STRONGLY $\lambda$–CONVEX FUNCTIONS AND SOME CHARACTERIZATION OF INNER PRODUCT SPACES

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Abstract. In this paper we show that each strongly $\lambda$-convex function $f : D \to \mathbb{R}$ with modulus $c > 0$, where $D$ is an nonempty convex subset of inner product space $X$ with norm $\|\cdot\|$, must by of the form $g + \|\cdot\|^2$, where $g$ is an $\lambda$-convex function. Moreover, involving the notion of strongly $\lambda$-convexity we get a new characterization of inner product space.

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space, $D$ stand for a convex subset of $X$ and $c$ be a positive constant. A function $f : D \to \mathbb{R}$ is called strongly convex with modulus $c$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2,$$

for all $x, y \in D$ and $t \in (0, 1)$.

Such functions play an important role in optimization theory and have been introduced by Polyak in [13]. It has been also investigated by many other authors, among other, see [9], [10], [14], [15].

A function $f : D \to \mathbb{R}$ is called strongly $\lambda$-convex with modulus $c$ ($c > 0$) if

$$f(\lambda(x,y)x + (1-\lambda(x,y))y)$$

$$\leq \lambda(x,y)f(x) + (1-\lambda(x,y))f(y) - c\lambda(x,y)(1-\lambda(x,y))\|x - y\|^2,$$

for all $x, y \in D$, where $\lambda : D^2 \to (0, 1)$ is a fixed function. If we take $c = 0$, then we get defining inequality of $\lambda$-convex functions. In particular, $\lambda$-convex functions have been investigated in [2] and [12]. Notice, that each strongly convex function with modulus $c$ is strongly $\lambda$-convex with modulus $c$ with arbitrary function $\lambda$, and for $\lambda \equiv 1/2$ we get strongly Jensen convex function with modulus $c$.

In [11] the authors present relations between strongly convex (strongly Jensen convex) and convex (Jensen convex) functions. They give also a new characterization of inner product space which enriches the large collection of such characterization (cf. [1], [4], [5], [6], [7], [8]). The following result gives a generalization of the results stated in [11].


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2. Main result

At the beginning we present three useful lemmas. Using elementary properties of an inner product we get Lemma 1.

**Lemma 1.** Let \((X, \|\cdot\|)\) be a real inner product space, then
\[
\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,
\]
for all \(x, y \in D\) and \(t \in \mathbb{R}\).

**Proof.** Observe that for all \(x, y \in D\) and \(t \in \mathbb{R}\) we have
\[
t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \\
= t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)(\|x\|^2 - 2\langle x, y \rangle + \|y\|^2) \\
= (t - t(1-t))\|x\|^2 + 2t(1-t)\langle x, y \rangle + ((1-t) - t(1-t))\|y\|^2 \\
= t^2\|x\|^2 + 2t(1-t)\langle x, y \rangle + (1-t)^2\|y\|^2 \\
= \|tx + (1-t)y\|^2.
\]
Which was to be proved. □

In the next lemma we have a characterization of strongly \(\lambda\)-convex functions defined on a convex subset of a real inner product space.

**Lemma 2.** Let \((X, \|\cdot\|)\) be a real inner product space, \(D\) be a convex subset of \(X\), \(c\) be a positive constant and \(\lambda : D^2 \to (0, 1)\) be a fixed function. A function \(f : D \to \mathbb{R}\) is strongly \(\lambda\)-convex with modulus \(c\) if and only if the function \(g = f - c\|\cdot\|^2\) is \(\lambda\)-convex.

**Proof.** Assume that \(f : D \to \mathbb{R}\) is strongly \(\lambda\)-convex with modulus \(c\). Multiplying (3), with \(t = \lambda(x, y)\), by \(-c\) and adding with both sides to the inequality (2), we get an equivalent inequality from which follows that function \(g = f - c\|\cdot\|^2\) is \(\lambda\)-convex. The proof is finished. □

In [2] the author presents an example of a \(\lambda\)-convex function which is not convex, nor is it Jensen convex. Therefore, considering additionally Lemma 2 we get an example of a strongly \(\lambda\)-convex function with modulus \(c\) which is not strongly convex function with modulus \(c\), nor is it strongly Jensen convex function with modulus \(c\). The following lemma presents some relation between strongly \(\lambda\)-convex function with modulus \(c\) and strongly convex function with modulus \(c\) and it is the analogous result that we can find in [3] for convex functions.

**Lemma 3.** Let \((X, \|\cdot\|)\) be a real inner product space, \(D\) be a convex subset of \(X\), \(c\) be a positive constant and \(\lambda : D^2 \to (0, 1)\) be a fixed function. If a function \(f : D \to \mathbb{R}\) is strongly \(\lambda\)-convex with modulus \(c\) and continuous, then it must be strongly convex with modulus \(c\).
Proof. Suppose that $f$ is not strongly convex with modulus $c$, i.e.

$$f(tx + (1-t)y) > tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2,$$  \hspace{1cm} (4) for some $x, y \in D$ and $t \in (0, 1)$.

Fix $x, y \in D$ such that (4) holds for some $t \in (0, 1)$. Define function $g : [0, 1] \to \mathbb{R}$ by the formula

$$g(t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) + ct(1-t)\|x-y\|^2.$$  

Define set $A$ as an inverse image of the set $(0, \infty)$ by function $g$,

$$A = \{ t \in [0, 1] : g(t) > 0 \} = g^{-1}((0, \infty)).$$

Because function $g$ is continuous, then $A$ is open in $[0, 1]$ and obviously nonempty. Moreover, $g(0) = g(1) = 0$. Thus there exist $t_1, t_2 \in [0, 1]$ ($t_1 < t_2$) such, that

$$g(t_1) = g(t_2) = 0$$  
and $$g(t) > 0,$$  \hspace{1cm} (5) for all $t \in (t_1, t_2)$. Fix $t_1, t_2$ such, that (5) hold and adopt the following notation

$$x_1 = t_1x + (1-t_1)y, \hspace{0.5cm} y_1 = t_2x + (1-t_2)y.$$  \hspace{1cm} (6) and

$$t_0 = \lambda (x_1, y_1)t_1 + (1-\lambda (x_1, y_1))t_2.$$  \hspace{1cm} (7)

Now notice that

$$t_0x + (1-t_0)y = \lambda (x_1, y_1)x_1 + (1-\lambda (x_1, y_1))y_1$$  \hspace{1cm} (8) and

$$g(t_0) > 0.$$  

From (5), (6) and the definition of function $g$ we conclude that

$$f(x_1) = t_1f(x) + (1-t_1)f(y) - ct_1(1-t_1)\|x-y\|^2$$  \hspace{1cm} (9) and

$$f(y_1) = t_2f(x) + (1-t_2)f(y) - ct_2(1-t_2)\|x-y\|^2.$$  

Using elementary calculations we get this equality

$$ab(1-b) + (1-a)c(1-c) + a(1-a)(b-c)^2 = (ab + (1-a)c)(1-(ab + (1-a)c)),$$  

for all $a, b, c \in \mathbb{R}$.
Finally, to simplify the notation we can write $\lambda$ instead of $\lambda(x,y)$ and from (8), (9), (10) we obtain

$$f(\lambda x_1 + (1 - \lambda)y_1) - \lambda f(x_1) - (1 - \lambda)f(y_1) + c\lambda(1 - \lambda)\|x_1 - y_1\|^2$$

$$= f(\lambda x_1 + (1 - \lambda)y_1) - (\lambda t_1 + (1 - \lambda)t_2)f(x) - (1 - (\lambda t_1 + (1 - \lambda)t_2))f(y)$$

$$+ c(\lambda t_1(1 - t_1) + (1 - \lambda)t_2(1 - t_2))\|x - y\|^2 + c\lambda(1 - \lambda)\|(t_1 - t_2)(x - y)\|^2$$

$$= f(t_0x + (1 - t_0)y) - t_0f(x) - (1 - t_0)f(y)$$

$$+ c(\lambda t_1(1 - t_1) + (1 - \lambda)t_2(1 - t_2) + \lambda(1 - \lambda)(t_1 - t_2)^2)\|x - y\|^2$$

$$= f(t_0x + (1 - t_0)y) - t_0f(x) - (1 - t_0)f(y) + ct_0(1 - t_0)\|x - y\|^2 = g(t_0) > 0.$$  

Which means that

$$f(\lambda x_1 + (1 - \lambda)y_1) > \lambda f(x_1) + (1 - \lambda)f(y_1) - c\lambda(1 - \lambda)\|x_1 - y_1\|^2$$

and we have a contradiction with strong $\lambda$-convexity of the function $f$. Thus the function $f$ must be strongly convex. This completes the proof. \qed

Similarly as in [11] we show, that the assumption that $X$ is an inner product space is necessary in Lemma 2. We conclude this from the following characterization of inner product spaces.

**Theorem 1.** Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent to each other:

1. For all $c > 0$, convex subsets $D$ of $X$, functions $\lambda : D^2 \to (0, 1)$ and for all functions $f : D \to \mathbb{R}$, $f$ is strongly $\lambda$-convex with modulus $c$ if and only if $g = f - c\|\cdot\|^2$ is $\lambda$-convex;

2. For all functions $\lambda : X^2 \to (0, 1)$ the function $\|\cdot\|^2 : X \to \mathbb{R}$ is strongly $\lambda$-convex with modulus 1;

3. $(X, \|\cdot\|)$ is an inner product space.

**Proof.** To proof implication $(1) \Rightarrow (2)$ we take $g = 0$. Then $f = c\|\cdot\|^2$ is strongly $\lambda$-convex with modulus $c$. Thus $\|\cdot\|^2 = \frac{1}{c}f$ is strongly $\lambda$-convex with modulus 1.

Assume (2). From Lemma 3 we have in particular the following inequality

$$\frac{\|x + y\|^2}{2} \leq \frac{\|x\|^2 + \|y\|^2}{2} - \frac{1}{4}\|x - y\|^2,$$

for all $x, y \in X$. Obviously, this inequality is equivalent to the parallelogram law, which implies that $(X, \|\cdot\|)$ is an inner product space.

Implication (3) $\Rightarrow$ (1) follows from Lemma 2. \qed

In the end, taking into account Lemma 2 and results stated in [2], we present two corollaries.
COROLLARY 1. Let $D$ be a nonempty, open, convex subset of $\mathbb{R}^n$ and $\lambda : D^2 \to (0, 1)$ be a fixed function continuous in both variables. If a function $f : D \to \mathbb{R}$ is strongly $\lambda$-convex and locally bounded from above at a point of $D$, then $f$ is strongly convex.

COROLLARY 2. Let $D$ be a nonempty, open, convex subset of $\mathbb{R}^n$ and $\lambda : D^2 \to (0, 1)$ be a fixed function continuously differentiable on $D^2$. If a function $f : D \to \mathbb{R}$ is strongly $\lambda$-convex and Lebesgue measurable, then $f$ is strongly convex.

REFERENCES