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STRONGLY λ–CONVEX FUNCTIONS AND SOME CHARACTERIZATION OF INNER PRODUCT SPACES

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Abstract. In this paper we show that each strongly λ-convex function $f : D \to \mathbb{R}$ with modulus $c > 0$, where D is an nonempty convex subset of inner product space X with norm $\|\cdot\|$, must by of the form $g + \|\cdot\|^2$, where g is an λ-convex function. Moreover, involving the notion of strongly λ-convexity we get a new characterization of inner product space.

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space, D stand for a convex subset of X and c be a positive constant. A function $f : D \to \mathbb{R}$ is called *strongly convex with modulus* c if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2, \tag{1}$$

for all $x, y \in D$ and $t \in (0,1)$.

Such functions play an important role in optimization theory and have been introduced by Polyak in [13]. It has been also investigated by many other authors, among other, see [9], [10], [14], [15].

A function $f : D \to \mathbb{R}$ is called *strongly λ-convex with modulus* c $(c > 0)$ if

$$\begin{aligned} &f(\lambda(x,y)x + (1-\lambda(x,y))y) \\ &\qquad \leqslant \lambda(x,y)f(x) + (1-\lambda(x,y))f(y) - c\lambda(x,y)(1-\lambda(x,y))\|x-y\|^2, \end{aligned} \tag{2}$$

for all $x, y \in D$, where $\lambda : D^2 \to (0,1)$ is a fixed function. If we take $c = 0$, then we get defining inequality of λ-convex functions. In particular, λ-convex functions have been investigated in [2] and [12]. Notice, that each strongly convex function with modulus c is strongly λ-convex with modulus c with arbitrary function λ, and for $\lambda \equiv 1/2$ we get *strongly Jensen convex function with modulus* c.

In [11] the authors present relations between strongly convex (strongly Jensen convex) and convex (Jensen convex) functions. They give also a new characterization of inner product space which enriches the large collection of such characterization (cf. [1], [4], [5], [6], [7], [8]). The following result gives a generalization of the results stated in [11].

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2. Main result

At the beginning we present three useful lemmas.

Using elementary properties of an inner product we get Lemma 1.

LEMMA 1. *Let* $(X, \|\cdot\|)$ *be a real inner product space, then*

$$\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \tag{3}$$

for all $x, y \in D$ *and* $t \in \mathbb{R}$.

Proof. Observe that for all $x, y \in D$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} &t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \\ &\quad = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)(\|x\|^2 - 2\langle x,y\rangle + \|y\|^2) \\ &\quad = (t - t(1-t))\|x\|^2 + 2t(1-t)\langle x,y\rangle + ((1-t) - t(1-t))\|y\|^2 \\ &\quad = t^2\|x\|^2 + 2t(1-t)\langle x,y\rangle + (1-t)^2\|y\|^2 \\ &\quad = \|tx+(1-t)y\|^2. \end{aligned}$$

Which was to be proved. □

In the next lemma we have a characterization of strongly λ-convex functions defined on a convex subset of a real inner product space.

LEMMA 2. *Let* $(X, \|\cdot\|)$ *be a real inner product space,* D *be a convex subset of* X*,* c *be a positive constant and* $\lambda : D^2 \to (0,1)$ *be a fixed function. A function* $f : D \to \mathbb{R}$ *is strongly* λ*-convex with modulus* c *if and only if the function* $g = f - c\|\cdot\|^2$ *is* λ*-convex.*

Proof. Assume that $f : D \to \mathbb{R}$ is strongly λ-convex with modulus c. Multiplying (3), with $t = \lambda(x,y)$, by $-c$ and adding with both sides to the inequality (2), we get an equivalent inequality from which follows that function $g = f - c\|\cdot\|^2$ is λ-convex. The proof is finished. □

In [2] the author presents an example of a λ-convex function which is not convex, nor is it Jensen convex. Therefore, considering additionally Lemma 2 we get an example of a strongly λ-convex function with modulus c which is not strongly convex function with modulus c, nor is it strongly Jensen convex function with modulus c. The following lemma presents some relation between strongly λ-convex function with modulus c and strongly convex function with modulus c and it is the analogous result that we can find in [3] for convex functions.

LEMMA 3. *Let* $(X, \|\cdot\|)$ *be a real inner product space,* D *be a convex subset of* X*,* c *be a positive constant and* $\lambda : D^2 \to (0,1)$ *be a fixed function. If a function* $f : D \to \mathbb{R}$ *is strongly* λ*-convex with modulus* c *and continuous, then it must be strongly convex with modulus* c.

Proof. Suppose that f is not strongly convex with modulus c, i.e.

$$f(tx+(1-t)y) > tf(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2, \tag{4}$$

for some $x,y \in D$ and $t \in (0,1)$.

Fix $x,y \in D$ such that (4) holds for some $t \in (0,1)$. Define function $g:[0,1] \to \mathbb{R}$ by the formula

$$g(t) = f(tx+(1-t)y)-tf(x)-(1-t)f(y)+ct(1-t)\|x-y\|^2.$$

Define set A as an inverse image of the set $(0,\infty)$ by function g,

$$A = \{t \in [0,1] : g(t) > 0\} = g^{-1}((0,\infty)).$$

Because function g is continuous , then A is open in $[0,1]$ and obviously nonempty. Moreover, $g(0) = g(1) = 0$. Thus there exist $t_1,t_2 \in [0,1]$ $(t_1 < t_2)$ such, that

$$\begin{gathered} g(t_1) = g(t_2) = 0 \\ \text{and} \\ g(t) > 0, \end{gathered} \tag{5}$$

for all $t \in (t_1,t_2)$. Fix t_1, t_2 such, that (5) hold and adopt the following notation

$$x_1 = t_1x+(1-t_1)y, \quad y_1 = t_2x+(1-t_2)y. \tag{6}$$

and

$$t_0 = \lambda(x_1,y_1)t_1+(1-\lambda(x_1,y_1))t_2. \tag{7}$$

Now notice that

$$t_0x+(1-t_0)y = \lambda(x_1,y_1)x_1+(1-\lambda(x_1,y_1))y_1 \tag{8}$$

and

$$g(t_0) > 0.$$

From (5), (6) and the definition of function g we conclude that

$$f(x_1) = t_1f(x)+(1-t_1)f(y)-ct_1(1-t_1)\|x-y\|^2 \tag{9}$$

and

$$f(y_1) = t_2f(x)+(1-t_2)f(y)-ct_2(1-t_2)\|x-y\|^2,$$

Using elementary calculations we get this equality

$$ab(1-b)+(1-a)c(1-c)+a(1-a)(b-c)^2 = (ab+(1-a)c)(1-(ab+(1-a)c)), \tag{10}$$

for all $a,b,c \in \mathbb{R}$.

Finally, to simplify the notation we can write λ instead of $\lambda(x,y)$ and from (8), (9), (10) we obtain

$$\begin{aligned} & f(\lambda x_1+(1-\lambda)y_1)-\lambda f(x_1)-(1-\lambda)f(y_1)+c\lambda(1-\lambda)\|x_1-y_1\|^2 \\ = & f(\lambda x_1+(1-\lambda)y_1)-(\lambda t_1+(1-\lambda)t_2)f(x)-(1-(\lambda t_1+(1-\lambda)t_2))f(y) \\ & +c(\lambda t_1(1-t_1)+(1-\lambda)t_2(1-t_2))\|x-y\|^2+c\lambda(1-\lambda)\|(t_1-t_2)(x-y)\|^2 \\ = & f(t_0x+(1-t_0)y)-t_0f(x)-(1-t_0)f(y) \\ & +c(\lambda t_1(1-t_1)+(1-\lambda)t_2(1-t_2)+\lambda(1-\lambda)(t_1-t_2)^2)\|x-y\|^2 \\ = & f(t_0x+(1-t_0)y)-t_0f(x)-(1-t_0)f(y)+ct_0(1-t_0)\|x-y\|^2=g(t_0)>0. \end{aligned}$$

Which means that

$$f(\lambda x_1+(1-\lambda)y_1)>\lambda f(x_1)+(1-\lambda)f(y_1)-c\lambda(1-\lambda)\|x_1-y_1\|^2$$

and we have a contradiction with strong λ-convexity of the function f. Thus the function f must by strongly convex. This completes the proof. □

Similarly as in [11] we show, that the assumption that X is an inner product space is necessary in Lemma 2. We conclude this from the following characterization of inner product spaces.

THEOREM 1. *Let $(X,\|\cdot\|)$ be a real normed space. The following conditions are equivalent to each other:*

1. *For all $c>0$, convex subsets D of X, functions $\lambda: D^2\to(0,1)$ and for all functions $f: D\to\mathbb{R}$, f is strongly λ-convex with modulus c if and only if $g=f-c\|\cdot\|^2$ is λ-convex;*
2. *For all functions $\lambda: X^2\to(0,1)$ the function $\|\cdot\|^2: X\to\mathbb{R}$ is strongly λ-convex with modulus 1;*
3. *$(X,\|\cdot\|)$ is an inner product space.*

Proof. To proof implication *(1)⇒(2)* we take $g=0$. Then $f=c\|\cdot\|^2$ is strongly λ-convex with modulus c. Thus $\|\cdot\|^2=\frac{1}{c}f$ is strongly λ-convex with modulus 1.

Assume *(2)*. From Lemma 3 we have in particular the following inequality

$$\left\|\frac{x+y}{2}\right\|^2\leqslant\frac{\|x\|^2+\|y\|^2}{2}-\frac{1}{4}\|x-y\|^2,$$

for all $x,y\in X$. Obviously, this inequality is equivalent to the parallelogram law, which implies that $(X,\|\cdot\|)$ is an inner product space.

Implication *(3)⇒(1)* follows from Lemma 2. □

In the end, taking into account Lemma 2 and results stated in [2], we present two corollaries.

COROLLARY 1. *Let D be a nonempty, open, convex subset of $\mathbb{R}^n$ and $\lambda : D^2 \to (0,1)$ be a fixed function continuous in both variables. If a function $f : D \to \mathbb{R}$ is strongly λ-convex and locally bounded from above at a point of D, then f is strongly convex.*

COROLLARY 2. *Let D be a nonempty, open, convex subset of $\mathbb{R}^n$ and $\lambda : D^2 \to (0,1)$ be a fixed function continuously differentiable on D^2. If a function $f : D \to \mathbb{R}$ is strongly λ-convex and Lebesgue measurable, then f is strongly convex.*

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