^c((a,b))$  instead of in  $\mathcal{K}_1^c((a,b))$ .

3) Similarly as in Definition 1, we can observe a class of functions  $\mathcal{K}_3^c(I)$ , when  $F$  is operator concave on  $I \cap (-\infty, c]$  and real convex function on  $I \cap [c, \infty)$ . So, we can obtain the following result:

Let mappings  $\Phi_i, \Psi_j$  and operators  $X_i, Y_j, X, Y, \bar{X}, \bar{Y}$  be as in Theorem 3. If  $f \in \mathcal{K}_3^c((a,b))$ ,  $\alpha X \leq \alpha Y$  and  $(m_Y, M_Y) \cap [m_{Y_j}, M_{Y_j}] = \emptyset, (j = 1, \dots, k)$  hold, then (5) is valid

In the same way, we can consider other combinations.

Next, we give a version of Levison's operator inequality with the scalar product.

THEOREM 4. Let  $(X_1, \dots, X_n)$  be an  $n$ -tuple and  $(Y_1, \dots, Y_k)$  be a  $k$ -tuple of self-adjoint operators  $X_i, Y_j \in \mathcal{B}_h(H)$  with spectra contained in  $[m_x, M_x]$  and  $[m_y, M_y]$ , respectively, such that  $a < m_x \leq M_x \leq c \leq m_y \leq M_y < b$  for some  $a, b, c \in \mathbb{R}$ . Let  $(z_1, \dots, z_n)$  be an  $n$ -tuple and  $(w_1, \dots, w_k)$  be a  $k$ -tuple of vectors  $z_i, w_j \in H$ , such that  $\sum_{i=1}^n \|z_i\|^2 = 1$  and  $\sum_{j=1}^k \|w_j\|^2 = 1$ .

If  $f \in \mathcal{K}_1^c((a,b))$  and  $\alpha x \leq \alpha y$ , then

$$\sum_{i=1}^n \langle f(X_i)z_i, z_i \rangle - f(\bar{x}) \leq \frac{\alpha}{2} x \leq \frac{\alpha}{2} y \leq \sum_{j=1}^k \langle f(Y_j)w_j, w_j \rangle - f(\bar{y}), \tag{11}$$

holds, where

$$\begin{aligned} \mathbf{x} &:= \sum_{i=1}^n \langle (X_i - \bar{x}1_H)^2 z_i, z_i \rangle, & \bar{x} &:= \sum_{i=1}^n \langle X_i z_i, z_i \rangle, \\ \mathbf{y} &:= \sum_{j=1}^k \langle (Y_j - \bar{y}1_H)^2 w_j, w_j \rangle, & \bar{y} &:= \sum_{j=1}^k \langle Y_j w_j, w_j \rangle. \end{aligned}$$

If  $f \in \mathcal{K}_2^c((a, b))$  and  $\alpha \mathbf{x} \geq \alpha \mathbf{y}$  hold, then reverse inequalities are valid in (11).

*Proof.* We use the same technique as in the proof of Theorem 1. For the sake of completeness, we give the proof.

Let  $f \in \mathcal{K}_1^c((a, b))$ . So, here is a constant  $\alpha$  such that  $F(t) = f(t) - \frac{\alpha}{2}t^2$  is concave on  $[m_x, c] \subset (a, c)$ . Then the converse of Jensen’s inequality for a concave function implies

$$0 \leq F(\bar{x}) - \sum_{i=1}^n \langle F(X_i) z_i, z_i \rangle = f(\bar{x}) - \frac{\alpha}{2} \bar{x}^2 - \sum_{i=1}^n \langle f(X_i) z_i, z_i \rangle + \frac{\alpha}{2} \sum_{i=1}^n \langle X_i^2 z_i, z_i \rangle.$$

So

$$\sum_{i=1}^n \langle f(X_i) z_i, z_i \rangle - f(\bar{x}) \leq \frac{\alpha}{2} \sum_{i=1}^n \langle X_i^2 z_i, z_i \rangle - \frac{\alpha}{2} \bar{x}^2 = \frac{\alpha}{2} \mathbf{x}. \tag{12}$$

Similarly, since  $F$  is convex on  $[c, M_y] \subset [c, b)$ , Jensen’s inequality implies

$$\frac{\alpha}{2} \mathbf{y} = \frac{\alpha}{2} \sum_{j=1}^k \langle Y_j^2 w_j, w_j \rangle - \frac{\alpha}{2} \bar{y}^2 \leq \sum_{j=1}^k \langle f(Y_j) w_j, w_j \rangle - f(\bar{y}). \tag{13}$$

Combining inequalities (12) and (13) and taking into account that  $\alpha \mathbf{x} \leq \alpha \mathbf{y}$ , we obtain (11).  $\square$

#### 4. Quasi-arithmetic means

We define the quasi-arithmetic mean of operators:

$$\mathfrak{M}_\varphi(\mathbf{X}, \Phi, n) := \varphi^{-1} \left( \sum_{i=1}^n \Phi_i(\varphi(X_i)) \right), \tag{14}$$

where  $(X_1, \dots, X_n)$  is an  $n$ -tuple of self-adjoint operators in  $\mathcal{B}_n(H)$  with spectra in  $I$ ,  $(\Phi_1, \dots, \Phi_n)$  is a unital  $n$ -tuple of positive linear mappings  $\Phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  and  $\varphi : I \rightarrow \mathbb{R}$  is a strictly monotone function. There have been many works devoted to observing the order among these means, see e.g. [7, 8, 4, 3].

The power mean is a special case of the quasi-arithmetic mean:

$$\mathfrak{M}_r(\mathbf{X}, \Phi, n) := \begin{cases} (\sum_{i=1}^n \Phi_i(X_i^r))^{1/r}, & r \in \mathbb{R} \setminus \{0\}, \\ \exp(\sum_{i=1}^n \Phi_i(\ln(X_i))) , & r = 0, \end{cases} \tag{15}$$

where  $X_1, \dots, X_n$  are positive operators.



### 4.1. Results with operator convexity and concavity

In this subsection we give order among quasi-arithmetic means under similar conditions as in Section 2.

First, as a generalization of [2, Corollary] on operators and quasi-arithmetic means, we obtain the following results.

**THEOREM 5.** *Let  $(X_1, \dots, X_n)$  be an  $n$ -tuple and  $(Y_1, \dots, Y_k)$  be a  $k$ -tuple of self-adjoint operators  $X_i, Y_j \in \mathcal{B}_h(H)$  with spectra contained in  $[m_x, M_x]$  and  $[m_y, M_y]$ , respectively, such that  $a < m_x \leq M_x \leq c_1 \leq m_y \leq M_y < b$  for some  $a, b, c_1 \in \mathbb{R}$ . Let  $(\Phi_1, \dots, \Phi_n)$  be a unital  $n$ -tuple and  $(\Psi_1, \dots, \Psi_k)$  be a unital  $k$ -tuple of positive linear mappings  $\Phi_i, \Psi_j : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ . Let  $\psi, \varphi : (a, b) \rightarrow \mathbb{R}$  be strictly monotone functions,  $c = \varphi(c_1)$  and  $I$  is the open interval between  $\varphi(a)$  and  $\varphi(b)$ .*

*If  $\psi \circ \varphi^{-1} \in \mathcal{K}_1^c(I)$  and  $\alpha X_\varphi \leq \alpha Y_\varphi$ , then*

$$\begin{aligned} & \psi(\mathfrak{M}_\psi(\mathbf{X}, \Phi, n)) - \psi(\mathfrak{M}_\varphi(\mathbf{X}, \Phi, n)) \\ & \leq \frac{\alpha}{2} X_\varphi \leq \frac{\alpha}{2} Y_\varphi \leq \psi(\mathfrak{M}_\psi(\mathbf{Y}, \Psi, k)) - \psi(\mathfrak{M}_\varphi(\mathbf{Y}, \Psi, k)), \end{aligned} \tag{16}$$

where

$$\begin{aligned} X_\varphi & := \sum_{i=1}^n \Phi_i(\varphi(X_i)^2) - \left( \sum_{i=1}^n \Phi_i(\varphi(X_i)) \right)^2, \\ Y_\varphi & := \sum_{j=1}^k \Psi_j(\varphi(Y_j)^2) - \left( \sum_{j=1}^k \Psi_j(\varphi(Y_j)) \right)^2. \end{aligned} \tag{17}$$

*If  $\psi \circ \varphi^{-1} \in \mathcal{K}_2^c(I)$  and  $\alpha X_\varphi \geq \alpha Y_\varphi$  hold, then reverse inequalities are valid in (16).*

*Proof.* Suppose that  $\varphi$  is a strictly increasing function in  $(a, b)$ .

Then  $m_x 1_H \leq X_i \leq M_x 1_H$  and  $m_y 1_H \leq Y_j \leq M_y 1_H$  implies  $\varphi(m_x) 1_H \leq \varphi(X_i) \leq \varphi(M_x) 1_H$  and  $\varphi(m_y) 1_H \leq \varphi(Y_j) \leq \varphi(M_y) 1_H$  for  $i = 1, \dots, n, j = 1, \dots, k$ . Furthermore,  $a < m_x \leq M_x \leq c_1 \leq m_y \leq M_y < b$  implies  $\varphi(a) < \varphi(m_x) \leq \varphi(M_x) \leq c \leq \varphi(m_y) \leq \varphi(M_y) < \varphi(b)$ .

So, for a function  $f \in \mathcal{K}_1^c((\varphi(a), \varphi(b)))$  there is a constant  $\alpha$  such that  $F(t) = f(t) - \frac{\alpha}{2} t^2$  is operator concave on  $[\varphi(m_x), c] \subset (\varphi(a), c]$ . Then the converse of Jensen's inequality for operators  $(\varphi(X_1), \dots, \varphi(X_n))$  gives

$$\sum_{i=1}^n \Phi_i(f(\varphi(X_i))) - f\left(\sum_{i=1}^n \Phi_i(\varphi(X_i))\right) \leq \frac{\alpha}{2} \left[ \sum_{i=1}^n \Phi_i(\varphi(X_i)^2) - \left(\sum_{i=1}^n \Phi_i(\varphi(X_i))\right)^2 \right].$$

Also, since  $F$  is operator convex on  $[c, \varphi(M_y)] \subset [c, \varphi(b))$ , Jensen's inequality for operators  $(\varphi(Y_1), \dots, \varphi(Y_k))$  gives

$$\frac{\alpha}{2} \left[ \sum_{j=1}^k \Psi_j(\varphi(Y_j)^2) - \left(\sum_{j=1}^k \Psi_j(\varphi(Y_j))\right)^2 \right] \leq \sum_{j=1}^k \Psi_j(f(\varphi(Y_j))) - f\left(\sum_{j=1}^k \Psi_j(\varphi(Y_j))\right).$$

Setting  $f = \psi \circ \varphi^{-1}$  in above two inequalities and taking into account that  $\alpha X_\varphi \leq \alpha Y_\varphi$  holds, we obtain

$$\begin{aligned} & \sum_{i=1}^n \Phi_i(\psi(X_i)) - \psi \circ \varphi^{-1} \left( \sum_{i=1}^n \Phi_i(\varphi(X_i)) \right) \leq \frac{\alpha}{2} X_\varphi \\ & \leq \frac{\alpha}{2} Y_\varphi \leq \sum_{j=1}^k \Psi_j(\psi(Y_j)) - \psi \circ \varphi^{-1} \left( \sum_{j=1}^k \Psi_j(\varphi(Y_j)) \right), \end{aligned}$$

which gives the desired inequality (16).

Analogously, we can prove (16) in the case when  $\varphi$  is a strictly decreasing function.  $\square$

Setting  $\psi$  equal to the identity function in Theorem 5, we can obtain inequality (18). We give this result with weakened assumptions.

**THEOREM 6.** *Let mappings  $\Phi_i, \Psi_j$ , operators  $X_i, Y_j$  and  $a, b, c_1$  be as in Theorem 5. Let  $f : (a, b) \rightarrow \mathbb{R}$  such that  $\varphi := f|_{(a, c_1]}$ ,  $\psi := f|_{[c_1, b)}$  be strictly monotone functions,  $c = \varphi(c_1)$  and  $I$  is the open interval between  $f(a)$  and  $f(b)$ .*

*If  $f^{-1} \in \mathcal{K}_1^c(I)$  and  $\alpha Y_\psi \leq \alpha X_\varphi$ , then*

$$\mathfrak{M}_\psi(\mathbf{Y}, \Psi, k) - \mathfrak{M}_1(\mathbf{Y}, \Psi, k) \leq \frac{\alpha}{2} Y_\psi \leq \frac{\alpha}{2} X_\varphi \leq \mathfrak{M}_\varphi(\mathbf{X}, \Phi, n) - \mathfrak{M}_1(\mathbf{X}, \Phi, n), \quad (18)$$

where

$$\begin{aligned} Y_\psi &:= \left( \sum_{j=1}^k \Psi_j(\psi(Y_j)) \right)^2 - \sum_{j=1}^k \Psi_j(\psi(Y_j)^2), \\ X_\varphi &:= \left( \sum_{i=1}^n \Phi_i(\varphi(X_i)) \right)^2 - \sum_{i=1}^n \Phi_i(\varphi(X_i)^2), \end{aligned}$$

*If  $f^{-1} \in \mathcal{K}_2^c(I)$  and  $\alpha Y_\psi \geq \alpha X_\varphi$  holds, then reverse inequalities are valid in (18).*

*Proof.* If both functions  $\varphi$  and  $\psi$  are strictly increasing or decreasing these results follow directly from Theorem 5. The remaining two cases can be proven by using the same technique as in the proof of Theorem 5.  $\square$

**REMARK 3.** Let  $\Phi_i, \Psi_j$  be mappings,  $X_i, Y_j$  be positive operators as in Theorem 5 and  $0 < m_x \leq M_x \leq c \leq m_y \leq M_y < b$ . Setting  $f(t) = t^s$ ,  $s \geq 1$  for  $t \in (0, c]$ ,  $f(t) = dt^r$ ,  $r \leq -1$  or  $\frac{1}{2} \leq r \leq 1$  for  $t \in [c, \infty)$ , where  $d = c^{s/r}$  and  $\alpha = 0$  in Theorem 6, we obtain order among power means as follows

$$\mathfrak{M}_r(\mathbf{Y}, \Psi, k) - \mathfrak{M}_1(\mathbf{Y}, \Psi, k) \leq 0 \leq \mathfrak{M}_s(\mathbf{X}, \Phi, n) - \mathfrak{M}_1(\mathbf{X}, \Phi, n). \quad (19)$$

We remark that (19) holds for all positive operators  $X_1, \dots, X_n, Y_1, \dots, Y_k$  without condition  $M_x \leq c \leq m_y$ . Really, LHS (resp. RHS) of (19) holds since  $t \mapsto t^s$  (resp.  $t \mapsto t^r$ ) is operator concave (resp. operator convex) on  $(0, \infty)$ , see [8, 4].

Setting  $\alpha \neq 0$  in Theorem 6, we can obtain a refinement of (19) for some  $r, s$ . In this way we obtain inequalities for instance as in the following corollary.

COROLLARY 7. Let  $\Phi_i, \Psi_j$  be mappings,  $X_i, Y_j$  be positive operators as in Theorem 5 and  $0 < m_x \leq M_x \leq c \leq m_y \leq M_y < b$ .

(i) If  $s \geq 1$  and  $C_{1/2} \leq C_s$ , then

$$\mathfrak{M}_{1/2}(\mathbf{Y}, \mathbf{\Psi}, k) - \mathfrak{M}_1(\mathbf{Y}, \mathbf{\Psi}, k) \leq \alpha C_{1/2} \leq \alpha C_s \leq \mathfrak{M}_s(\mathbf{X}, \mathbf{\Phi}, n) - \mathfrak{M}_1(\mathbf{X}, \mathbf{\Phi}, n) \quad (20)$$

for every  $\alpha \in (0, 2c^{1-2s})$ , where

$$C_{1/2} := \left( \sum_{j=1}^k \Psi_j(\sqrt{Y_j}) \right)^2 - \sum_{j=1}^k \Psi_j(Y_j), \quad C_s := \left( \sum_{i=1}^n \Phi_i(X_i^s) \right)^2 - \sum_{i=1}^n \Phi_i(X_i^{2s}).$$

(ii) If  $C_{1/2} \leq C_{\text{exp}}$ , then

$$\mathfrak{M}_{1/2}(\mathbf{Y}, \mathbf{\Psi}, k) - \mathfrak{M}_1(\mathbf{Y}, \mathbf{\Psi}, k) \leq \alpha C_{1/2} \leq \alpha C_{\text{exp}} \leq \mathfrak{M}_{\text{exp}}(\mathbf{X}, \mathbf{\Phi}, n) - \mathfrak{M}_1(\mathbf{X}, \mathbf{\Phi}, n) \quad (21)$$

for every  $\alpha \in (0, 2c^{1-2s})$ , where

$$C_{\text{exp}} := \left( \sum_{i=1}^n \Phi_i(\exp X_i) \right)^2 - \sum_{i=1}^n \Phi_i((\exp X_i)^2).$$

*Proof.* (i) We set  $f(t) = t^s$ ,  $s \geq 1$  for  $t \in (0, c]$  and  $f(t) = d\sqrt{t}$  for  $t \in [c, \infty)$ , where  $d = c^{s-1/2}$ . Then  $f$  is strictly monotone on  $(0, \infty)$  and  $f^{-1} \in \mathcal{K}_1^c((0, \infty))$  for every  $\alpha \in (0, 2/a^2)$ . So,  $\alpha Y_\psi \leq \alpha X_\phi$  and (18) give  $\alpha C_{1/2} \leq \alpha C_s$  and (20), since the power means are positively homogeneous.

(iii) Setting  $f(t) = \exp(t)$  for  $t \in (0, c]$ ,  $f(t) = d\sqrt{t}$  for  $t \in [c, \infty)$ , where  $d = \exp(c)/\sqrt{c}$ , and using the same technique as above, we obtain  $C_{1/2} \leq C_{\text{exp}}$  and (21).  $\square$

### 4.2. Results without operator convexity and concavity

For wider application it is interesting to consider inequalities involving quasi-arithmetic means under similar conditions as in Section 3. Thus, if spectra conditions hold, then (16) is valid for all strictly monotone functions  $\varphi, \psi : (a, b) \rightarrow \mathbb{R}$  such that  $\psi \circ \varphi^{-1} \in \mathcal{K}_1^c(I)$  and (18) is valid for every strictly monotone function  $f : (a, b) \rightarrow \mathbb{R}$  such that  $f^{-1} \in \mathcal{K}_1^c(I)$ . Now we give these results.

THEOREM 8. Let  $(X_1, \dots, X_n)$  be an  $n$ -tuple and  $(Y_1, \dots, Y_k)$  be a  $k$ -tuple of self-adjoint operators  $X_i, Y_j \in \mathcal{B}_h(H)$ . Let  $m_{X_i}, M_{X_i}$  be bounds of  $X_i$  and  $m_{Y_j}, M_{Y_j}$  be bounds of  $Y_j$ , such that  $a < m_{X_i} \leq M_{X_i} \leq c_1 \leq m_{Y_j} \leq M_{Y_j} < b$  for some  $a, b, c \in \mathbb{R}$  and every  $i = 1, \dots, n, j = 1, \dots, k$ . Let  $(\Phi_1, \dots, \Phi_n)$  be a unital  $n$ -tuple and  $(\Psi_1, \dots, \Psi_k)$  be a unital  $k$ -tuple of positive linear mappings  $\Phi_i, \Psi_j : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ . Let  $m_{\Phi_X}, M_{\Phi_X}$  and  $m_{\Phi_Y}, M_{\Phi_Y}$  be bounds of  $\mathfrak{M}_\Phi(\mathbf{X}, \mathbf{\Phi}, n)$  and  $\mathfrak{M}_\Phi(\mathbf{Y}, \mathbf{\Psi}, k)$ , respectively, such that

$$\begin{aligned} (m_{\Phi_X}, M_{\Phi_X}) \cap [m_{X_i}, M_{X_i}] &= \emptyset, & i = 1, \dots, n, \\ (m_{\Phi_Y}, M_{\Phi_Y}) \cap [m_{Y_j}, M_{Y_j}] &= \emptyset, & j = 1, \dots, k. \end{aligned} \quad (22)$$

Let  $\varphi, \psi : (a, b) \rightarrow \mathbb{R}$  be strictly monotone functions,  $c = \varphi(c_1)$  and  $I$  is the open interval between  $\varphi(a)$  and  $\varphi(b)$ .

If  $\psi \circ \varphi^{-1} \in \mathcal{K}_1^c(I)$  and  $\alpha X_\varphi \leq \alpha Y_\varphi$ , then

$$\begin{aligned} & \psi(\mathfrak{M}_\psi(\mathbf{X}, \Phi, n)) - \psi(\mathfrak{M}_\varphi(\mathbf{X}, \Phi, n)) \\ & \leq \frac{\alpha}{2} X_\varphi \leq \frac{\alpha}{2} Y_\varphi \leq \psi(\mathfrak{M}_\psi(\mathbf{Y}, \Psi, k)) - \psi(\mathfrak{M}_\varphi(\mathbf{Y}, \Psi, k)), \end{aligned} \tag{**}$$

is valid (i.e. (16) is valid), where  $X_\varphi, Y_\varphi$  are defined by (17).

If  $\psi \circ \varphi^{-1} \in \mathcal{K}_2^c(I)$  and  $\alpha X_\varphi \geq \alpha Y_\varphi$  hold, then reverse inequalities are valid in (\*\*).

*Proof.* Suppose that  $\varphi$  is a strictly increasing function in  $(a, b)$ .

Then  $m_{X_i} 1_H \leq X_i \leq M_{X_i} 1_H$  and  $m_{\varphi_X} 1_K \leq \mathfrak{M}_\varphi(\mathbf{X}, \Phi, n) \leq M_{\varphi_X} 1_K$ , implies

$$\begin{aligned} & \varphi(m_{X_i}) 1_H \leq \varphi(X_i) \leq \varphi(M_{X_i}) 1_H, \quad i = 1, \dots, n, \\ \text{and} \quad & \varphi(m_{\varphi_X}) 1_K \leq \varphi(\mathfrak{M}_\varphi(\mathbf{X}, \Phi, n)) = \sum_{i=1}^n \Phi_i(\varphi(A_i)) \leq \varphi(M_{\varphi_X}) 1_K. \end{aligned}$$

It follows that

$$\begin{aligned} & (m_{\varphi_X}, M_{\varphi_X}) \cap [m_{X_i}, M_{X_i}] = \emptyset, \quad i = 1, \dots, n, \\ \Rightarrow \quad & (\varphi(m_{\varphi_X}), \varphi(M_{\varphi_X})) \cap [\varphi(m_{X_i}), \varphi(M_{X_i})] = \emptyset, \quad i = 1, \dots, n. \end{aligned} \tag{23}$$

Similarly, we can prove that

$$\begin{aligned} & (m_{\varphi_Y}, M_{\varphi_Y}) \cap [m_{Y_j}, M_{Y_j}] = \emptyset, \quad j = 1, \dots, k, \\ \Rightarrow \quad & (\varphi(m_{\varphi_Y}), \varphi(M_{\varphi_Y})) \cap [\varphi(m_{Y_j}), \varphi(M_{Y_j})] = \emptyset, \quad j = 1, \dots, k. \end{aligned} \tag{24}$$

Furthermore,  $a < m_{X_i} \leq M_{X_i} \leq c_1 \leq m_{Y_j} \leq M_{Y_j} < b$  implies

$$\varphi(a) < \varphi(m_{X_i}) \leq \varphi(M_{X_i}) \leq c \leq \varphi(m_{Y_j}) \leq \varphi(M_{Y_j}) < \varphi(b)$$

for every  $i = 1, \dots, n, j = 1, \dots, k$ .

Now, we use a similar technique as in the proof of Theorem 5: Since  $\psi \circ \varphi^{-1} \in \mathcal{K}_1^c(I)$ , there is a constant  $\alpha$  such that  $F(t) = \psi \circ \varphi^{-1}(t) - \frac{\alpha}{2} t^2$  is concave on  $(\varphi(a), c]$ . Then the converse of Jensen’s inequality for operators  $(\varphi(X_1), \dots, \varphi(X_n))$  and with spectra condition (23) gives (see [9])

$$\sum_{i=1}^n \Phi_i(\psi(X_i)) - \psi \circ \varphi^{-1} \left( \sum_{i=1}^n \Phi_i(\varphi(X_i)) \right) \leq X_\varphi. \tag{25}$$

Also, since  $F$  is convex on  $[c, \varphi(b))$ , Jensen’s inequality for operators  $(\varphi(Y_1), \dots, \varphi(Y_k))$  and spectra condition (24) gives

$$Y_\varphi \leq \sum_{j=1}^k \Psi_j(\psi(Y_j)) - \psi \circ \varphi^{-1} \left( \sum_{j=1}^k \Psi_j(\varphi(Y_j)) \right), \tag{26}$$

Combining (25) and (26) and taking into account that  $\alpha X_\varphi \leq \alpha Y_\varphi$ , we obtain the desired inequality (\*\*).

Analogously, we can prove (\*\*) in the case when  $\varphi$  is a strictly decreasing function.  $\square$

Applying Theorem 8, we obtain a generalization and refining of Bullen's result [2, Corollary] for power means.

**COROLLARY 9.** *Let  $\Phi_i, \Psi_j$  be mappings,  $X_i, Y_j$  be positive operators as in Theorem 8 and  $0 < m_{X_i} \leq M_{X_i} \leq c \leq m_{Y_j} \leq M_{Y_j} < b$  for some  $a, b, c \in \mathbb{R}$  and every  $i = 1, \dots, n, j = 1, \dots, k$ . Let  $m_{\varphi_X}, M_{\varphi_X}$  and  $m_{\varphi_Y}, M_{\varphi_Y}$  be bounds of  $\mathfrak{M}_\varphi(\mathbf{X}, \Phi, n)$  and  $\mathfrak{M}_\varphi(\mathbf{Y}, \Psi, k)$ , respectively. Let*

$$C_{s_X} := \begin{cases} \left( \sum_{i=1}^n \Phi_i(X_i^s) \right)^2 - \sum_{i=1}^n \Phi_i(X_i^{2s}), & s \neq 0 \\ \left( \sum_{i=1}^n \Phi_i(\ln X_i) \right)^2 - \sum_{i=1}^n \Phi_i(\ln^2(X_i)), & s = 0, \end{cases}$$

and  $C_{s_Y}, C_{0_Y}$  be analogous notations for operators  $Y_1, \dots, Y_k$ . Let  $\alpha = \frac{r}{s} \left( \frac{r}{s} - 1 \right) c^{\frac{r}{s}-2}$  if  $r s \neq 0$ ,  $\alpha = r^2 \exp(cr)$  if  $s = 0$  and  $\alpha = -\frac{1}{s} c^{-2}$  if  $r = 0$ .

(i) *If  $r < 0 < s, 2s \leq r \leq s < 0, s < 0 < r$  or  $0 < s \leq r \leq 2s, C_{s_X} \leq C_{s_Y}$  and spectra conditions (22) hold, then*

$$\mathfrak{M}_s(\mathbf{X}, \Phi, n)^r - \mathfrak{M}_r(\mathbf{X}, \Phi, n)^r \leq \alpha C_{s_X} \leq \alpha C_{s_Y} \leq \mathfrak{M}_s(\mathbf{Y}, \Psi, k)^r - \mathfrak{M}_r(\mathbf{Y}, \Psi, k)^r. \tag{27}$$

(ii) *If  $r \leq 2s < 0, 0 < 2s \leq r, C_{s_X} \geq C_{s_Y}$  or  $s \leq r < 0$  or  $0 < r \leq s, C_{s_X} \leq C_{s_Y}$  and (22) holds, then reverse inequalities are valid in (27).*

(iii) *If  $r < 0, C_{0_X} \leq C_{0_Y}$  and (22) holds, then (27) is valid for  $s = 0$ .*

*But, if  $r > 0, C_{0_X} \geq C_{0_Y}$  and (22) holds, then reverse inequalities are valid in (27) for  $s = 0$ .*

(iv) *If  $s > 0, C_{s_X} \leq C_{s_Y}$  and (22) holds, then*

$$\ln \left( \frac{\mathfrak{M}_0(\mathbf{X}, \Phi, n)}{\mathfrak{M}_s(\mathbf{X}, \Phi, n)} \right) \leq \alpha C_{s_X} \leq \alpha C_{s_Y} \leq \ln \left( \frac{\mathfrak{M}_0(\mathbf{Y}, \Psi, k)}{\mathfrak{M}_s(\mathbf{Y}, \Psi, k)} \right). \tag{28}$$

*If  $s < 0, C_{s_X} \geq C_{s_Y}$  and spectra conditions:*

$$(m_{\varphi_Y}, M_{\varphi_Y}) \cap [m_{Y_i}, M_{Y_i}] = \emptyset, \quad i = 1, \dots, k, \tag{29}$$

*hold, then reverse inequalities are valid in (28).*

*Proof.* (i)–(ii): We set  $\varphi(t) = t^s, \psi(t) = t^r$  and  $f(t) = t^{\frac{r}{s}}, r, s \neq 0$ . Let us consider a function  $F(t) = t^{\frac{r}{s}} - \frac{\alpha}{2} t^2$  for  $\alpha = \frac{r}{s} \left( \frac{r}{s} - 1 \right) c^{\frac{r}{s}-2}$ . Since  $F''(t) = \frac{r}{s} \left( \frac{r}{s} - 1 \right) (t^{\frac{r}{s}-2} - c^{\frac{r}{s}-2})$ , then  $c$  is inflection point of  $F$ .

If  $\frac{r}{s} < 0$  or  $1 \leq \frac{r}{s} \leq 2$ , then  $f \in \mathcal{H}_2^c((0, \infty))$  and  $\alpha > 0$ . So, applying Theorem 8 we obtain (27) in the case (i).

If  $\frac{t}{s} \geq 2$ , then  $\alpha > 0$  or if  $0 < \frac{t}{s} \leq 1$ , then  $\alpha < 0$ . Also,  $f \in \mathcal{K}_1^c((0, \infty))$  and reverses of inequalities (27) hold in the case (ii).

(iii): If  $s = 0$ , we set  $\varphi(t) = \ln t$ ,  $\psi(t) = t^r$  and  $f(t) = \exp(tr)$ ,  $r \neq 0$ . Let  $F(t) = \exp(tr) - \frac{\alpha}{2}t^2$  for  $\alpha = r^2 \exp(cr)$ . Then  $f \in \mathcal{K}_1^c((0, \infty))$  for  $r > 0$ , or  $f \in \mathcal{K}_2^c((0, \infty))$  for  $r < 0$ . So, applying Theorem 8 we obtain (27) under conditions  $r > 0$ ,  $C_{0X} \geq C_{0Y}$  and (22). Or, we obtain (27) under conditions  $r < 0$ ,  $C_{0X} \leq C_{0Y}$  and (22).

(iv): If  $r = 0$ , we set  $\varphi(t) = t^s$ ,  $\psi(t) = \ln t$  and  $f(t) = \frac{1}{s} \ln t$ ,  $s \neq 0$ . Let  $F(t) = \frac{1}{s} \ln t - \frac{\alpha}{2}t^2$  for  $\alpha = -\frac{1}{sc^2}$ . Then  $f \in \mathcal{K}_1^c((0, \infty))$  for  $s > 0$ , or  $f \in \mathcal{K}_3^c((0, \infty))$  for  $s < 0$ . So, applying Theorem 8 we obtain (28) under conditions  $s < 0$ ,  $C_{sX} \geq C_{sY}$  and (22). Or, we obtain (28) under conditions  $s > 0$ ,  $C_{sX} \leq C_{sY}$  and (29).  $\square$

Finally, we give version of Theorem 6 without operator convexity or concavity. The proof is similar to the one for Theorem 8 and we omit it.

**THEOREM 10.** *Let  $\Phi_i, \Psi_j$  be mappings,  $X_i, Y_j$  be operators as in Theorem 8 and  $a < m_{X_i} \leq M_{X_i} \leq c_1 \leq m_{Y_j} \leq M_{Y_j} < b$  for some  $a, b, c \in \mathbb{R}$  and every  $i = 1, \dots, n, j = 1, \dots, k$ . Let  $m_{\varphi_X}, M_{\varphi_X}$  and  $m_{\varphi_Y}, M_{\varphi_Y}$  be bounds of  $\mathfrak{M}_\varphi(\mathbf{X}, \Phi, n)$  and  $\mathfrak{M}_\varphi(\mathbf{Y}, \Psi, k)$ , respectively, such that spectra conditions (22) hold. Let  $f : (a, b) \rightarrow \mathbb{R}$  such that  $\varphi := f|_{(a, c_1]}$ ,  $\psi := f|_{[c_1, b)}$  be strictly monotone functions,  $c = \varphi(c_1)$  and  $I$  is the open interval between  $f(a)$  and  $f(b)$ .*

*If  $f^{-1} \in \mathcal{K}_1^c(I)$  and  $\alpha Y_\psi \leq \alpha X_\varphi$ , then (18) is valid, where  $X_\varphi, Y_\psi$  are defined by (17).*

*If  $f^{-1} \in \mathcal{K}_2^c(I)$  and  $\alpha Y_\psi \geq \alpha X_\varphi$  holds, then reverse inequalities are valid in (18).*

**REMARK 4.** By setting  $r = 1$  in Corollary 9, we obtain order between  $\mathfrak{M}_s(\mathbf{Y}, \Psi, k)$  and  $\mathfrak{M}_s(\mathbf{X}, \Phi, n)$ . Applying Theorem 10, we can obtain another order among power means as follows. Setting  $f(t) = t^s$ ,  $s \geq 1$  for  $t \in (0, c]$ ,  $f(t) = dt^r$ ,  $r \leq 1$  for  $t \in [c, \infty)$ , where  $d = c^{s/r}$  and  $\alpha = 0$  in this theorem, we obtain obvious inequality

$$\mathfrak{M}_r(\mathbf{Y}, \Psi, k) - \mathfrak{M}_1(\mathbf{Y}, \Psi, k) \leq 0 \leq \mathfrak{M}_s(\mathbf{X}, \Phi, n) - \mathfrak{M}_1(\mathbf{X}, \Phi, n)$$

under spectra conditions (22).

But, for some  $\alpha > 0$ , we can obtain refining of the above inequalities as follows:

*Let  $C_{sX}, C_{rY}$  and  $C_{0Y}$  as in Corollary 6. If  $s \geq 1, 0 \leq r \leq 1, C_{rY} \leq C_{sX}$  and spectra conditions (29) hold, then*

$$\mathfrak{M}_r(\mathbf{Y}, \Psi, k) - \mathfrak{M}_1(\mathbf{Y}, \Psi, k) \leq \alpha C_{sY} \leq \alpha C_{sX} \leq \mathfrak{M}_s(\mathbf{X}, \Phi, n) - \mathfrak{M}_1(\mathbf{X}, \Phi, n)$$

*is valid for every  $\alpha \in (0, c^{-2+(1-r)/r^2} (1-r)/r^2)$ .*

*Acknowledgement.* This work has been fully supported by Croatian Science Foundation under the project 5435.

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(Received November 1, 2014)

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