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GENERALIZED JENSEN–STEFFENSEN AND RELATED INEQUALITIES

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Abstract. We introduce a new tool for comparing two linear functionals that are positive on convex functions. We generalize Jensen-Steffensen and related inequalities.

1. Introduction

Jensen-Steffensen inequality is proved by Steffensen in [9]:

THEOREM 1.1. *If $\varphi : I \to \mathbb{R}$ is a convex function, $\mathbf{x}$ is a real monotonic n-tuple such that $x_i \in I$ $(i = 1, \dots, n)$, and $\mathbf{p}$ is a real n-tuple such that*

$$0 \leqslant P_k \leqslant P_n \quad (k = 1, \dots, n), \qquad P_n > 0$$

is satisfied, then

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i), \tag{1}$$

where $P_k = \sum_{i=1}^{k} p_i$. If φ is strictly convex, then inequality (1) is strict unless $x_1 = x_2 = \dots = x_n$.

For later purpose, let us introduce the *Jensen-Steffensen functional* $A_{\mathbf{x},\mathbf{p}}$ with

$$A_{\mathbf{x},\mathbf{p}}(\varphi) = \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) - \varphi\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right).$$

Obviously, (1) can be expressed with $A_{\mathbf{x},\mathbf{p}}(\varphi) \geqslant 0$.

An integral analogue of (1) was also given by Steffensen. Here we consider the integral analogue of Jensen-Steffensen's inequality given by Boas [2].

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THEOREM 1.2. *Let $\varphi : I \to \mathbb{R}$ be a continuous convex function where I is the range of the continuous monotonic function (either increasing or decreasing) $f : [a,b] \to \mathbb{R}$ and $\lambda : [a,b] \to \mathbb{R}$ be either continuous or of bounded variation satisfying*

$$\lambda(a) \leqslant \lambda(x) \leqslant \lambda(b) \quad \text{for all} \quad x \in [a,b], \quad \lambda(b) > \lambda(a).$$

Then

$$\varphi\left(\int_a^b f(x)d\lambda(x) \Big/ \int_a^b d\lambda(x)\right) \leqslant \int_a^b \varphi(f(x))d\lambda(x) \Big/ \int_a^b d\lambda(x). \tag{2}$$

Here, similarly, we define the *Jensen-Boas functional*

$$B_{a,b}^{f,\lambda}(\varphi) = \frac{\int_a^b \varphi(f(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \varphi\left(\frac{\int_a^b f(x)d\lambda(x)}{\int_a^b d\lambda(x)}\right).$$

Boas [2] also proved a generalization of Theorem 1.2, the so called *Jensen-Boas inequality* (see [6] and [7] pp. 59.).

THEOREM 1.3. *Let $f : [a,b] \to \mathbb{R}$ be continuous with range I and $\lambda : [a,b] \to \mathbb{R}$ be continuous or of bounded variation such that*

$$\lambda(a) \leqslant \lambda(x_1) \leqslant \lambda(y_1) \leqslant \lambda(x_2) \leqslant \cdots \leqslant \lambda(y_{n-1}) \leqslant \lambda(x_n) \leqslant \lambda(b)$$

for all $x_k \in (y_{k-1}, y_k)$ $(y_0 = a, y_n = b)$ and $\lambda(b) > \lambda(a)$. If f is continuous and monotonic (either increasing or decreasing) in each of the n intervals (y_{k-1}, y_k), then for every continuous convex function $\varphi : I \to \mathbb{R}$ inequality (2) holds.

The following generalization of Jensen-Steffensen's inequality is also known as the *Jensen-Brunk inequality* (see [3]).

THEOREM 1.4. *Let $f : [a,b] \to \mathbb{R}$ be a continuous and increasing function with range I, $\lambda : [a,b] \to \mathbb{R}$ be continuous or of bounded variation and $\lambda(b) > \lambda(a)$. Then inequality (2) holds for every convex function $\varphi : I \to \mathbb{R}$ if and only if*

$$\int_a^x (f(x) - f(t))\, d\lambda(t) \geqslant 0 \text{ and } \int_x^b (f(x) - f(t))\, d\lambda(t) \leqslant 0$$

for all $x \in [a,b]$.

We have the same inequality with different conditions (see [7] pp. 62).

THEOREM 1.5. *Let $f:[a,b]\to\mathbb{R}$ be continuous with range I and $\lambda:[a,b]\to\mathbb{R}$ be either continuous or of bounded variation with $\lambda(b)>\lambda(a)$. Then* (2) *holds for every continuous convex function $\varphi: I\to\mathbb{R}$ if and only if*

$$\left|\int_a^b (f(x)-f(t))\,d\lambda(t)\right| \leqslant \int_a^b |f(x)-f(t)|\,d\lambda(t)$$

for every $x\in[a,b]$.

We close this section with one companion inequality to Jensen's inequality (see [5]).

THEOREM 1.6. *Let $(\Omega,\mathscr{A},\mu)$ be a measure space with $0<\mu(\Omega)<\infty$ and let $\varphi: I\to\mathbb{R}$ be a convex function defined on an open interval I. If $f:\Omega\to I$ is such that $f,\varphi(f),\varphi'_+(f)$ and $f\varphi'_+(f)$ are all in $L^1(\mu)$, then for any $u,v\in I$ we have*

$$\varphi(u)+\varphi'_+(u)(\bar{x}-u)\leqslant \bar{y}\leqslant \varphi(v)+\frac{1}{\mu(\Omega)}\int_\Omega (f-v)\varphi'_+(f)d\mu, \tag{3}$$

where

$$\bar{x}=\frac{1}{\mu(\Omega)}\int_\Omega f d\mu,\quad \bar{y}=\frac{1}{\mu(\Omega)}\int_\Omega \varphi(f)d\mu.$$

Further, when φ is strictly convex, we have equality in the left inequality in (3) *if and only if $f(t)=u$ almost everywhere on Ω, while we have equality in the right inequality in* (3) *if and only if $f(t)=v$ almost everywhere on Ω.*

Using Theorem 1.6 we can define the two *Matić-Pečarić functionals*:

$$M_{f,\mu,u}(\varphi)=\frac{1}{\mu(\Omega)}\int_\Omega \varphi(f)d\mu-\varphi(u)-\varphi'_+(u)(\bar{x}-u) \tag{4}$$

and

$$M^{f,\mu,v}(\varphi)=\varphi(v)+\frac{1}{\mu(\Omega)}\int_\Omega (f-v)\varphi'_+(f)d\mu-\frac{1}{\mu(\Omega)}\int_\Omega \varphi(f)d\mu. \tag{5}$$

REMARK 1.7. As is pointed out in [5], if $\int_\Omega \varphi'_+(f)d\mu\neq 0$ and if

$$v=\int_\Omega f\varphi'_+(f)d\mu\Big/\int_\Omega \varphi'_+(f)d\mu$$

belongs to I, then the right inequality in (3) reduces to *Slater's inequality* (see [8]):

$$\frac{1}{\mu(\Omega)}\int_\Omega \varphi(f)d\mu\leqslant \varphi\left(\int_\Omega f\varphi'_+(f)d\mu\Big/\int_\Omega \varphi'_+(f)d\mu\right).$$

REMARK 1.8. If we take $u = v = \bar{x}$ in (3) then we get *Dragomir-Goh inequality* (see [4]):

$$0 \leqslant \bar{y} - \varphi(\bar{x}) \leqslant \frac{1}{\mu(\Omega)} \int_{\Omega} (f - v)\varphi'_{+}(f) d\mu.$$

REMARK 1.9. Minimization of the functional (5) over $v \in I$ is also considered in [5]. To be quite precise, we have to assume, additionally, that the function $\varphi : I \to \mathbb{R}$ is differentiable on I. Then

$$v \mapsto \varphi(v) + \frac{1}{\mu(\Omega)} \int_{\Omega} (f - v)\varphi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi(f) d\mu$$

attains its global minimum for at least one $v \in I$ which satisfies

$$\varphi'(v) = \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi'(f) d\mu.$$

When φ is strictly convex, v is, obviously, uniquely defined with

$$v = (\varphi')^{-1} \left(\frac{1}{\mu(\Omega)} \int_{\Omega} \varphi'(f) d\mu \right).$$

In this paper we will compare the aforementioned functionals on the class of 3-convex functions at a point that was introduced in [1]. From the obtained inequalities we will construct linear functionals for which we will prove mean value results and use them to construct new families of exponentially convex functions.

2. Main results

Throughout this section, I denotes an interval (open, closed or semi-open in either direction) in $\mathbb{R}$. Also, for further convenience, we denote by e_n the functions $e_n(t) = t^n$, $n \in \mathbb{N}$. The following is a class of functions introduced in [1].

DEFINITION 2.1. Let $\varphi : I \to \mathbb{R}$ and $c \in I^{\circ}$, where I° is the interior of I. We say that $\varphi \in \mathscr{K}_1^c(I)$ (resp. $\varphi \in \mathscr{K}_2^c(I)$) if there exists a constant K_{φ} such that the function $\Phi(x) = \varphi(x) - \frac{K_{\varphi}}{2} e_2(x)$ is concave (resp. convex) on $I \cap (-\infty, c]$ and convex (resp. concave) on $I \cap [c, \infty)$.

REMARK 2.2. A function $\varphi \in \mathscr{K}_1^c(I)$ is said to be 3-convex at point c. It was shown in [1] that a function φ is 3-convex on an interval if and only if it is 3-convex at every point of the interval. It was also shown in [1] that if $\varphi''(c)$ exists, then $K_{\varphi} = \varphi''(c)$.

The following theorem is our main result.

THEOREM 2.3. *Let* $\mathbf{x} = (x_1,\dots,x_n), \mathbf{y} = (y_1,\dots,y_m)$ *be two real monotonic tuples from* I *such that*

$$\max(x_1,\dots,x_n) \leqslant c \leqslant \min(y_1,\dots,y_m), \tag{6}$$

and $\mathbf{p} = (p_1,\dots,p_n), \mathbf{q} = (q_1,\dots,q_m)$ *are real tuples such that*

$$\begin{aligned} &0 \leqslant P_k \leqslant P_n, \quad P_n > 0, \quad (k = 1,\dots,n); \\ &0 \leqslant Q_i \leqslant Q_m, \quad Q_m > 0 \quad (i = 1,\dots,m). \end{aligned} \tag{7}$$

If $A_{\mathbf{x},\mathbf{p}}(e_2) = A_{\mathbf{y},\mathbf{q}}(e_2)$, *i. e.*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^2 = \frac{1}{Q_m}\sum_{i=1}^{m} q_i y_i^2 - \left(\frac{1}{Q_m}\sum_{i=1}^{m} q_i y_i\right)^2, \tag{8}$$

then for every $\varphi \in \mathscr{K}_1^c(I)$

$$A_{\mathbf{x},\mathbf{p}}(\varphi) \leqslant A_{\mathbf{y},\mathbf{q}}(\varphi). \tag{9}$$

Proof. Let $\Phi(x) = \varphi(x) - \frac{K_\varphi}{2} e_2(x)$, where K_φ is the constant from Definition 2.1. Since $\Phi : I \cap (-\infty, c] \to \mathbb{R}$ is concave, Jensen-Steffensen's inequality (1) implies

$$0 \leqslant A_{\mathbf{x},\mathbf{p}}(-\Phi) = -A_{\mathbf{x},\mathbf{p}}(\varphi) + \frac{K_\varphi}{2} A_{\mathbf{x},\mathbf{p}}(e_2). \tag{10}$$

Similarly, $\Phi : I \cap [c, \infty) \to \mathbb{R}$ is convex, so

$$0 \leqslant A_{\mathbf{y},\mathbf{q}}(\Phi) = A_{\mathbf{y},\mathbf{q}}(\varphi) - \frac{K_\varphi}{2} A_{\mathbf{y},\mathbf{q}}(e_2). \tag{11}$$

Adding up (10) and (11) we obtain

$$0 = \frac{K_\varphi}{2}\left(A_{\mathbf{y},\mathbf{q}}(e_2) - A_{\mathbf{x},\mathbf{p}}(e_2)\right) \leqslant A_{\mathbf{y},\mathbf{q}}(\varphi) - A_{\mathbf{x},\mathbf{p}}(\varphi),$$

which completes the proof. □

REMARK 2.4. It is obvious from the proof that we have, in fact, proven

$$A_{\mathbf{x},\mathbf{p}}(\varphi) \leqslant \frac{K_\varphi}{2} A_{\mathbf{x},\mathbf{p}}(e_2) = \frac{K_\varphi}{2} A_{\mathbf{y},\mathbf{q}}(e_2) \leqslant A_{\mathbf{y},\mathbf{q}}(\varphi).$$

Furthermore, inequality (9) holds if equality (8) is replaced by the weaker condition

$$K_\varphi\left(A_{\mathbf{y},\mathbf{q}}(e_2) - A_{\mathbf{x},\mathbf{p}}(e_2)\right) \geqslant 0.$$

Since $\varphi''_-(c) \leqslant K_\varphi \leqslant \varphi''_+(c)$ (see [1]), if, additionally, φ is convex (resp. concave), this condition can be further weakened to $A_{\mathbf{y},\mathbf{q}}(e_2) - A_{\mathbf{x},\mathbf{p}}(e_2) \geqslant 0$ (resp. $\leqslant 0$).

By the same reasoning we can compare the Jensen-Steffensen functionals under the Jensen-Boas conditions.

THEOREM 2.5. *Let $c \in I^{\circ}$ and let $f:[a_1,b_1] \to \mathbb{R}$ and $g:[a_2,b_2] \to \mathbb{R}$ be continuous monotonic functions (either increasing or decreasing) with ranges $I \cap (-\infty,c]$ and $I \cap [c,\infty)$ respectively. Let $\lambda:[a_1,b_1] \to \mathbb{R}$ and $\mu:[a_2,b_2] \to \mathbb{R}$ be continuous or of bounded variation satisfying*

$$\lambda(a_1) \leqslant \lambda(x_1) \leqslant \lambda(y_1) \leqslant \lambda(x_2) \leqslant \cdots \leqslant \lambda(y_{n-1}) \leqslant \lambda(x_n) \leqslant \lambda(b_1)$$

for all $x_k \in (y_{k-1},y_k)$ $(y_0 = a_1, y_n = b_1)$ and $\lambda(b_1) > \lambda(a_1)$ and

$$\mu(a_2) \leqslant \mu(u_1) \leqslant \mu(v_1) \leqslant \mu(u_2) \leqslant \cdots \leqslant \mu(v_{m-1}) \leqslant \mu(u_m) \leqslant \mu(b_2)$$

for all $u_k \in (v_{k-1},v_k)$ $(v_0 = a_2, v_m = b_2)$ and $\mu(b_2) > \mu(a_2)$.

If $\varphi \in \mathscr{K}_1^c(I)$ is continuous and $B_{a_1,b_1}^{f,\lambda}(e_2) = B_{a_2,b_2}^{g,\mu}(e_2)$, i. e.

$$\frac{\int_{a_1}^{b_1} d\lambda(x) \int_{a_1}^{b_1} f^2(x)d\lambda(x) - \left(\int_{a_1}^{b_1} f(x)d\lambda(x)\right)^2}{\left(\int_{a_1}^{b_1} d\lambda(x)\right)^2} = \frac{\int_{a_2}^{b_2} d\mu(x) \int_{a_2}^{b_2} g^2(x)d\mu(x) - \left(\int_{a_2}^{b_2} g(x)d\mu(x)\right)^2}{\left(\int_{a_2}^{b_2} d\mu(x)\right)^2} \tag{12}$$

then

$$B_{a_1,b_1}^{f,\lambda}(\varphi) \leqslant B_{a_2,b_2}^{g,\mu}(\varphi). \tag{13}$$

Using the Jensen-Brunk inequality we have the following theorem.

THEOREM 2.6. *Let $c \in I^{\circ}$ and let $f:[a_1,b_1] \to \mathbb{R}$ and $g:[a_2,b_2] \to \mathbb{R}$ be continuous increasing functions with ranges $I \cap (-\infty,c]$ and $I \cap [c,\infty)$ respectively. Let $\lambda:[a_1,b_1] \to \mathbb{R}$ and $\mu:[a_2,b_2] \to \mathbb{R}$ be continuous or of bounded variation satisfying $\lambda(b_1) > \lambda(a_1)$, $\mu(b_2) > \mu(a_2)$. Let*

$$\int_{a_1}^{x} (f(x)-f(t))\,d\lambda(t) \geqslant 0 \text{ and } \int_{x}^{b_1} (f(x)-f(t))\,d\lambda(t) \leqslant 0$$

for all $x \in [a_1,b_1]$, and let

$$\int_{a_2}^{x} (g(x)-g(t))\,d\mu(t) \geqslant 0 \text{ and } \int_{x}^{b_2} (g(x)-g(t))\,d\mu(t) \leqslant 0$$

for all $x \in [a_2,b_2]$.

If (12) is satisfied, then for any continuous $\varphi \in \mathscr{K}_1^c(I)$ inequality (13) holds.

Using Theorem 1.5 we have the same comparison for functionals.

THEOREM 2.7. *Let $c \in I^\circ$ and let $f : [a_1,b_1] \to \mathbb{R}$ and $g : [a_2,b_2] \to \mathbb{R}$ be continuous functions with ranges $I \cap (-\infty,c]$ and $I \cap [c,\infty)$ respectively. Let $\lambda : [a_1,b_1] \to \mathbb{R}$ and $\mu : [a_2,b_2] \to \mathbb{R}$ be continuous or of bounded variation satisfying $\lambda(b_1) > \lambda(a_1)$, $\mu(b_2) > \mu(a_2)$. Let*

$$\left| \int_{a_1}^{b_1} (f(x) - f(t))\, d\lambda(t) \right| \leqslant \int_{a_1}^{b_1} |f(x) - f(t)|\, d\lambda(t)$$

for all $x \in [a_1,b_1]$ and

$$\left| \int_{a_2}^{b_2} (g(x) - g(t))\, d\mu(t) \right| \leqslant \int_{a_2}^{b_2} |g(x) - g(t)|\, d\mu(t) \leqslant 0$$

for all $x \in [a_2,b_2]$.

If (12) is satisfied, then for any continuous $\varphi \in \mathscr{K}_1^c(I)$ inequality (13) holds.

Now we compare the Matić-Pečarić functionals.

THEOREM 2.8. *Let $(\Omega_1, \mathscr{A}_1, \mu_1)$ and $(\Omega_2, \mathscr{A}_2, \mu_2)$ be two measure spaces with $0 < \mu_i(\Omega_i) < \infty$, $i = 1,2$ and let $\varphi \in \mathscr{K}_1^c(I)$, I an open interval. Let $f : \Omega_1 \to I \cap (-\infty, c)$ be a function such that $f, \varphi(f), \varphi'_+(f)$ and $f\varphi'_+(f)$ are all in $L^1(\mu_1)$, let $g : \Omega_2 \to I \cap (c,\infty)$ be a function such that $g, \varphi(g), \varphi'_+(g)$ and $g\varphi'_+(g)$ are all in $L^1(\mu_2)$ and let $u \in I \cap (-\infty,c)$ and $v \in I \cap (c,\infty)$. If*

$$u^2 + \frac{1}{\mu_1(\Omega_1)} \int_{\Omega_1} (f - 2u) f\, d\mu_1 = v^2 + \frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} (g - 2v) g\, d\mu_2, \tag{14}$$

then

$$M_{f,\mu_1,u}(\varphi) \leqslant M_{g,\mu_2,v}(\varphi).$$

and

$$M^{f,\mu_1,u}(\varphi) \leqslant M^{g,\mu_2,v}(\varphi).$$

Proof. Notice that

$$M_{f,\mu_1,u}(e_2) = M^{f,\mu_1,u}(e_2) = u^2 + \frac{1}{\mu_1(\Omega_1)} \int_{\Omega_1} (f - 2u) f\, d\mu_1$$

and

$$M_{g,\mu_2,v}(e_2) = M^{g,\mu_2,v}(e_2) = v^2 + \frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} (g - 2v) g\, d\mu_2.$$

Therefore, condition (14) is equivalent to the condition $M_{f,\mu_1,u}(e_2) = M_{g,\mu_2,v}(e_2)$ and to the condition $M^{f,\mu_1,u}(e_2) = M^{g,\mu_2,v}(e_2)$. The rest of the proof follows the lines of the proof of Theorem 2.3. □

REMARK 2.9. Conditions (14) has a more general form than condition (12). Indeed, if we put $u = \frac{1}{\mu_1(\Omega_1)} \int_{\Omega_1} f d\mu_1$ and $v = \frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} g d\mu_2$ in (14), then we get (12).

Using Theorem 2.8 and Remark 1.7 we get the following corollary.

COROLLARY 2.10. *Let all of the assumptions of Theorem 2.8 be satisfied with*

$$u = \int_{\Omega_1} f\varphi'_+(f)d\mu_1 \Big/ \int_{\Omega_1} \varphi'_+(f)d\mu_1,$$

$$v = \int_{\Omega_2} g\varphi'_+(g)d\mu_2 \Big/ \int_{\Omega_2} \varphi'_+(g)d\mu_2,$$

where we assume $\int_{\Omega_1} \varphi'_+(f)d\mu_1 \neq 0$, $\int_{\Omega_2} \varphi'_+(g)d\mu_2 \neq 0$. *Then*

$$\varphi\left(\int_{\Omega_1} f\varphi'_+(f)d\mu_1 \Big/ \int_{\Omega_1} \varphi'_+(f)d\mu_1\right) - \frac{1}{\mu_1(\Omega_1)} \int_{\Omega_1} \varphi(f)d\mu_1$$
$$\leqslant \varphi\left(\int_{\Omega_2} g\varphi'_+(g)d\mu_2 \Big/ \int_{\Omega_2} \varphi'_+(g)d\mu_2\right) - \frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} \varphi(f)d\mu_2. \tag{15}$$

Similarly, using Theorem 2.8 and Remark 1.9 we get the following corollary.

COROLLARY 2.11. *Let all of the assumptions of Theorem 2.8 be satisfied with* φ *strictly convex and*

$$u = (\varphi')^{-1}\left(\frac{1}{\mu_1(\Omega_1)} \int_{\Omega_1} \varphi'(f)d\mu_1\right),$$

$$v = (\varphi')^{-1}\left(\frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} \varphi'(g)d\mu_2\right).$$

If $\tilde{u} \in I \cap (-\infty, c)$ *and* $\tilde{v} \in I \cap (c, \infty)$ *are any other two points satisfying* $M^{f,\mu_1,\tilde{u}}(e_2) = M^{g,\mu_2,\tilde{v}}(e_2)$, *then*

$$\frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} \varphi(g)d\mu_2 - \frac{1}{\mu_1(\Omega_1)} \int_{\Omega_1} \varphi(f)d\mu_1$$
$$\leqslant \varphi(v) + \frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} (g-v)\varphi'_+(g)d\mu_2 - \varphi(u) - \frac{1}{\mu_1(\Omega_1)} \int_{\Omega_1} (f-u)\varphi'_+(f)d\mu_1$$
$$\leqslant \varphi(\tilde{v}) + \frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} (g-\tilde{v})\varphi'_+(g)d\mu_2 - \varphi(\tilde{u}) - \frac{1}{\mu_1(\Omega_1)} \int_{\Omega_1} (f-\tilde{u})\varphi'_+(f)d\mu_1. \tag{16}$$

REMARK 2.12. We observe that inequalities (15) and (16) are not linear over φ anymore.

3. Mean value results

In this section we will construct new linear functionals as certain differences of the linear functionals from the previous section. We will make use of the linearity of these newly introduced functionals to derive two mean value results.

For tuples $\mathbf{x},\mathbf{p}\in\mathbb{R}^n$ and $\mathbf{y},\mathbf{q}\in\mathbb{R}^m$ that satisfy the assumptions of Theorem 2.3, i. e. such that $\mathbf{x}\in I^n$ and $\mathbf{y}\in I^m$ are monotonic and (6), (7) and (8) hold, we define the linear functional

$$\Lambda_1(\varphi)=\Lambda_1(\varphi;\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})=A_{\mathbf{y},\mathbf{q}}(\varphi)-A_{\mathbf{x},\mathbf{p}}(\varphi). \tag{17}$$

The linear functional Λ_1 depends on the choice of the tuples $\mathbf{x},\mathbf{p},\mathbf{y}$ and $\mathbf{q}$, but we will omit them from the notation when they are clear from the context. By Theorem 2.3, for every continuous $\varphi\in\mathscr{K}_1^c(I)$ we have $\Lambda_1(\varphi)\geqslant 0$.

Similarly, under the assumptions of Theorem 2.5 (or Theorem 2.6 or Theorem 2.7), we define the linear operator

$$\Lambda_2(\varphi)=\Lambda_2(\varphi;f,\lambda,a_1,b_1,g,\mu,a_2,b_2)=B^{g,\mu}_{a_2,b_2}(\varphi)-B^{f,\lambda}_{a_1,b_1}(\varphi);$$

under the assumptions of Theorem 2.8, we define the linear operators

$$\Lambda_3(\varphi)=\Lambda_3(\varphi;f,\mu_1,u_1,g,\mu_2,u_2)=M_{g,\mu_2,u_2}(\varphi)-M_{f,\mu_1,u_1}(\varphi)$$

and

$$\Lambda_4(\varphi)=\Lambda_4(\varphi;f,\mu_1,v_1,g,\mu_2,v_2)=M^{g,\mu_2,v_2}(\varphi)-M^{f,\mu_1,v_1}(\varphi).$$

We will state the mean value results for the linear functional Λ_1, but analogous results hold for the remaining linear functionals Λ_i, $i=2,3,4$. The first mean value result is of Lagrange type.

THEOREM 3.1. *Let $-\infty<a<c<b<\infty$ and $I=[a,b]$, let $\mathbf{x},\mathbf{p},\mathbf{y}$ and $\mathbf{q}$ be as in Theorem 2.3 and let Λ_1 be given by (17). Then for $\varphi\in C^3([a,b])$ there exists $\xi\in[a,b]$ such that*

$$\Lambda_1(\varphi)=\frac{\varphi'''(\xi)}{6}\left[\frac{1}{Q_m}\sum_{i=1}^m q_iy_i^3-\left(\frac{1}{Q_m}\sum_{i=1}^m q_iy_i\right)^3\right.$$
$$\left.-\frac{1}{P_n}\sum_{i=1}^n p_ix_i^3+\left(\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right)^3\right]. \tag{18}$$

Proof. Since $\varphi \in C^3([a,b])$, there exist $m = \min_{x\in[a,b]} \varphi'''(x)$ and $M = \max_{x\in[a,b]} \varphi'''(x)$. The functions

$$\varphi_1(x) = \varphi(x) - \frac{m}{6}e_3(x),$$
$$\varphi_2(x) = \frac{M}{6}e_3(x) - \varphi(x)$$

satisfy $\varphi_i'''(x) \geqslant 0$, $i = 1,2$, so they are three times differentiable 3-convex functions. Therefore, $\varphi_1, \varphi_2 \in \mathscr{K}_1^c(I)$ (see Remark 2.2) and by Theorem 2.3 we have $\Lambda_1(\varphi_i) \geqslant 0$, $i = 1,2$, so

$$\frac{m}{6}\Lambda_1(e_3) \leqslant \Lambda_1(\varphi) \leqslant \frac{M}{6}\Lambda_1(e_3). \tag{19}$$

Since e_3 is 3-convex, by Theorem 2.3 we have

$$0 \leqslant \Lambda_1(e_3) = \frac{1}{Q_m}\sum_{i=1}^{m} q_i y_i^3 - \left(\frac{1}{Q_m}\sum_{i=1}^{m} q_i y_i\right)^3 - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^3 + \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)^3.$$

If $\Lambda_1(e_3) = 0$, then (19) implies $\Lambda_1(\varphi) = 0$ and (18) holds for every $\xi \in [a,b]$. Otherwise, dividing (19) by $0 < \Lambda_1(e_3)/6$ we get

$$m \leqslant \frac{6\Lambda_1(\varphi)}{\Lambda_1(e_3)} \leqslant M,$$

so continuity of φ''' insures existence of $\xi \in [a,b]$ satisfying (18). □

The next theorem is a Cauchy type mean value result.

THEOREM 3.2. *Let $c, I, \mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}$ and Λ_1 be as in Theorem 3.1 and let $\varphi, \rho \in C^3([a,b])$. If $\Lambda_1(\rho) \neq 0$, then there exists $\xi \in [a,b]$ such that either*

$$\frac{\Lambda_1(\varphi)}{\Lambda_1(\rho)} = \frac{\varphi'''(\xi)}{\rho'''(\xi)},$$

or $\varphi'''(\xi) = \rho'''(\xi) = 0$.

Proof. Define $\tau \in C^3([a,b])$ by $\tau(x) = \alpha\varphi(x) - \beta\rho(x)$, where $\alpha = \Lambda_1(\rho)$, $\beta = \Lambda_1(\varphi)$. Due to the linearity of Λ_1 we have $\Lambda_1(\tau) = 0$. Now, by Theorem 3.1 there exist $\xi, \xi_1 \in [a,b]$ such that

$$0 = \Lambda_1(\tau) = \frac{\tau'''(\xi)}{6}\Lambda_1(e_3),$$
$$0 \neq \Lambda_1(\rho) = \frac{\rho'''(\xi_1)}{6}\Lambda_1(e_3).$$

Therefore, $\Lambda_1(e_3) \neq 0$ and

$$0 = \tau'''(\xi) = \alpha\varphi'''(\xi) - \beta\rho'''(\xi),$$

which gives the claim of the theorem. □

4. Exponential convexity

The linear functionals introduced in the previous section will be used in the construction of new families of exponentially convex functions and some related results will be derived. We will first start with some basic definitions and results on exponential convexity.

DEFINITION 4.1. A function $\psi : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is k-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{k} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, k$.

A function $\psi : I \to \mathbb{R}$ is k-exponentially convex on I if it is k-exponentially convex in the Jensen sense and continuous on I.

REMARK 4.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions.

Also, k-exponentially convex functions in the Jensen sense are n-exponentially convex in the Jensen sense for every $n \leqslant k$, $n \in \mathbb{N}$.

DEFINITION 4.3. A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is k-exponentially convex in the Jensen sense on I for every $k \in \mathbb{N}$. A function $\psi : I \to \mathbb{R}$ is exponentially convex on I if it is exponentially convex in the Jensen sense and continuous on I.

REMARK 4.4. A function $\psi : I \to \mathbb{R}$ is log-convex in the Jensen sense, i. e.

$$\psi\left(\frac{x_1 + x_2}{2}\right)^2 \leqslant \psi(x_1)\psi(x_2), \quad \text{for all } x_1, x_2 \in I, \tag{20}$$

if and only if

$$\xi_1^2 \psi(x_1) + 2\xi_1\xi_2 \psi\left(\frac{x_1 + x_2}{2}\right) + \xi_2^2 \psi(x_2) \geqslant 0$$

holds for every $\xi_1, \xi_2 \in \mathbb{R}$ and $x_1, x_2 \in I$, i. e., if and only if ψ is 2-exponentially convex in the Jensen sense. If $\psi(x_1) = 0$ for some x_1 and $[a,b] \subset I$ is an arbitrary interval containing x_1, then it follows from (20) and non-negativity of ψ (see Remark 4.2) that ψ vanishes on $[a_1, b_1]$, where $a_1 = (a + x_1)/2$ and $b_1 = (x_1 + b)/2$. Applying the same reasoning to intervals $[a, a_1]$ and $[b_1, b]$ we obtain sequences $a_n \searrow a$ and $b_n \nearrow b$ with ψ vanishing on $[a_n, b_n]$. Thus ψ is zero on (a,b) and a function that is 2-exponentially convex in the Jensen sense is either identically equal to zero or it is strictly positive and log-convex in the Jensen sense.

The following lemma is equivalent to the definition of convex functions (see [7, pg 2]).

LEMMA 4.5. *A function* $\psi : I \to \mathbb{R}$ *is convex if and only if the inequality*

$$(x_3 - x_2)\psi(x_1) + (x_1 - x_3)\psi(x_2) + (x_2 - x_1)\psi(x_3) \geqslant 0$$

holds for all $x_1, x_2, x_3 \in I$ *such that* $x_1 < x_2 < x_3$.

We will also need the following result (see [7, pg 2]).

LEMMA 4.6. *If* ψ *is a convex function on an interval* I *and if* $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$ *and* $y_1 \neq y_2$, $x_i, y_i \in I$ *for* $i = 1,2$, *then the following inequality holds*

$$\frac{\psi(x_2) - \psi(x_1)}{x_2 - x_1} \leqslant \frac{\psi(y_2) - \psi(y_1)}{y_2 - y_1}. \tag{21}$$

If the function ψ *is concave then the sign of the above inequality is reversed.*

For the rest of the paper we will state results for the linear functional Λ_1, but analogous results hold for the remaining linear functionals Λ_i, $i = 2,3,4$. In what follows, the nth order divided difference of a function φ at $n+1$ mutually distinct points $x_0, x_1, \ldots, x_n$ is denoted by $[x_0, x_1, \ldots, x_n]\varphi$ and defined recursively by

$$[x_i]\varphi = \varphi(x_i), \qquad i = 0,1,\ldots,n$$

and

$$[x_0, x_1, \ldots, x_n]\varphi = \frac{[x_1, x_2, \ldots, x_n]\varphi - [x_0, x_1, \ldots, x_{n-1}]\varphi}{x_n - x_0}.$$

A function $\varphi : I \to \mathbb{R}$ is said to be n-convex if $[x_0, x_1, \ldots, x_n]\varphi \geqslant 0$ for all choices of $n+1$ distinct points $x_0, \ldots, x_n \in I$. In case $n = 2$ we obtain convex functions, i. e. 2-convex functions are just convex functions. The next theorem will enable us to construct families of exponentially convex functions.

THEOREM 4.7. *Let* $c, I, \mathbf{x}, \mathbf{p}, \mathbf{y}$ *and* $\mathbf{q}$ *be as in Theorem 2.3 and let* Λ_1 *be given by* (17). *Furthermore, let* $\Upsilon = \{\varphi_t : I \to \mathbb{R} \mid t \in J\} \subset \mathscr{K}_1^c(I)$, *where* J *is an interval in* $\mathbb{R}$, *be a family of* 3*-convex functions at point* $c \in I^\circ$ *such that for every three mutually different points* $x_0, x_1, x_2 \in I \cap (-\infty, c]$ *and* $y_0, y_1, y_2 \in I \cap [c, \infty)$ *the mappings*

$$t \mapsto -[x_0, x_1, x_2]\varphi_t + \frac{1}{2}K_{\varphi_t}$$

and

$$t \mapsto [y_0, y_1, y_2]\varphi_t - \frac{1}{2}K_{\varphi_t}$$

are k*-exponentially convex. Then the mapping* $t \mapsto \Lambda_1(\varphi_t)$ *is* k*-exponentially convex in the Jensen sense on* J. *If the mapping* $t \mapsto \Lambda_1(\varphi_t)$ *is continuous on* J, *then it is* k*-exponentially convex on* J.

Proof. For $\xi_i \in \mathbb{R}$ and $t_i \in J$, $i = 1,\dots,k$, we define the function

$$\varphi(x) = \sum_{i,j=1}^{k} \xi_i \xi_j \varphi_{\frac{t_i+t_j}{2}}(x)$$

and let

$$K_\varphi = \sum_{i,j=1}^{k} \xi_i \xi_j K_{\varphi_{\frac{t_i+t_j}{2}}}, \qquad \Phi(x) = \varphi(x) - \frac{K_\varphi}{2} x^2.$$

Due to the assumptions, for $x_0,x_1,x_2 \in I \cap (-\infty, c]$ the mapping $t \mapsto -[x_0,x_1,x_2]\varphi_t + \frac{1}{2}K_{\varphi_t} = -[x_0,x_1,x_2]\Phi_t$ is k-exponentially convex in the Jensen sense. Therefore

$$\begin{aligned}[x_0,x_1,x_2]\Phi &= [x_0,x_1,x_2]\varphi - \frac{1}{2}K_\varphi \\ &= \sum_{i,j=1}^{k} \xi_i\xi_j \left([x_0,x_1,x_2]\varphi_{\frac{t_i+t_j}{2}} - \frac{1}{2}K_{\varphi_{\frac{t_i+t_j}{2}}} \right) = \sum_{i,j=1}^{k} \xi_i\xi_j [x_0,x_1,x_2]\Phi_{\frac{t_i+t_j}{2}} \leqslant 0.\end{aligned}$$

Similarly, $[y_0,y_1,y_2]\Phi \geqslant 0$ for $y_0,y_1,y_2 \in I \cap [c,\infty)$ and this implies that φ is 3-convex at c. Therefore, by Theorem 2.3

$$0 \leqslant \Lambda_1(\varphi) = \sum_{i,j=1}^{k} \xi_i\xi_j \Lambda_1(\varphi_{\frac{t_i+t_j}{2}}).$$

Hence, the mapping $t \mapsto \Lambda_1(\varphi_t)$ is k-exponentially convex. If it is also continuous, it is k-exponentially convex by definition. □

If the assumptions of Theorem 4.7 hold for all $k \in \mathbb{N}$, then we immediately get the following corollary.

COROLLARY 4.8. *Let $c, I, \mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}$ and Λ_1 be as in Theorem 4.7. Furthermore, let $\Upsilon = \{\varphi_t : I \to \mathbb{R} \mid t \in J\} \subset \mathscr{K}_1^c(I)$, where J is an interval in $\mathbb{R}$, be a family of 3-convex functions at point $c \in I^\circ$ such that for every three mutually different points $x_0,x_1,x_2 \in I \cap (-\infty,c]$ and $y_0,y_1,y_2 \in I \cap [c,\infty)$ the mappings*

$$t \mapsto -[x_0,x_1,x_2]\varphi_t + \frac{1}{2}K_{\varphi_t}$$

and

$$t \mapsto [y_0,y_1,y_2]\varphi_t - \frac{1}{2}K_{\varphi_t}$$

are exponentially convex. Then the mapping $t \mapsto \Lambda_1(\varphi_t)$ is exponentially convex in the Jensen sense on J. If the mapping $t \mapsto \Lambda_1(\varphi_t)$ is continuous on J, then it is exponentially convex on J.

Another corollary of Theorem 4.7 is the following.

COROLLARY 4.9. *Let $c, I, \mathbf{x}, \mathbf{p}, \mathbf{y}, \mathbf{q}$ and Λ_1 be as in Theorem 4.7. Furthermore, let $\Upsilon = \{\varphi_t : I \to \mathbb{R} \mid t \in J\} \subset \mathscr{K}_1^c(I)$, where J is an interval in $\mathbb{R}$, be a family of 3-convex functions at point $c \in I^\circ$ such that for every three mutually different points $x_0, x_1, x_2 \in I \cap (-\infty, c]$ and $y_0, y_1, y_2 \in I \cap [c, \infty)$ the mappings*

$$t \mapsto -[x_0, x_1, x_2]\varphi_t + \frac{1}{2}K_{\varphi_t}$$

and

$$t \mapsto [y_0, y_1, y_2]\varphi_t - \frac{1}{2}K_{\varphi_t}$$

are 2-exponentially convex in the Jensen sense. Then the following statements hold:

(i) If the mapping $t \mapsto \Lambda_1(\varphi_t)$ is continuous on J, then for $r, s, t \in J$ such that $r < s < t$ we have

$$\Lambda_1(\varphi_s)^{t-r} \leqslant \Lambda_1(\varphi_r)^{t-s} \Lambda_1(\varphi_t)^{s-r}. \tag{22}$$

(ii) If the mapping $t \mapsto \Lambda_1(\varphi_t)$ is strictly positive and differentiable on J, then for all $s, t, u, v \in J$ such that $s \leqslant u$ and $t \leqslant v$ we have

$$\mu_{s,t}(\Upsilon) \leqslant \mu_{u,v}(\Upsilon),$$

where

$$\mu_{s,t}(\Upsilon) = \begin{cases} \left(\frac{\Lambda_1(\varphi_s)}{\Lambda_1(\varphi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}(\Lambda_1(\varphi_s))}{\Lambda_1(\varphi_s)}\right), & s = t. \end{cases} \tag{23}$$

Proof. (i) By Theorem 4.7 the mapping $t \mapsto \Lambda_1(\varphi_t)$ is 2-exponentially convex. Hence, by Remark 4.4, this mapping is either identically equal to zero, in which case inequality (22) holds trivially with zeros on both sides, or it is strictly positive and log-convex. Therefore, for $r, s, t \in J$ such that $r < s < t$ Lemma 4.5 with $g(t) = \log \Lambda_1(\varphi_t)$ gives

$$(t-s)\log\Lambda_1(\varphi_r) + (r-t)\log\Lambda_1(\varphi_s) + (s-r)\log\Lambda_1(\varphi_t) \geqslant 0.$$

This is equivalent to inequality (22).

(ii) By (i), the mapping $t \mapsto \Lambda_1(\varphi_t)$ is log-convex on J, which means that the function $t \mapsto \log \Lambda_1(\varphi_t)$ is convex on J. Hence, by using Lemma 4.6 with $s \leqslant u$, $t \leqslant v$, $s \neq t$, $u \neq v$, we obtain

$$\frac{\log\Lambda_1(\varphi_s) - \log\Lambda_1(\varphi_t)}{s-t} \leqslant \frac{\log\Lambda_1(\varphi_u) - \log\Lambda_1(\varphi_v)}{u-v},$$

that is,

$$\mu_{s,t}(\Upsilon) \leqslant \mu_{u,v}(\Upsilon).$$

Finally, the limiting cases $s = t$ are $u = v$ are obtained by applying standard continuity argument. $\square$

As an example of application of the above results, consider now the family of functions

$$\Upsilon_1 = \{\varphi_t : I \to \mathbb{R} \mid t \in \mathbb{R}\}, \qquad I \subset (0,\infty),$$

defined by

$$\varphi_t(x) = \begin{cases} \frac{1}{t(t-1)(t-2)} x^t, & t \neq 0,1,2 \\ \frac{1}{2}\ln x, & t = 0, \\ -x\ln x, & t = 1, \\ \frac{1}{2}x^2 \ln x, & t = 2. \end{cases} \tag{24}$$

Since $\varphi_t'''(x) = x^{t-3} \geqslant 0$, the functions φ_t are three times differentiable 3-convex functions. Therefore, $\varphi_t \in \mathscr{K}_1^c(I)$ for every $c \in I^\circ$ and $K_{\varphi_t} = \varphi_t''(c)$ (see Remark 2.2). Moreover, the function

$$\varphi(x) = \sum_{i,j=1}^{k} \xi_i \xi_j \varphi_{\frac{t_i+t_j}{2}}(x)$$

satisfies

$$\varphi'''(x) = \sum_{i,j=1}^{k} \xi_i \xi_j \varphi'''_{\frac{t_i+t_j}{2}}(x) = \left(\sum_{i=1}^{k} \xi_i e^{\frac{t_i-3}{2}\ln x} \right)^2 \geqslant 0,$$

so φ is also in $\mathscr{K}_1^c(I)$ with $K_\varphi = \varphi''(c)$. Furthermore, since $\Phi = \varphi - \frac{K_\varphi}{2} e_2$ is convex on $I \cap [c,\infty)$ and $K_\varphi = \sum_{i,j=1}^{k} \xi_i \xi_j K_{\varphi_{\frac{t_i+t_j}{2}}}$, for every three mutually different points $y_0, y_1, y_2 \in I \cap [c,\infty)$ we have

$$0 \leqslant [y_0,y_1,y_2]\Phi = [y_0,y_1,y_2]\varphi - \frac{1}{2}K_\varphi = \sum_{i,j=1}^{k} \xi_i \xi_j \left([y_0,y_1,y_2]\varphi_{\frac{t_i+t_j}{2}} - \frac{1}{2} K_{\varphi_{\frac{t_i+t_j}{2}}} \right).$$

Therefore, the mapping $t \mapsto [y_0,y_1,y_2]\varphi_t - \frac{1}{2}K_{\varphi_t}$ is k-exponentially convex in the Jensen sense. Analogously one can show that the same holds for the mapping $t \mapsto -[x_0,x_1,x_2]\varphi_t + \frac{1}{2}K_{\varphi_t}$. As this holds for all $k \in \mathbb{N}$, we see that the family Υ_1 satisfies the assumptions of Corollary 4.8. Hence, by Corollary 4.8, the mapping $t \mapsto \Lambda_1(\varphi_t)$ is exponentially convex in the Jensen sense. It is straightforward to check that it is also continuous, so the mapping $t \mapsto \Lambda_1(\varphi_t)$ is exponentially convex.

Applying Theorem 3.2 for the functions $\varphi = \varphi_s$ and $\rho = \varphi_t$ given by (24) and defined on a segment $I = [a,b] \subset (0,\infty)$, we conclude that there exist $\xi \in I$ such that

$$\xi = \left(\frac{\varphi_s'''}{\varphi_t'''} \right)^{-1} \left(\frac{\Lambda_1(\varphi_s)}{\Lambda_1(\varphi_t)} \right) = \left(\frac{\Lambda_1(\varphi_s)}{\Lambda_1(\varphi_t)} \right)^{\frac{1}{s-t}}, \quad s \neq t.$$

Moreover, $\mu_{s,t}(\Upsilon_1)$ given by (23) for the family Υ_1 can be calculated in the limiting cases $s \to t$ as well and it is equal to

$$\mu_{s,t}(\Upsilon_1) = \begin{cases} \left(\frac{\Lambda_1(\varphi_s)}{\Lambda_1(\varphi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{2\Lambda_1(\varphi_s\varphi_0)}{\Lambda_1(\varphi_0)} - \frac{3s^2-6s+2}{s(s-1)(s-2)}\right), & s = t \neq 0,1,2, \\ \exp\left(\frac{\Lambda_1(\varphi_0^2)}{\Lambda_1(\varphi_0)} + \frac{3}{2}\right), & s = t = 0, \\ \exp\left(\frac{\Lambda_1(\varphi_0\varphi_1)}{\Lambda_1(\varphi_1)}\right), & s = t = 1, \\ \exp\left(\frac{\Lambda_1(\varphi_0\varphi_2)}{\Lambda_1(\varphi_2)} - \frac{3}{2}\right), & s = t = 2. \end{cases}$$

By Corollary 4.9 (ii), $\mu_{s,t}(\Upsilon_1)$ are monotone in parameters s and t.

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