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ESTIMATIONS OF HERON MEANS FOR POSITIVE OPERATORS

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Abstract. The arithmetic-geometric mean inequality induces the path of Heron means through these two means by $H_r^\mu(A,B) = r(A\sharp_\mu B) + (1-r)(A\nabla_\mu B)$ for each $\mu \in [0,1]$, $r \in \mathbb{R}$ and positive operators A, B on a Hilbert space. In this note, we estimate $H_r^\mu(A,B)$ by the harmonic mean. As an application of this method, we refine the arithmetic-geometric mean inequality under the assumption of the strict order $A - B \geqslant m > 0$.

1. Introduction

The arithmetic-geometric mean inequality has been discussed in various extensions. One of them is the Heron mean, which interpolates between the arithmetic mean and the geometric one, see Bhatia [1]. That is, for a fixed $\mu \in [0,1]$, the Heron mean for positive operators A and B is defined by

$$H_r^\mu(A,B) = rA \sharp_\mu B + (1-r)(A \nabla_\mu B)$$

for $r \in \mathbb{R}$. Here $A \nabla_\mu B = (1-\mu)A + \mu B$ and $A \sharp_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}}$, the μ-geometric mean. We also denote the μ-harmonic mean by $A\ !_\mu\ B = ((1-\mu)A^{-1} + \mu B^{-1})^{-1}$ and $A\ !\ B = A\ !_{\frac{1}{2}}\ B$, simply. Similarly we denote by $A \nabla B = A \nabla_{\frac{1}{2}} B$ and $A\sharp B = A\sharp_{\frac{1}{2}} B$.

Recently, one of the authors [4] proved the following proposition for the Heron mean.

PROPOSITION 1.1. ([4]) *Let A,B be invertible positive operators and $r \in \mathbb{R}$. Then the following inequalities hold.*

(i) If $r \geqslant 2$, then $rA\sharp B + (1-r)A \nabla B \leqslant A\ !\ B$.

(ii) If $r \leqslant 1$, then $rA\sharp B + (1-r)A \nabla B \geqslant A\ !\ B$.

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As a weighted version of (ii) in Proposition 1.1, we can easily show that

$$rA\sharp_\mu B + (1-r)A\,\nabla_\mu\, B \geqslant A\;!_\mu\; B \quad (r \leqslant 1,\; 0 \leqslant \mu \leqslant 1).$$

As a matter of fact, we have

$$rA\sharp_\mu B + (1-r)A\,\nabla_\mu\, B \geqslant rA\sharp_\mu B + (1-r)A\sharp_\mu B = A\sharp_\mu B \geqslant A!_\mu B.$$

On the other hand, a similar generalization of (i) in Proposition 1.1 does not hold: We take $r = 2$ and $\mu = \frac{2}{3}$. If $A = 1$ and $B = 2$, then it is arranged as the numerical inequality

$$2t^\mu \leqslant 1-\mu+\mu t + (1-\mu+\mu t^{-1})^{-1}. \tag{1}$$

It is false as follows:

$$(t^\mu)^3 = 4 \text{ and } \left(\frac{1}{2}\left(1-\mu+\mu t+(1-\mu+\mu t^{-1})^{-1}\right)\right)^3 = \left(\frac{19}{12}\right)^3 \simeq 3.9693\cdots.$$

Therefore our interest is to find a constant r_μ such that

$$H_r^\mu(A,B) \leqslant A\;!_\mu\; B \quad for \quad r \geqslant r_\mu.$$

We remark that the inequality (1) holds for $0 \leqslant \mu \leqslant 1/2$ and $t \geqslant 1$, or $1/2 \leqslant \mu \leqslant 1$ and $0 < t \leqslant 1$, [4, Lemma 2.3].

In the rest of this section, we discuss Proposition 1.1 itself.

PROPOSITION 1.2. *The conditions on r in Proposition F are optimal, i.e.,*

$$\inf\{r \mid rA\sharp B + (1-r)A\,\nabla\, B \leqslant A!B\} = 2$$

and

$$\sup\{r \mid rA\sharp B + (1-r)A\,\nabla\, B \geqslant A!B\} = 1.$$

Proof. We note that $rA\sharp B + (1-r)A\,\nabla\, B \leqslant A!B$ (resp. $\geqslant$) is equivalent to

$$r \geqslant \frac{(1+\sqrt{a})^2}{1+a}\; (resp. \leqslant).$$

Since $1 \leqslant \frac{(1+\sqrt{a})^2}{1+a} \leqslant 2$ and

$$\lim_{a\to 0}\frac{(1+\sqrt{a})^2}{1+a} = 1;\; \lim_{a\to 1}\frac{(1+\sqrt{a})^2}{1+a} = 2,$$

we have the conclusion. $\square$

Moreover we investigate it from another viewpoint: Let $R(t)$ be the ratio of $\frac{t+1}{2} - \frac{2t}{t+1}$ by $\frac{t+1}{2} - \sqrt{t}$, i.e., $R(t) = \frac{t+1-\frac{4t}{t+1}}{t+1-2\sqrt{t}}$. Then we have $R(0) = 1 \leqslant R(t) \leqslant 2 = R(1)$ for $t \geqslant 0$ and $R'(t) = \frac{1-t}{\sqrt{t}(t+1)^2}$. As a matter of fact, $R(t) = 1 + \frac{2\sqrt{t}}{t+1} = \frac{(1+\sqrt{t})^2}{1+t}$ and

$$r \geqslant \frac{(1+\sqrt{a})^2}{1+a} \Leftrightarrow r \geqslant R(a).$$

If we put $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, then $R(C) \leqslant r$ if and only if $H_r(A,B) \leqslant A \,!\, B$.
We have another variant of Proposition 1.1, cf. [4, Corollary 2.7].

PROPOSITION 1.3. *If either (i)* $0 \leqslant \mu \leqslant \frac{1}{2}$ *and* $0 < A \leqslant B$ *or (ii)* $\frac{1}{2} \leqslant \mu \leqslant 1$ *and* $0 < B \leqslant A$, *then*

$$2A\sharp_\mu B \leqslant A \nabla_\mu B + B \,!_\mu\, A.$$

If either (i′) $0 \leqslant \mu \leqslant \frac{1}{2}$ *and* $0 < B \leqslant A$ *or (ii′)* $\frac{1}{2} \leqslant \mu \leqslant 1$ *and* $0 < A \leqslant B$, *then*

$$2A\sharp_{1-\mu} B \leqslant A \nabla_\mu B + B \,!_\mu\, A$$

and

$$A\sharp B \leqslant A\sharp_\mu B.$$

The proof is reduced to the following lemma:

LEMMA 1.4. *Put* $f(t) = (1-\mu) + \mu t + \left((1-\mu)t^{-1} + \mu\right)^{-1} - 2t^\mu$. *Then*

$$f(t) = (1-\mu) + \mu t + \frac{t}{(1-\mu) + \mu t} - 2t^\mu \geqslant 2\sqrt{t} - 2t^\mu = 2\sqrt{t}\left(1 - t^{\mu - \frac{1}{2}}\right).$$

If either (i) $0 \leqslant \mu \leqslant \frac{1}{2}$ *and* $t \geqslant 1$ *or (ii)* $\frac{1}{2} \leqslant \mu \leqslant 1$ *and* $t \leqslant 1$, *then*

$$2t^\mu \leqslant (1-\mu) + \mu t + ((1-\mu)t^{-1} + \mu)^{-1}.$$

If either (i′) $0 < \mu \leqslant \frac{1}{2}$ *and* $0 < t \leqslant 1$ *or (ii′)* $\frac{1}{2} \leqslant \mu \leqslant 1$ *and* $t \geqslant 1$, *then*

$$2t^{1-\mu} \leqslant (1-\mu) + \mu t + ((1-\mu)t^{-1} + \mu)^{-1}$$

and

$$\sqrt{t} \leqslant t^\mu.$$

Proof of Proposition 1.3. As an easy consequence of the first half, if (i) $0 \leqslant \mu \leqslant \frac{1}{2}$ and $t \geqslant 1$ or (ii) $\frac{1}{2} \leqslant \mu \leqslant 1$ and $0 < t \leqslant 1$, then $f(t) \geqslant 0$.

The second and third inequalities are obtained by $2\sqrt{t} - 2t^{1-\mu} = 2\sqrt{t}(1 - t^{\frac{1}{2}-\mu})$ and $t^\mu - t^{\frac{1}{2}} = t^\mu(1 - t^{\frac{1}{2}-\mu})$ respectively. □

REMARK 1.5. We find that the sign of $f(t)$ is not definite for the following cases

(a) $\mu \in [0, \frac{1}{2})$ and $0 \leqslant t \leqslant 1$,

(b) $\mu \in (\frac{1}{2}, 1]$ and $t \geqslant 1$,

since we actually have $f\left(\frac{1}{2}\right) \simeq -0.235364$ and $f\left(\frac{1}{50}\right) \simeq 0.0293694$ when $\mu = \frac{1}{4}$. We also have $f(2) \simeq -0.470729$ and $f(50) \simeq 1.46847$ when $\mu = \frac{3}{4}$, by numerical computations.

2. Generalization

Next we consider a generalization of $R(t)$ by defining $R_\mu(t)$ for $\mu \in (0,1)$;

$$R_\mu(t) = \frac{1-\mu+\mu t - \frac{t}{(1-\mu)t+\mu}}{1-\mu+\mu t - t^\mu} \quad (t \geqslant 0).$$

It is clear that $R_{\frac{1}{2}}(t) = R(t)$, and that $R_\mu(0) = 1$, $R_\mu(1) = 2$ for all $\mu \in (0,1)$ and $\max\{R(t); t \geqslant 0\} = 2$.

PROBLEM 2.1. $\max\{R_\mu(t); t \geqslant 0\} = ?$

For this problem, we pose an "answer" as a upper bound of $R_\mu(t)$ for $t \geqslant 0$.
For this, we note the following lemma mentioned in [4, Lemma 2.3].

LEMMA 2.2. *If either (i)* $0 \leqslant \mu \leqslant \frac{1}{2}$ *and* $t \geqslant 1$*, or (ii)* $\frac{1}{2} \leqslant \mu \leqslant 1$ *and* $0 \leqslant t \leqslant 1$*, then* $2t^\mu \leqslant 1-\mu+\mu t + (1-\mu+\frac{\mu}{t})^{-1}$.

To solve Problem 2.1, we define the function $f_r(t)$ for $r \geqslant 2$ and $\mu \in [\frac{1}{2}, 1]$ by

$$\begin{aligned} f_r(t) &= (r-1)(1-\mu+\mu t) + (1-\mu+\mu t^{-1})^{-1} - rt^\mu \\ &= (r-1)(1-\mu+\mu t) + \frac{t}{(1-\mu)t+\mu} - rt^\mu. \end{aligned}$$

It is easily seen that for a fixed $t > 0$, $f_r(t)$ is increasing for $r \geqslant 2$. As a consequence, it follows that if $\mu \in [\frac{1}{2}, 1]$, then

$$f_r(t) \geqslant 0 \ \ for\ r \geqslant 2 \ and\ 0 < t \leqslant 1. \tag{2}$$

We have the following generalized result.

THEOREM 2.3. *For a fixed* $\mu \in [\frac{1}{2}, 1)$*, if* $r \geqslant r_\mu \equiv \frac{2(2-\mu)}{3(1-\mu)}$*, then*

$$rA \sharp_\mu B + (1-r)A \nabla_\mu B \leqslant A \,!_\mu\, B$$

for all $A, B > 0$.

Proof. Put $f(t) = f_r(t) = (r-1)(1-\mu+\mu t) + \frac{t}{(1-\mu)t+\mu} - rt^\mu$. Thus it suffices to show that if $r \geqslant \frac{2(2-\mu)}{3(1-\mu)}$, then $f_r(t) \geqslant 0$ for $t \geqslant 1$. Because it follows from (ii) in Lemma 2.2 and monotonicity of $f_r(t)$ on r that $f_r(t) \geqslant 0$ for $0 < t \leqslant 1$. Now we have

$$f'(t) = (r-1)\mu + \frac{\mu}{((1-\mu)t+\mu)^2} - r\mu t^{\mu-1}$$

and

$$f''(t) = -\frac{2\mu(1-\mu)}{((1-\mu)t+\mu)^3} - r\mu(\mu-1)t^{\mu-2}$$
$$= \frac{\mu(1-\mu)}{((1-\mu)t+\mu)^3 t^{2-\mu}}\{r((1-\mu)t+\mu)^3 - 2t^{2-\mu}\}.$$

Next we put $g(t) = r((1-\mu)t+\mu)^3 - 2t^{2-\mu}$, the right half of $f''(t)$. Then we have

$$g'(t) = 3r(1-\mu)((1-\mu)t+\mu)^2 - 2(2-\mu)t^{1-\mu},$$
$$g''(t) = 6r(1-\mu)^2((1-\mu)t+\mu) - 2(2-\mu)(1-\mu)t^{-\mu}$$

and

$$g'''(t) = 6r(1-\mu)^3 + 2\mu(2-\mu)(1-\mu)t^{-\mu-1}.$$

Since $g'''(t) \geqslant 0$, $g''(t)$ is increasing and

$$g''(1) = 6r(1-\mu)^2 - 2(2-\mu)(1-\mu) = 2(1-\mu)(3r(1-\mu)-(2-\mu))$$
$$\geqslant 2(1-\mu)(3r(1-\mu)-2(2-\mu)) \geqslant 0.$$

Therefore $g''(t) \geqslant 0$ for $t \geqslant 1$, and so $g'(t)$ is increasing on $[1,\infty)$. Since

$$g'(1) = 3r(1-\mu) - 2(2-\mu) \geqslant 0,$$

we have $g'(t) \geqslant 0$ for $t \geqslant 1$. Furthermore, since $g(t)$ is increasing and $g(1) = r-2 \geqslant 0$, it implies that $g(t) \geqslant 0$ for $t \geqslant 1$. Hence $f''(t) \geqslant 0$ and so $f'(t)$ is increasing. Moreover, since $f'(1) = 0$, we have $f'(t) \geqslant 0$, that is, $f(t)$ is increasing for $t \geqslant 1$. Finally, $f(1) = 0$ implies the desired conclusion $f(t) \geqslant 0$ for $t \geqslant 1$. □

REMARK 2.4. Related to the assumption $r \geqslant r_\mu$ in Theorem 2.3, we have $r_\mu \geqslant r_{\frac{1}{2}} = 2$, since $r_\mu = \frac{2}{3}\left(1+\frac{1}{1-\mu}\right)$ is increasing as a function of $\mu \in [\frac{1}{2},1)$.

3. Strict order

Next we consider the above argument under the strict operator order. For convenience, we denote by $X > 0$ for invertivle positive operators X, $\sigma(X)$ for the spectrum of X and $m_X = \min\sigma(X) = \|X^{-1}\|^{-1}$ for $X > 0$.

THEOREM 3.1. *For given $A,B > 0$, put $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$.*

(i) If $0 \leqslant \mu \leqslant \frac{1}{2}$ and $B - A \geqslant m > 0$, then

$$c_1 \leqslant A \nabla_\mu B + B!_\mu A - 2A\sharp_\mu B$$

where

$$c_1 \equiv 2m_A\{(1+m\|A\|^{-1})^{\frac{1}{2}} - (1+m\|A\|^{-1})^\mu\} \geqslant 0.$$

(ii) *If* $\frac{1}{2} \leqslant \mu \leqslant 1$ *and* $A - B \geqslant m > 0$, *then*

$$\min\{c_2, c_3\} \leqslant A \nabla_\mu B + B!_\mu A - 2A \sharp_\mu B.$$

where

$$c_2 \equiv 2m_A \left\{ \left(1 - m\|A\|^{-1}\right)^{\frac{1}{2}} - \left(1 - m\|A\|^{-1}\right)^{\mu} \right\} \geqslant 0$$

and

$$c_3 \equiv 2m_A \left\{ (m_C)^{\frac{1}{2}} - (m_C)^{\mu} \right\} \geqslant 0.$$

To prove it, we prepare the following lemma. The proofs are not difficult computations so that we omit them.

LEMMA 3.2. *For a fixed* $\mu \in [0,1]$, *define* $g_\mu(t) = 2\left(\sqrt{t} - t^\mu\right)$ *for* $t \geqslant 0$. *Then*

(i) *If* $\mu \in [0, \frac{1}{2}]$, *then* g_μ *is increasing on* $[1, \infty)$.

(ii) *If* $\mu \in [\frac{1}{2}, 1]$, *then* g_μ *is decreasing on* $[1, \infty)$.

(iii) *If* $\mu \in [\frac{1}{2}, 1]$, *then* g_μ *is concave on* $[0, 1]$.

(iv) *If* $\mu \in [0, \frac{1}{2}]$, *then* g_μ *is increasing on* $[0, t_0]$ *and decreasing on* $[t_0, 1]$ *for some* $t_0 \in (0, 1)$.

(v) $g_\mu(t) \geqslant 0$ *for the following conditions (c) or (d).*

(c) $\mu \in [0, \frac{1}{2}]$ *and* $t \geqslant 1$

(d) $\mu \in [\frac{1}{2}, 1]$ *and* $0 \leqslant t \leqslant 1$.

Proof of Theorem 3.1.

(i) Since $B \geqslant A + m$, we have

$$C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \geqslant A^{-\frac{1}{2}} (A + m) A^{-\frac{1}{2}} = 1 + mA^{-1} \geqslant 1 + m\|A\|^{-1}.$$

So it follows from (i) of Lemma 3.2 that if $t \in \sigma(C)$, then g_μ is increasing for $t \geqslant 1$ by $0 \leqslant \mu \leqslant \frac{1}{2}$. Hence we have

$$g_\mu(t) \geqslant g_\mu(1 + m\|A\|^{-1}),$$

so that by Lemma 1.4

$$(1-\mu) + \mu t + \frac{t}{(1-\mu) + \mu t} - 2t^\mu \geqslant g_\mu(t) \geqslant g_\mu(1 + m\|A\|^{-1}).$$

Thus we have

$$(1-\mu) + \mu C + \frac{C}{(1-\mu) + \mu C} - 2C^\mu \geqslant g_\mu(1 + m\|A\|^{-1}),$$

That is,

$$c_1 \leqslant A \nabla_\mu B + B!_\mu A - 2A \sharp_\mu B.$$

(ii) Since $A-m \geqslant B$, we have

$$m_C \leqslant C \leqslant A^{-\frac{1}{2}}(A-m)A^{-\frac{1}{2}} = 1-mA^{-1} \leqslant 1-m\|A\|^{-1}.$$

Now g_μ is concave on $[0,1]$ by (iii) of Lemma 3.2, so that if $t \in \sigma(C)$, then

$$\min\{g_\mu(m_C), g_\mu(1-m\|A\|^{-1})\} \leqslant g_\mu(t).$$

Therefore, as in the proof of (i), we have

$$\min\{g_\mu(m_C), g_\mu(1-m\|A\|^{-1})\} m_A \leqslant A \nabla_\mu B + B!_\mu A - 2A\sharp_\mu B. \quad \square$$

REMARK 3.3. Since $C \leqslant ||C||$ and g_μ is decreasing on $[1,\infty)$ by (ii) of Lemma 3.2, we have $g_\mu(t) \geqslant g_\mu(||C||)$. This implies the following result.

If $\frac{1}{2} \leqslant \mu \leqslant 1$ and $B-A \geqslant 0$ (these conditions correspond to (b) in Remark 1.5), then

$$c_4 \leqslant A \nabla_\mu B + B!_\mu A - 2A\sharp_\mu B$$

where

$$c_4 \equiv 2m_A \left(||C||^{\frac{1}{2}} - ||C||^{\mu}\right) \leqslant 0.$$

To study the bounds for $A \nabla_\mu B + A!_\mu B - 2A\sharp_\mu B$ instead of $A \nabla_\mu B + B!_\mu A - 2A\sharp_\mu B$, we give the following lemma.

LEMMA 3.4. *For a fixed $\mu \in [0,1]$, define*

$$f_\mu(t) \equiv (1-\mu) + \mu t + \frac{t}{(1-\mu)t+\mu} - 2t^\mu$$

for $t \geqslant 0$. $g_\mu(t)$ is defined for $t \geqslant 0$ in Lemma 3.2. We set $t_\mu \equiv \frac{\mu(2-\mu)}{(1-\mu)(1+\mu)}$. Then we have the following properties.

(i) $f_\mu(1) = 0$. In addition, $f_\mu(t) \geqslant 0$ for the following conditions (c) or (d).

(c) $\mu \in [0,\frac{1}{2}]$ and $t \geqslant 1$

(d) $\mu \in [\frac{1}{2},1]$ and $0 \leqslant t \leqslant 1$.

(ii) If $\mu \in [0,\frac{1}{2}]$, then f_μ is increasing on $[1,\infty)$.

(iii) If $\mu \in [\frac{1}{2},1]$, then f_μ is decreasing on $[0,1]$.

(iv) If $\mu \in [0,\frac{1}{2}]$, then f_μ is increasing on $[t_\mu,1)$

(v) If $\mu \in [\frac{1}{2},1)$, then f_μ is decreasing on $(1,t_\mu]$.

(vi) If $\mu \in [0,\frac{1}{2}]$, then f_μ is convex on $[1,\infty)$.

(vii) If $\mu \in [\frac{1}{2},1]$, then f_μ is convex on $[0,1]$.

(viii) If $\mu \in [0, \frac{1}{2}]$, then f_μ is concave on $[t_\mu, 1)$.

(ix) If $\mu \in [\frac{1}{2}, 1)$, then f_μ is concave on $(1, t_\mu]$.

(x) If $\mu \in [0, \frac{1}{2}]$ and $0 \leqslant t \leqslant 1$, then $g_\mu(t) \leqslant \min\{f_\mu(t), 0\}$.

(xi) If $\mu \in [\frac{1}{2}, 1]$ and $t \geqslant 1$, then $g_\mu(t) \leqslant \min\{f_\mu(t), 0\}$.

Proof. First of all, we note the following facts. Since we have

$$f'_\mu(t) = \frac{\mu}{\{(1-\mu)t+\mu\}^2}\left(\left(1-2t^{\mu-1}\right)\{(1-\mu)t+\mu\}^2+1\right),$$

we put $d_\mu(t) \equiv \left(1-2t^{\mu-1}\right)\{(1-\mu)t+\mu\}^2+1$. Then we have

$$d'_\mu(t) = 2(1-\mu)\{(1-\mu)t+\mu\}\left(\{(1-\mu)t+\mu\}t^{\mu-2}+(1-2t^{\mu-1})\right),$$

so we put $h_\mu(t) \equiv -(\mu+1)t^{\mu-1}+\mu t^{\mu-2}+1$. Then we have $h'_\mu(t) = t^{\mu-3}k_\mu(t)$, where $k_\mu(t) \equiv (1-\mu^2)t+\mu(\mu-2)$. Then we have $k'_\mu(t) = 1-\mu^2$.

Since we also have

$$f''_\mu(t) = \frac{2\mu(1-\mu)}{\{(1-\mu)t+\mu\}^3}\left(t^{\mu-2}\{(1-\mu)t+\mu\}^3-1\right),$$

we put $l_\mu(t) \equiv t^{\mu-2}\{(1-\mu)t+\mu\}^3-1$. Then we have $l'_\mu(t) = t^{\mu-3}\{(1-\mu)t+\mu\}^2 k_\mu(t)$.

In addition, we note that $t_\mu \leqslant 1$ is equivalent to $\mu \leqslant \frac{1}{2}$.

(i) $f_\mu(1) = 0$ is trivial. The non-negativity of $f_\mu(t)$ has given in the proof of [4, Lemma2.3].

(ii) For the case $\mu \in [0, \frac{1}{2}]$ and $t \geqslant 1$, we prove $f'_\mu(t) \geqslant 0$. Firstly $k'_\mu(t) \geqslant 0$ and $k_\mu(1) = 1-2\mu \geqslant 0$ imply $k_\mu(t) \geqslant 0$, that is, $h'_\mu(t) \geqslant 0$. Since $h_\mu(1) = 0$, we have $h_\mu(t) \geqslant 0$, that is, $d'_\mu(t) \geqslant 0$. Since $d_\mu(1) = 0$, we have $d_\mu(t) \geqslant 0$, that is, $f'_\mu(t) \geqslant 0$.

(iii) For the case $\mu \in [\frac{1}{2}, 1]$ and $0 < t \leqslant 1$, we prove $f'_\mu(t) \leqslant 0$. Firstly $k'_\mu(t) \geqslant 0$ and $k_\mu(0) = \mu(\mu-2) < 0$, $k_\mu(1) = 1-2\mu \leqslant 0$ imply $k_\mu(t) \leqslant 0$, that is, $h_\mu(t) \leqslant 0$. Since $h_\mu(1) = 0$ (and $\lim_{t\to 0} h_\mu(t) = \infty$), we have $h_\mu(t) \geqslant 0$, that is, $d'_\mu(t) \geqslant 0$. Since $d_\mu(1) = 0$ (and $\lim_{t\to 0} d_\mu(t) = -\infty$), we have $d_\mu(t) \leqslant 0$, that is, $f'_\mu(t) \leqslant 0$.

(iv) For a fixed $\mu \in [0, \frac{1}{2}]$, if $t_\mu \leqslant t \leqslant 1$, then $k_\mu(t) \geqslant 0$, that is, $h'_\mu(t) \geqslant 0$. Since $h_\mu(1) = 0$, we have $h_\mu(t) \leqslant 0$, that is, $d'_\mu(t) \leqslant 0$. Since $d_\mu(1) = 0$, we have $d_\mu(t) \geqslant 0$, that is, $f'_\mu(t) \geqslant 0$.

(v) For a fixed $\mu \in [\frac{1}{2}, 1)$, if $1 \leqslant t \leqslant t_\mu$, then $k_\mu(t) \leqslant 0$, that is, $h'_\mu(t) \leqslant 0$. Since $h_\mu(1) = 0$, we have $h_\mu(t) \leqslant 0$, that is, $d'_\mu(t) \leqslant 0$. Since $d_\mu(1) = 0$, we have $d_\mu(t) \leqslant 0$, that is, $f'_\mu(t) \leqslant 0$.

(vi) For the case $\mu \in [0, \frac{1}{2}]$ and $t \geqslant 1$, we prove $f''_\mu(t) \geqslant 0$. Since $k_\mu(t) \geqslant 0$ (by (ii) above), we have $l'_\mu(t) \geqslant 0$. Since $l_\mu(1) = 0$, we have $l_\mu(t) \geqslant 0$, that is, $f''_\mu(t) \geqslant 0$.

(vii) For the case $\mu \in [\frac{1}{2}, 1]$ and $0 < t \leqslant 1$, we prove $f''_\mu(t) \geqslant 0$. Since $k_\mu(t) \leqslant 0$ (by (iii) above), we have $l'_\mu(t) \leqslant 0$. Since $l_\mu(1) = 0$ (and $\lim_{t \to 0} l_\mu(t) = \infty$), we have $l_\mu(t) \geqslant 0$, that is, $f''_\mu(t) \geqslant 0$.

(viii) For a fixed $\mu \in [0, \frac{1}{2}]$, if $t_\mu \leqslant t \leqslant 1$, then $k_\mu(t) \geqslant 0$, that is, $l'_\mu(t) \geqslant 0$. Since $l_\mu(1) = 0$, we have $l_\mu(t) \leqslant 0$, that is, $f''_\mu(t) \leqslant 0$.

(ix) For a fixed $\mu \in [\frac{1}{2}, 1)$, if $1 \leqslant t \leqslant t_\mu$, then $k_\mu(t) \leqslant 0$, that is, $l'_\mu(t) \leqslant 0$. Since $l_\mu(1) = 0$, we have $l_\mu(t) \leqslant 0$, that is, $f''_\mu(t) \leqslant 0$.

(x) For the condition $\mu \in [0, \frac{1}{2}]$ and $0 \leqslant t \leqslant 1$, we easily find that $g_\mu(t) \leqslant 0$. We note

$$f_\mu(t) \geqslant 2\sqrt{\frac{\{(1-\mu)+\mu t\}t}{(1-\mu)t+\mu}} - 2t^\mu. \tag{3}$$

If $\mu \in [0, \frac{1}{2}]$ and $0 < t \leqslant 1$, then we have

$$\sqrt{\frac{\{(1-\mu)+\mu t\}t}{(1-\mu)t+\mu}} \geqslant \sqrt{t} \tag{4}$$

Thus the inequalities (3) and (4) imply $g_\mu(t) \leqslant f_\mu(t)$.

(xi) If $\mu \in [\frac{1}{2}, 1]$ and $t \geqslant 1$, then we also have $g_\mu(t) \leqslant 0$ and the inequality (4) so that we have $g_\mu(t) \leqslant f_\mu(t)$. $\square$

REMARK 3.5. We find that the sign of $f_\mu(t)$ is not definite for the following cases

(a) $\mu \in [0, \frac{1}{2})$ and $0 \leqslant t \leqslant 1$,

(b) $\mu \in (\frac{1}{2}, 1]$ and $t \geqslant 1$,

since we actually have $f_{\frac{1}{4}}\left(\frac{1}{5}\right) \simeq -0.0374806$, $f_{\frac{1}{4}}\left(\frac{1}{50}\right) \simeq 0.0783511$, $f_{\frac{3}{4}}(10) \simeq -0.419903$ and $f_{\frac{3}{4}}(50) \simeq 3.91755$, by numerical computations.

THEOREM 3.6. *For $A, B > 0$, put $C \equiv A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$. $f_\mu(t)$ is defined in Lemma 3.4. Then we have the following inequalities.*

(i) If $\mu \in [0, \frac{1}{2}]$ and $B - A \geqslant m > 0$, then

$$0 \leqslant f_\mu(1 + m||A||^{-1}) m_A \leqslant A \nabla_\mu B + A !_\mu B - 2A \sharp_\mu B \leqslant f_\mu(||C||)||A||.$$

(ii) If $\mu \in [\frac{1}{2}, 1]$ and $A - B \geqslant m > 0$, then

$$0 \leqslant f_\mu(1 - m||A||^{-1})m_A \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B \leqslant f_\mu(m_C)||A||.$$

Proof.

(i) Since $B \geqslant A + m$, we have

$$||C|| \geqslant C \equiv A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geqslant A^{-\frac{1}{2}}(A+m)A^{-\frac{1}{2}} \geqslant 1 + mA^{-1} \geqslant 1 + m||A||^{-1}.$$

From (ii) of Lemma 3.4, $f_\mu(t)$ is increasing for $t \geqslant 1$ when $\mu \in [0, \frac{1}{2}]$ and $t \geqslant 1$ for $t \in \sigma(C)$, we have $f_\mu(1 + m||A||^{-1}) \leqslant f_\mu(t) \leqslant f_\mu(||C||)$. Thus we have

$$f_\mu(1 + m||A||^{-1})m_A \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B \leqslant f_\mu(||C||)||A||.$$

(ii) Since $A - m \geqslant B$, we have $m_C \leqslant C \leqslant 1 - m||A||^{-1}$. From (iii) of Lemma 3.4, $f_\mu(t)$ is decreasing for $0 < t \leqslant 1$ when $\mu \in [\frac{1}{2}, 1]$ and $0 < t \leqslant 1$ for $t \in \sigma(C)$, we have $f_\mu(1 - m||A||^{-1}) \leqslant f_\mu(t) \leqslant f_\mu(m_C)$. Thus we have

$$f_\mu(1 - m||A||^{-1})m_A \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B \leqslant f_\mu(m_C)||A||. \quad \square$$

We note that Theorem 3.6 give the refinements for the second inequality in [4, Theorem2.1]. We also show the following results by the similar way to the proof of Theorem 3.6. The conditions in (i) and (ii) of the following proposition correspond to those in (a) and (b) of Remark 3.5.

PROPOSITION 3.7. *For $A, B > 0$, put $C \equiv A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. $f_\mu(t)$ and t_μ are defined in Lemma 3.4. $g_\mu(t)$ is also defined in Lemma 3.2. c_3 and c_4 are given in Theorem 3.1 and Remark 3.3. Then we have the following inequalities.*

(i) For a given $\mu \in [0, \frac{1}{2}]$, if $t_\mu A \leqslant B \leqslant A - m$ with $m > 0$, then

$$f_\mu(m_C)m_A \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B \leqslant f_\mu(1 - m||A||^{-1})||A||.$$

In particular,

$$0 \leqslant c_3 \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B.$$

(ii) For a given $\mu \in [\frac{1}{2}, 1)$, if $A + m \leqslant B \leqslant t_\mu A$ with $m > 0$, then

$$f_\mu(||C||)m_A \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B \leqslant f_\mu(1 + m||A||^{-1})||A||.$$

In particular,

$$c_4 \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B.$$

Proof.

(i) The condition $t_\mu A \leqslant B \leqslant A - m$ implies $t_\mu \leqslant C \leqslant 1$ and $m_C \leqslant C \leqslant 1 - m||A||^{-1}$ as $t_\mu \leqslant m_C$. From (iv) of Lemma 3.4, f_μ is increasing for $t_\mu \leqslant t \leqslant 1$ so that we have $f_\mu(m_C) \leqslant f_\mu(t) \leqslant f_\mu(1 - m||A||^{-1})$, since $t_\mu \leqslant t \leqslant 1$ if $t \in \sigma(C)$. Thus we have

$$f_\mu(m_C)m_A \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B \leqslant f_\mu(1 - m||A||^{-1})||A||.$$

From (x) of Lemma 3.4, we especially have, $c_3 \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B$.

(ii) The condition $A + m \leqslant B \leqslant t_\mu A$ implies $1 \leqslant C \leqslant t_\mu$ and $1 + m||A||^{-1} \leqslant C \leqslant ||C|| \leqslant t_\mu$. From (v) of Lemma 3.4, f_μ is decreasing for $1 \leqslant t \leqslant t_\mu$ so that we have $f_\mu(||C||) \leqslant f_\mu(t) \leqslant f_\mu(1 + m||A||^{-1})$, since $1 \leqslant t \leqslant t_\mu$ if $t \in \sigma(C)$. Thus we have

$$f_\mu(||C||)m_A \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B \leqslant f_\mu(1 + m||A||^{-1})||A||.$$

From (xi) of Lemma 3.4, we especially have, $c_4 \leqslant A\nabla_\mu B + A!_\mu B - 2A\sharp_\mu B$. □

Related to the strict positivity of operators, the arithmetic-geometric mean inequality is refined as follows:

THEOREM 3.8. *If $A - B$ is invertible for $A, B > 0$, then for each $0 < \mu < 1$*

$$A\ \nabla_\mu\ B - A\ \sharp_\mu\ B > 0.$$

In particular, if $A - B \geqslant m > 0$, then

$$s_\mu\left(1 - \frac{m}{\|A\|}\right)m_A \leqslant A\ \nabla_\mu\ B - A\ \sharp_\mu\ B \leqslant s_\mu\left(\frac{m_B}{\|A\|}\right)\|A\|,$$

where $s_\mu(x) = 1 - \mu + \mu x - x^\mu$.

Proof. Put $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Since 1 does not belong to the spectrum $\sigma(C)$ of C, we have

$$A\nabla_\mu B - A\sharp_\mu B = A^{\frac{1}{2}}s_\mu(C)A^{\frac{1}{2}} \geqslant \varepsilon A \geqslant \varepsilon m_A$$

for some $\varepsilon > 0$.

Next, if $A - B \geqslant m > 0$, then C has bounds such that

$$\frac{m_B}{\|A\|} \leqslant m_B A^{-1} \leqslant C \leqslant (A - m)A^{-1} = 1 - mA^{-1} \leqslant 1 - \frac{m}{\|A\|} < 1.$$

Noting that s_μ is convex, decreasing and $s_\mu(x) > 0$ on $[0, 1)$, we have

$$s_\mu\left(\frac{m_B}{\|A\|}\right) \geqslant s_\mu(C) \geqslant s_\mu\left(1 - \frac{m}{\|A\|}\right).$$

Since $A\ \nabla_\mu\ B - A\ \sharp_\mu\ B = A^{\frac{1}{2}}s_\mu(C)A^{\frac{1}{2}}$,

$$s_\mu\left(1 - \frac{m}{\|A\|}\right)m_A \leqslant A\ \nabla_\mu\ B - A\ \sharp_\mu\ B \leqslant s_\mu\left(\frac{m_B}{\|A\|}\right)\|A\|,$$

as desired. □

LEMMA 3.9. *If $A - B \geqslant m$ for some $m > 0$, then $B^{-1} - A^{-1} \geqslant \frac{m}{(\|B\|+m)\|B\|} := m_1$.*

It is easily proved as

$$B^{-1} - A^{-1} \geqslant B^{-1} - (B+m)^{-1} = mB^{-1}(B+m)^{-1} \geqslant m_1.$$

See [3] and [2].

By the use of Lemma 3.9, we have a refinement of the geometric-harmonic mean inequality.

COROLLARY 3.10. *Notation as in above. If $A - B \geqslant m$ for some $m > 0$ and $0 < \mu < 1$, then*

$$A \sharp_\mu B - A \,!_\mu\, B \geqslant \frac{m_2}{(\|B_1\| + m_2)\|B_1\|},$$

where $B_1 = A \sharp_\mu B$ and $m_2 = s_{1-\mu}\left(1 - \frac{m_1}{\|B^{-1}\|}\right) m_B$.

Proof. Combining Lemma 3.9 with Theorem 3.8, we have

$$B^{-1} \nabla_{1-\mu} A^{-1} - B^{-1} \sharp_{1-\mu} A^{-1} \geqslant s_{1-\mu}\left(1 - \frac{m_1}{\|B^{-1}\|}\right) m_B = m_2.$$

If we put $B_1 = (A \sharp_\mu B)^{-1} = B^{-1} \sharp_{1-\mu} A^{-1}$, then it follows from Lemma 3.9 that

$$A \sharp_\mu B - A \,!_\mu\, B = (B^{-1} \sharp_{1-\mu} A^{-1})^{-1} - (B^{-1} \nabla_{1-\mu} A^{-1})^{-1} \geqslant \frac{m_2}{(\|B_1\| + m_2)\|B_1\|},$$

as desired. □

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