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SOME SHARP INEQUALITIES ON MIXED–NORM SPACES ON THE UNIT BALL

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Abstract. We give some sharp inequalities on the mixed-norm spaces of holomorphic and harmonic functions on the unit balls in $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively. Several interesting corollaries and applications of the inequalities are also given.

1. Introduction and preliminaries

Let $\mathbb{B}$ be the open unit ball in the complex-vector space $\mathbb{C}^n$, $\mathbb{S}=\partial\mathbb{B}$ its boundary, $d\sigma$ the normalized surface measure on $\mathbb{S}$, i.e. $\sigma(\mathbb{S})=1$, dv the normalized Lebesgue measure on $\mathbb{B}$, i.e. $v(\mathbb{B})=1$, and $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$.

For an $f\in H(\mathbb{B})$ with the Taylor expansion $f(z)=\sum_{|\beta|\geqslant 0}a_\beta z^\beta$, let

$$\Re f(z)=\sum_{|\beta|\geqslant 0}|\beta|a_\beta z^\beta$$

be the radial derivative of f, where $\beta=(\beta_1,\beta_2,\dots,\beta_n)$ is a multi-index, $|\beta|=\beta_1+\cdots+\beta_n$ and $z^\beta=z_1^{\beta_1}\cdots z_n^{\beta_n}$ ([9]). It is easy to see that

$$\Re f(z)=\langle\nabla f(z),\bar{z}\rangle,$$

where ∇f is the complex gradient of function f, that is,

$$\nabla f=\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right),$$

and $\langle z,w\rangle=\sum_{j=1}^n z_j\overline{w}_j$ is the standard scalar product in $\mathbb{C}^n$.

A function continuous on the interval $[0,1)$ which is positive on $(0,1)$ is called *weight*.

Let $0<p,q<\infty$ and ω be a weight on $[0,1)$. The *mixed norm space* $H(p,q,\omega)(\mathbb{B})$ $=H(p,q,\omega)$, consists of all $f\in H(\mathbb{B})$ such that

$$\|f\|^p_{H(p,q,\omega)}=\int_0^1 M_q^p(f,r)\omega(r)dr<\infty, \tag{1}$$

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where

$$M_q(f,r)=\left(\int_{\mathbb{S}}|f(r\zeta)|^q d\sigma(\zeta)\right)^{1/q}.$$

Typical examples of the mixed norm spaces are obtained for

$$\omega(r)=\frac{\phi^p(r)}{1-r},\quad r\in[0,1),$$

where $p>0$, and ϕ is a weight function on the interval $[0,1)$, and there are $\delta\in[0,1)$ and a and b, $0<a<b$, such that

$$\frac{\phi(r)}{(1-r)^a}\ \text{is decreasing on } [\delta,1),\ \text{and}\ \lim_{r\to 1^-}\frac{\phi(r)}{(1-r)^a}=0;$$
$$\frac{\phi(r)}{(1-r)^b}\ \text{is increasing on } [\delta,1),\ \text{and}\ \lim_{r\to 1^-}\frac{\phi(r)}{(1-r)^b}=+\infty.$$

Such weight function ϕ is called *normal* ([10]), and the corresponding mixed norm space is denoted by $H(p,q,\phi)(\mathbb{B})=H(p,q,\phi)$. For $\phi(r)=(1-r)^{(\alpha+1)/p}$, $\alpha>-1$, the mixed norm space is reduced to the classical one, i.e. $H(p,q,\alpha)(\mathbb{B})=H(p,q,\alpha)$. Mixed norm spaces and some operators on them have been studied considerably recently (see, e.g. [3, 5, 6, 17, 19, 21, 23, 24] and the references therein, where, among others, were studied the operators introduced in [3, 16, 18, 22]; some related results can be found also in [1, 4, 7, 20]).

For $p\in(0,\infty)$ and $\alpha>-1$, the *weighted Bergman space* $A^p_\alpha(\mathbb{B})=A^p_\alpha$ consists of all $f\in H(\mathbb{B})$ such that

$$\|f\|^p_{A^p_\alpha}=\int_{\mathbb{B}}|f(z)|^p dv_\alpha(z)<\infty,$$

where $dv_\alpha(z)=c_{n,\alpha}(1-|z|^2)^\alpha dv(z)$ is the normalized weighted Lebesgue measure on $\mathbb{B}$, that is, constant $c_{n,\alpha}$ is chosen such that $v_\alpha(\mathbb{B})=1$. When $\alpha=0$, $A^p_0(\mathbb{B})=A^p(\mathbb{B})$ is the standard (unweighted) Bergman space. For some information on the space see, for example, [7, 26, 27]. By using the polar coordinates it is easy to see that the weighted Bergman space is the special case of the mixed norm space $H(p,q,\omega)$ with $p=q$ and $\omega(r)=2nc_{n,\alpha}(1-r^2)^\alpha r^{2n-1}$.

When $a\neq 0$, by φ_a we denote the involutive holomorphic automorphism of the unit ball such that $\varphi_a(0)=a$. It is known that

$$\varphi_a(z)=\frac{a-P_a(z)-s_aQ_a(z)}{1-\langle z,a\rangle},\quad z\in\mathbb{B},$$

where $s_a=\sqrt{1-|a|^2}$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one-dimensional subspace $[a]$ generated by a, and $Q_a=I-P_a$ (i.e. the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n\ominus[a]$). It is easy to see that

$$P_a(z)=\frac{\langle z,a\rangle}{\langle a,a\rangle}a,\quad z\in\mathbb{C}^n.$$

If $a=0$, then the involution is simply defined by $\varphi_0(z)=-z$.

It is also known that the following relation holds

$$1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2},\quad z\in\mathbb{B}. \tag{2}$$

For more information about automorphisms of the unit ball in $\mathbb{C}^n$, see, for example, [9] or [27].

Let $\mathbb{B}_{\mathbb{R}}$ be the open unit ball in the real-vector space $\mathbb{R}^n$, $\mathbb{S}_{\mathbb{R}}=\partial\mathbb{B}_{\mathbb{R}}$ its boundary, dS the normalized surface measure on $\mathbb{S}_{\mathbb{R}}$, i.e. $S(\mathbb{S}_{\mathbb{R}})=1$, dV the normalized Lebesgue measure on $\mathbb{B}_{\mathbb{R}}$, i.e. $V(\mathbb{B}_{\mathbb{R}})=1$, and $\mathscr{H}(\mathbb{B}_{\mathbb{R}})$ the class of all harmonic functions on $\mathbb{B}_{\mathbb{R}}$.

Let $0<p,q<\infty$ and ω be a weight on $[0,1)$. The *harmonic mixed norm space* $\mathscr{H}(p,q,\omega)(\mathbb{B}_{\mathbb{R}})=\mathscr{H}(p,q,\omega)$, consists of all $u\in\mathscr{H}(\mathbb{B}_{\mathbb{R}})$ such that

$$\|u\|^p_{H(p,q,\omega)}=\int_0^1 M_q^p(u,r)\omega(r)dr<\infty, \tag{3}$$

where

$$M_q(u,r)=\left(\int_{\mathbb{S}_{\mathbb{R}}}|u(r\zeta)|^q dS(\zeta)\right)^{1/q}.$$

For $p\in(0,\infty)$ and $\alpha>-1$, the *weighted harmonic Bergman space* $\mathscr{A}^p_\alpha(\mathbb{B}_{\mathbb{R}})=\mathscr{A}^p_\alpha$ consists of all $u\in\mathscr{H}(\mathbb{B}_{\mathbb{R}})$ such that

$$\|u\|^p_{\mathscr{A}^p_\alpha}=\int_{\mathbb{B}_{\mathbb{R}}}|u(x)|^p dV_\alpha(x)<\infty,$$

where $dV_\alpha(x)=d_{n,\alpha}(1-|x|^2)^\alpha dV(x)$ is the normalized weighted Lebesgue measure on $\mathbb{B}_{\mathbb{R}}$, that is, constant $d_{n,\alpha}$ is chosen such that $V_\alpha(\mathbb{B}_{\mathbb{R}})=1$. When $\alpha=0$, $A^p_0(\mathbb{B}_{\mathbb{R}})=A^p(\mathbb{B}_{\mathbb{R}})$ is the standard (unweighted) harmonic Bergman space. By using the polar coordinates ([2]) it is easy to see that the weighted Bergman space is the special case of the mixed norm space $\mathscr{H}(p,q,\omega)$ with $p=q$ and $\omega(r)=nd_{n,\alpha}(1-r^2)^\alpha r^{n-1}$.

A frequent situation in working with mixed norm spaces is that during some calculations under the signs of integrals in (1) and (3) is appeared a power function, that is, $g(r)=r^\beta$, for some $\beta\in\mathbb{R}\setminus\{0\}$, which for some technical reasons should be eliminated or replaced by another, more suitable, power function.

Hence, a natural question is to compare the integral in (1) with the following

$$\|f\|^p_{H(p,q,\omega,\beta)}:=\int_0^1 M_q^p(f,r)\omega(r)r^\beta dr. \tag{4}$$

Note that, if $\beta<0$, then obviously holds

$$\int_0^1 M_q^p(f,r)\omega(r)dr\leqslant\int_0^1 M_q^p(f,r)\omega(r)r^\beta dr,$$

while if $\beta>0$, then we have

$$\int_0^1 M_q^p(f,r)r^\beta\omega(r)dr\leqslant\int_0^1 M_q^p(f,r)\omega(r)dr.$$

Hence, of some interest are the reverse type inequalities. There are some tricks which produce such inequalities, but of a special interest are the sharp ones.

Motivated by nice paper [25], here we give some sharp inequalities. We will also present their several important consequences. To do this we will need the following result, which is called *Čebišev's integral inequality* (see, e.g. [8, p. 40]).

LEMMA 1. *Assume that f_1 and f_2 are both nondecreasing or nonincreasing integrable functions, and h is a nonnegative integrable function on the interval $[a,b]$. Then the following inequality holds*

$$\int_a^b f_1(t)h(t)dt \int_a^b f_2(t)h(t)dt \leqslant \int_a^b f_1(t)f_2(t)h(t)dt \int_a^b h(t)dt, \tag{5}$$

where the equality in (5) holds if and only if one of the functions f_1 and f_2 is constant.

REMARK 1. By using the linear change of variables $t = a + s(b-a)$, the monotonicity and integrability of the functions in Lemma 1 are not changed, so it is easy to see that we may assume $a = 0$ and $b = 1$, the case which will be used in the rest of the paper. Beside this, a simple limit argument shows that the closed interval $[a,b]$ in Lemma 1 can be replaced by any of the intervals $(a,b]$, $[a,b)$ and (a,b).

2. Main results

In this section we state and prove our main results and present numerous consequences of them.

THEOREM 1. *Assume that $p,q > 0$, $\beta > 0$, and ω is a weight function on $[0,1)$ such that*

$$\lim_{r\to+0} \frac{\omega(r)}{r^\sigma} = c > 0, \tag{6}$$

for some $\sigma > 0$, and

$$\int_{1/2}^1 \frac{\omega(r)}{r^\sigma} < \infty. \tag{7}$$

Then for every $f \in H(p,q,\omega)$ and $\beta \in (0,\sigma+1)$, the following inequality holds

$$\int_0^1 M_q^p(f,r)\frac{\omega(r)}{r^\beta}dr \leqslant \frac{\int_0^1 \frac{\omega(r)}{r^\beta}dr}{\int_0^1 \omega(r)dr} \int_0^1 M_q^p(f,r)\omega(r)dr, \tag{8}$$

where the equality in (8) holds if and only if function f is constant.

Proof. First note that condition (6) implies that for $\beta \in (0,\sigma+1)$ the integral

$$I := \int_0^1 \frac{\omega(r)}{r^\beta}dr$$

is finite. Indeed, from (6) we have that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\omega(r)| < (c+\varepsilon)r^{\sigma}$, when $r \in (0,\delta)$. Hence

$$\int_0^{\delta} \frac{\omega(r)}{r^{\beta}} dr < (c+\varepsilon)\int_0^{\delta} r^{\sigma-\beta} dr < \infty,$$

due to the condition $\beta \in (0,\sigma+1)$. From the continuity of ω and condition (7) it easily follows that

$$\int_{\delta}^{1} \frac{\omega(r)}{r^{\beta}} dr < \infty.$$

From these two facts the claim follows.

Since

$$\int_0^1 \omega(r)dr \leqslant \int_0^1 \frac{\omega(r)}{r^{\beta}} dr,$$

for $\beta > 0$, we have also proved that

$$\int_0^1 \omega(r)dr < \infty.$$

It is well known that the function $f_1(r) = M_q^p(f,r)$ is nondecreasing on $[0,1)$, for every $f \in H(\mathbb{B})$ and $p,q > 0$. On the other hand, the function $f_2(r) = r^{\beta}$ is increasing on the interval $[0,1)$ for every $\beta > 0$. Applying Lemma 1 to these two functions on the interval $(0,1)$, with $h(r) = \omega(r)/r^{\beta}$, inequality (8) follows.

Since function f_2 is not constant, according to the second part of Lemma 1, the equality holds if and only if $M_q^p(f,r)$ is constant on $[0,1)$. If $f(0) = 0$, then from the equality

$$M_q(f,r) = |f(0)| = 0, \quad r \in (0,1),$$

we get $f(z) \equiv 0$, $z \in \mathbb{B}$. If $f(0) \neq 0$, then from the continuity of f it follows that there is a $\delta > 0$ such that $f(z) \neq 0$ on the ball $\delta\mathbb{B}$.

Now, if we use the equality

$$r\frac{d}{dr}M_q^q(f,r) = \frac{q^2}{2n}\int_{r\mathbb{B}} |\Re f(z)|^2 |f(z)|^{q-2}|z|^{-2n} d\nu(z), \quad r \in [0,1), \tag{9}$$

(see, e.g. [27, Theorem 4.20]), we get

$$\int_{r\mathbb{B}} |\Re f(z)|^2 |f(z)|^{q-2}|z|^{-2n} d\nu(z) = 0, \quad r \in [0,\delta),$$

from which it follows that $\Re f(z) = 0$, $z \in \delta\mathbb{B}$.

From this and since

$$f(z) - f(0) = \int_0^1 \frac{\Re f(tz)}{t} dt,$$

it follows that $f(z) = f(0)$, $z \in \delta\mathbb{B}$, which along with the uniqueness theorem for holomorphic functions implies that $f(z) = f(0)$, $z \in \mathbb{B}$, as claimed. □

REMARK 2. That $M_q(f,r) = const.$ implies $f(z) = f(0)$ is a known fact. We give the proof above to show the usefulness of formula (9), for the completeness, and since this proof might be new or interesting to the reader.

THEOREM 2. *Assume that* $p,q>0$, $\beta>0$, *and* ω *is a weight function on* $[0,1)$ *such that*

$$\int_0^1 \omega(r)dr<\infty. \tag{10}$$

Then for every $f\in H(p,q,\omega)$, *the following inequality holds*

$$\int_0^1 M_q^p(f,r)\omega(r)dr \leqslant \frac{\int_0^1 \omega(r)dr}{\int_0^1 r^\beta \omega(r)dr}\int_0^1 M_q^p(f,r)r^\beta\omega(r)dr, \tag{11}$$

where the equality in (11) *holds if and only if function* f *is constant.*

Proof. First note that condition (10) obviously implies

$$\int_0^1 r^\beta\omega(r)dr<\infty.$$

As in the proof of Theorem 1, we apply Lemma 1 to the functions $f_1(r)=M_q^p(f,r)$ and $f_2(r)=r^\beta$ on the interval $(0,1)$, but with $h(r)=\omega(r)$. The proof of the part concerning the equality in (11) is the same as in Theorem 1, so is omitted. □

For the case $\min\{p,q\}\geqslant 1$, and $\beta\in\mathbb{R}\setminus\{0\}$, the quantities $\|\cdot\|_{H(p,q,\omega)}$ and $\|\cdot\|_{H(p,q,\omega,\beta)}$ are norms on the space $H(p,q,\omega)$. Hence, Theorems 1 and 2 claim that the identity operators

$$I:(H(p,q,\omega),\|\cdot\|_{H(p,q,\omega)})\to(H(p,q,\omega),\|\cdot\|_{H(p,q,\omega,-\beta)})$$

when $\beta\in(0,\sigma+1)$ and

$$I:(H(p,q,\omega),\|\cdot\|_{H(p,q,\omega,\beta)})\to(H(p,q,\omega),\|\cdot\|_{H(p,q,\omega)})$$

when $\beta>0$, are bounded, and that their norms are

$$\|I\|_{(H(p,q,\omega),\|\cdot\|_{H(p,q,\omega)})\to(H(p,q,\omega),\|\cdot\|_{H(p,q,\omega,-\beta)})}=\left(\frac{\int_0^1\frac{\omega(r)}{r^\beta}dr}{\int_0^1\omega(r)dr}\right)^{1/p}$$

and

$$\|I\|_{(H(p,q,\omega),\|\cdot\|_{H(p,q,\omega,\beta)})\to(H(p,q,\omega),\|\cdot\|_{H(p,q,\omega)})}=\left(\frac{\int_0^1\omega(r)dr}{\int_0^1 r^\beta\omega(r)dr}\right)^{1/p}.$$

Now we state and prove the corresponding results for the space of harmonic functions $\mathscr{H}(p,q,\omega)$ on the unit ball $\mathbb{B}_{\mathbb{R}}$.

THEOREM 3. *Assume that* $p>0$ $q>1$, $\beta>0$, *and* ω *is a weight function on* $[0,1)$ *satisfying conditions* (6) *and* (7). *Then for every* $u\in\mathscr{H}(p,q,\omega)$ *and* $\beta\in(0,\sigma+1)$, *the inequality*

$$\int_0^1 M_q^p(u,r)\frac{\omega(r)}{r^\beta}dr\leqslant\frac{\int_0^1\frac{\omega(r)}{r^\beta}dr}{\int_0^1\omega(r)dr}\int_0^1 M_q^p(u,r)\omega(r)dr, \tag{12}$$

holds, where the equality in (12) *holds if and only if function* u *is constant.*

Proof. The proof is similar to the proof of Theorem 1 where Lemma 1 is applied to the nondecreasing functions $f_1(r) = M_q^p(u,r)$ and $f_2(r) = r^\beta$ on the interval $(0,1)$ with the weight function $h(r) = \omega(r)/r^\beta$, from which (12) follows.

Since function f_2 is not constant then the equality in (12) holds if and only if $M_q(u,r)$ is constant on $[0,1)$. Hence, if $u(0) = 0$, then from the equality

$$M_q(u,r) = |u(0)| = 0, \quad r \in (0,1),$$

we get $u(x) \equiv 0$, $x \in \mathbb{B}_\mathbb{R}$. If $u(0) \neq 0$, then from the continuity of u it follows that there is a $\delta > 0$ such that $u(x) \neq 0$ on the ball $\delta\mathbb{B}_\mathbb{R}$.

For $q > 1$ and $n \geqslant 3$, the following equality

$$M_q^q(u,r) = |u(0)|^q + \frac{q(q-1)}{n(n-2)} \int_{r\mathbb{B}_\mathbb{R}} |u(x)|^{q-2} |\nabla u(x)|^2 (|x|^{2-n} - r^{2-n}) dV(x), \tag{13}$$

was essentially proved in [11] (although it was explicitly formulated in [12]), and subsequently used in several papers of ours (see, e.g. [13], [14] and [15])). By using (13), we get

$$\int_{r\mathbb{B}_\mathbb{R}} |u(x)|^{q-2} |\nabla u(x)|^2 (|x|^{2-n} - r^{2-n})\, dV(x) = 0,$$

for every $r \in [0,1)$, from which it follows that $\nabla u(x) = 0$, $x \in \delta\mathbb{B}_\mathbb{R}$, and consequently $u(x) = u(0)$, $x \in \delta\mathbb{B}_\mathbb{R}$, which along with the uniqueness theorem for harmonic functions implies that $u(x) = u(0)$, $x \in \mathbb{B}_\mathbb{R}$.

For the case $q > 1$ and $n = 2$ the result can be obtained by the Stein's formula

$$\int_{\mathbb{S}_\mathbb{R}} |u(r\zeta)|^q d\sigma(\zeta) = |u(0)|^q + \frac{q(q-1)}{2} \int_{r\mathbb{B}_\mathbb{R}} |u(x)|^{q-2} |\nabla u(x)|^2 \ln \frac{r}{|x|} dV(x), \tag{14}$$

$r \in [0,1)$, which was also essentially proved in [11, Theorem 4]. □

REMARK 3. That $M_q(u,r) = \mathit{const}$. implies $u(x) = u(0)$, $x \in \mathbb{B}_\mathbb{R}$, is a known result (see, e.g. [2, p. 138]), but we give the proof above to show the usefulness of our formula (13), the completeness, and since the proof might be new or interesting to the reader.

The proof of the next result is a combination of the proofs of Theorems 2 and 3, so is omitted.

THEOREM 4. *Assume that* $p > 0$ $q > 1$, $\beta > 0$, *and* ω *is a weight function on* $[0,1)$ *satisfying condition* (10). *Then for every* $u \in \mathscr{H}(p,q,\omega)$, *the following inequality*

$$\int_0^1 M_q^p(u,r)\omega(r)dr \leqslant \frac{\int_0^1 \omega(r)dr}{\int_0^1 r^\beta \omega(r)dr} \int_0^1 M_q^p(u,r) r^\beta \omega(r)dr, \tag{15}$$

holds, where the equality in (15) *holds if and only if function* u *is constant.*

Note that, for the case $p \geqslant 1$, $q > 1$, and $\beta \in \mathbb{R} \setminus \{0\}$, the quantities $\|\cdot\|_{\mathscr{H}(p,q,\omega)}$ and $\|\cdot\|_{\mathscr{H}(p,q,\omega,\beta)}$ are norms on the space of harmonic functions $\mathscr{H}(p,q,\omega)$. Hence, Theorems 3 and 4 claim that the identity operators

$$I : (\mathscr{H}(p,q,\omega), \|\cdot\|_{\mathscr{H}(p,q,\omega)}) \to (\mathscr{H}(p,q,\omega), \|\cdot\|_{\mathscr{H}(p,q,\omega,-\beta)})$$

when $\beta \in (0, \sigma+1)$, and

$$I : (\mathscr{H}(p,q,\omega), \|\cdot\|_{\mathscr{H}(p,q,\omega,\beta)}) \to (\mathscr{H}(p,q,\omega), \|\cdot\|_{\mathscr{H}(p,q,\omega)})$$

when $\beta > 0$ are bounded, and that their norms are

$$\|I\|_{(\mathscr{H}(p,q,\omega),\|\cdot\|_{\mathscr{H}(p,q,\omega)})\to(\mathscr{H}(p,q,\omega),\|\cdot\|_{\mathscr{H}(p,q,\omega,-\beta)})} = \left(\frac{\int_0^1 \frac{\omega(r)}{r^\beta}dr}{\int_0^1 \omega(r)dr}\right)^{1/p}$$

and

$$\|I\|_{(\mathscr{H}(p,q,\omega),\|\cdot\|_{\mathscr{H}(p,q,\omega,\beta)})\to(\mathscr{H}(p,q,\omega),\|\cdot\|_{\mathscr{H}(p,q,\omega)})} = \left(\frac{\int_0^1 \omega(r)dr}{\int_0^1 r^\beta \omega(r)dr}\right)^{1/p}.$$

Now we will present some important corollaries and applications of our main results.

COROLLARY 1. *Assume that* $p,q > 0$, $\gamma \geqslant 0$, $\alpha > -1$ *and* $\beta \in (0, \gamma+1)$. *Then for every* $f \in H(p,q,\alpha)$ *the following inequality holds*

$$\int_0^1 M_q^p(f,r)\frac{(1-r)^\alpha}{r^\beta} r^\gamma dr \leqslant \frac{\Gamma(\gamma-\beta+1)\Gamma(\alpha+\gamma+2)}{\Gamma(\alpha+\gamma+2-\beta)\Gamma(\gamma+1)} \int_0^1 M_q^p(f,r)(1-r)^\alpha r^\gamma dr, \tag{16}$$

where the equality in (16) *holds if and only if function* f *is constant.*

Proof. If we apply Theorem 1 with $\omega(r) = (1-r)^\alpha r^\gamma$, and note that ω satisfies conditions (6) and (7) with $\sigma = \gamma$ and $c = 1$, then, we have

$$\int_0^1 M_q^p(f,r)\frac{(1-r)^\alpha}{r^\beta} r^\gamma dr \leqslant \frac{\int_0^1 (1-r)^\alpha r^{\gamma-\beta} dr}{\int_0^1 (1-r)^\alpha r^\gamma dr} \int_0^1 M_q^p(f,r)(1-r)^\alpha r^\gamma dr. \tag{17}$$

On the other hand, we have that

$$\int_0^1 (1-r)^\alpha r^{\gamma-\beta} dr = \frac{\Gamma(\alpha+1)\Gamma(\gamma-\beta+1)}{\Gamma(\alpha+\gamma+2-\beta)} \tag{18}$$

and

$$\int_0^1 (1-r)^\alpha r^\gamma dr = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+2)}. \tag{19}$$

By using (18) and (19) in (17), we obtain (16). According to Theorem 1 the equality in (16) holds if and only if function f is constant. □

COROLLARY 2. *Assume that* $p > 0$, $s \geqslant 0$, $s + \alpha > -1$ *and* $\beta \in (0, 2n)$. *Then for every* $f \in A^p_\alpha$ *the following inequality holds*

$$\int_{\mathbb{B}} \frac{|f(z)|^p}{|z|^\beta}(1-|\varphi_a(z)|^2)^s d\nu_\alpha(z) \leqslant \frac{\Gamma(n-\beta/2)\Gamma(\alpha+s+n+1)}{\Gamma(n+\alpha+s+1-\beta/2)\Gamma(n)} \times \int_{\mathbb{B}} |f(z)|^p (1-|\varphi_a(z)|^2)^s d\nu_\alpha(z), \qquad (20)$$

where the equality in (20) *holds if and only if*

$$f(z) = w(1-\langle z, a\rangle)^{2s/p} \qquad (21)$$

for some $w \in \mathbb{C}$.

Proof. By using the polar coordinates and (2) we see that inequality (20) is equivalent to the following

$$\int_0^1 M_p^p(f/g, r)(1-r^2)^{\alpha+s} r^{2n-1-\beta} dr \leqslant \frac{\Gamma(n-\beta/2)\Gamma(\alpha+s+n+1)}{\Gamma(n+\alpha+s+1-\beta/2)\Gamma(n)} \times \int_0^1 M_p^p(f/g, r)(1-r^2)^{\alpha+s} r^{2n-1} dr, \qquad (22)$$

where $g(z) = (1-\langle z, a\rangle)^{2s/p}$. Note that $f/g \in H(\mathbb{B})$, since $g(z) \neq 0$ for every $z \in \mathbb{B}$.

If we apply Theorem 1 with $p = q$ and $\omega(r) = (1-r^2)^{\alpha+s} r^{2n-1}$, and note that function ω satisfies conditions (6) and (7) with $\sigma = 2n-1$ and $c = 1$, we get

$$\int_0^1 M_p^p(f/g, r)(1-r^2)^{\alpha+s} r^{2n-1-\beta} dr \leqslant \frac{\int_0^1 (1-r^2)^{\alpha+s} r^{2n-1-\beta} dr}{\int_0^1 (1-r^2)^{\alpha+s} r^{2n-1} dr} \times \int_0^1 M_p^p(f/g, r)(1-r^2)^{\alpha+s} r^{2n-1} dr. \qquad (23)$$

By using the change of variables $t = r^2$, we have that

$$\int_0^1 (1-r^2)^{\alpha+s} r^{2n-1-\beta} dr = \frac{1}{2}\int_0^1 (1-t)^{\alpha+s} t^{n-1-\beta/2} dt = \frac{\Gamma(\alpha+s+1)\Gamma(n-\beta/2)}{2\Gamma(\alpha+s+n+1-\beta/2)} \qquad (24)$$

and

$$\int_0^1 (1-r^2)^{\alpha+s} r^{2n-1} dr = \frac{\Gamma(\alpha+s+1)\Gamma(n)}{2\Gamma(\alpha+s+n+1)}. \qquad (25)$$

By using (24) and (25) into (23), we get (22), that is, (20). According to Theorem 1, the equality in (20) holds if and only if function f/g is constant, from which (21) follows. □

For $s = 0$ from Corollary 2 we get the following result.

COROLLARY 3. *Assume that* $p>0$, $\alpha>-1$ *and* $\beta\in(0,2n)$. *Then for every* $f\in A^p_\alpha$ *the following inequality holds*

$$\int_{\mathbb{B}}\frac{|f(z)|^p}{|z|^\beta}d\nu_\alpha(z)\leqslant\frac{\Gamma(n-\beta/2)\Gamma(\alpha+n+1)}{\Gamma(n+\alpha+1-\beta/2)\Gamma(n)}\int_{\mathbb{B}}|f(z)|^p d\nu_\alpha(z), \tag{26}$$

where the equality in (26) *holds if and only if function* f *is constant.*

COROLLARY 4. *Assume that* $p>1$, $\alpha>-1$ *and* $\beta\in(0,n)$. *Then for every* $u\in\mathscr{A}^p_\alpha$ *the following inequality holds*

$$\int_{\mathbb{B}_{\mathbb{R}}}\frac{|u(x)|^p}{|x|^\beta}dV_\alpha(x)\leqslant\frac{\Gamma((n-\beta)/2)\Gamma(\alpha+1+n/2)}{\Gamma(\alpha+1+(n-\beta)/2)\Gamma(n/2)}\int_{\mathbb{B}_{\mathbb{R}}}|u(x)|^p dV_\alpha(x), \tag{27}$$

where the equality in (27) *holds if and only if function* u *is constant.*

Proof. By using the polar coordinates we see that inequality (27) is equivalent to the following

$$\int_0^1 M_p^p(u,r)(1-r^2)^\alpha r^{n-1-\beta}dr\leqslant\frac{\Gamma((n-\beta)/2)\Gamma(\alpha+1+n/2)}{\Gamma(\alpha+1+(n-\beta)/2)\Gamma(n/2)}\times\int_0^1 M_p^p(u,r)(1-r^2)^\alpha r^{n-1}dr. \tag{28}$$

If we apply Theorem 3 with $p=q$, $\omega(r)=(1-r^2)^\alpha r^{n-1}$, and note that function ω satisfies conditions (6) and (7) with $\sigma=n-1$ and $c=1$, we get

$$\int_0^1 M_p^p(u,r)(1-r^2)^\alpha r^{n-1-\beta}dr\leqslant\frac{\int_0^1(1-r^2)^\alpha r^{n-1-\beta}dr}{\int_0^1(1-r^2)^\alpha r^{n-1}dr}\int_0^1 M_p^p(u,r)(1-r^2)^\alpha r^{n-1}dr. \tag{29}$$

By using the change of variables $t=r^2$, we have that

$$\int_0^1(1-r^2)^\alpha r^{n-1-\beta}dr=\frac12\int_0^1(1-t)^\alpha t^{(n-\beta)/2-1}dt=\frac{\Gamma(\alpha+1)\Gamma((n-\beta)/2)}{2\Gamma(\alpha+1+(n-\beta)/2)} \tag{30}$$

and

$$\int_0^1(1-r^2)^\alpha r^{n-1}dr=\frac{\Gamma(\alpha+1)\Gamma(n/2)}{2\Gamma(\alpha+1+n/2)}. \tag{31}$$

By using (30) and (31) into (29), we get (28), that is, (27). According to Theorem 3, the equality in (27) holds if and only if function u is constant. □

COROLLARY 5. *Assume that* $p,q>0$, $\gamma\geqslant0$, $\alpha>-1$ *and* $\beta>0$. *Then for every* $f\in H(p,q,\alpha)$ *the following inequality holds*

$$\int_0^1 M_q^p(f,r)(1-r)^\alpha r^\gamma dr\leqslant\frac{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+2)}{\Gamma(\alpha+\gamma+2)\Gamma(\gamma+\beta+1)}\int_0^1 M_q^p(f,r)(1-r)^\alpha r^{\beta+\gamma}dr, \tag{32}$$

where the equality in (32) holds if and only if function f is constant.

Proof. If we apply Theorem 2 with $\omega(r) = (1-r)^{\alpha} r^{\gamma}$, and note that it satisfies condition (10), we obtain

$$\int_0^1 M_q^p(f,r)(1-r)^{\alpha} r^{\gamma} dr \leqslant \frac{\int_0^1 (1-r)^{\alpha} r^{\gamma} dr}{\int_0^1 (1-r)^{\alpha} r^{\gamma+\beta} dr} \int_0^1 M_q^p(f,r)(1-r)^{\alpha} r^{\beta+\gamma} dr. \quad (33)$$

On the other hand, we have that

$$\int_0^1 (1-r)^{\alpha} r^{\gamma} dr = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+2)} \quad (34)$$

and

$$\int_0^1 (1-r)^{\alpha} r^{\gamma+\beta} dr = \frac{\Gamma(\alpha+1)\Gamma(\gamma+\beta+1)}{\Gamma(\alpha+\beta+\gamma+2)}. \quad (35)$$

By using (34) and (35) in (33), we obtain (32). According to Theorem 2, the equality in (32) holds if and only if function f is constant. □

COROLLARY 6. *Assume that $p>0$, $s \geqslant 0$, $s+\alpha > -1$ and $\beta > 0$. Then for every $f \in A_{\alpha}^p$ the following inequality holds*

$$\int_{\mathbb{B}} |f(z)|^p (1-|\varphi_a(z)|^2)^s d\nu_{\alpha}(z) \leqslant \frac{\Gamma(n)\Gamma(\alpha+s+n+1+\beta/2)}{\Gamma(\alpha+s+n+1)\Gamma(n+\beta/2)} \times \int_{\mathbb{B}} |f(z)|^p |z|^{\beta} (1-|\varphi_a(z)|^2)^s d\nu_{\alpha}(z), \quad (36)$$

where the equality in (36) holds if and only if

$$f(z) = w(1-\langle z,a\rangle)^{2s/p} \quad (37)$$

for some $w \in \mathbb{C}$.

Proof. By using the polar coordinates and (2) we see that inequality (36) is equivalent to the following one

$$\int_0^1 M_p^p(f/g,r)(1-r^2)^{\alpha+s} r^{2n-1} dr \leqslant \frac{\Gamma(n)\Gamma(\alpha+s+n+1+\beta/2)}{\Gamma(\alpha+s+n+1)\Gamma(n+\beta/2)} \times \int_0^1 M_p^p(f/g,r)(1-r^2)^{\alpha+s} r^{2n-1+\beta} dr, \quad (38)$$

where $g(z) = (1-\langle z,a\rangle)^{2s/p}$. Recall that $f/g \in H(\mathbb{B})$.

If we apply Theorem 2 with $p=q$ and $\omega(r) = (1-r^2)^{\alpha+s} r^{2n-1}$, and note that it satisfies condition (10), we get

$$\int_0^1 M_p^p(f/g,r)(1-r^2)^{\alpha+s} r^{2n-1} dr \leqslant \frac{\int_0^1 (1-r^2)^{\alpha+s} r^{2n-1} dr}{\int_0^1 (1-r^2)^{\alpha+s} r^{2n-1+\beta} dr} \times \int_0^1 M_p^p(f/g,r)(1-r^2)^{\alpha+s} r^{2n-1+\beta} dr. \quad (39)$$

By using the change of variables $t = r^2$, we have that

$$\int_0^1 (1-r^2)^{\alpha+s} r^{2n-1+\beta} dr = \frac{1}{2}\int_0^1 (1-t)^{\alpha+s} t^{n+\frac{\beta}{2}-1} dt = \frac{\Gamma(\alpha+s+1)\Gamma(n+\beta/2)}{2\Gamma(\alpha+s+n+1+\beta/2)} \tag{40}$$

and consequently

$$\int_0^1 (1-r^2)^{\alpha+s} r^{2n-1} dr = \frac{\Gamma(\alpha+s+1)\Gamma(n)}{2\Gamma(\alpha+s+n+1)}. \tag{41}$$

By using (40) and (41) into (39), we get (38), that is, (36). According to Theorem 2, the equality in (36) holds if and only if function f/g is constant, from which (37) follows. □

For $s = 0$, from Corollary 6 we get the following result.

COROLLARY 7. *Assume that* $p > 0$, $\alpha > -1$ *and* $\beta > 0$. *Then for every* $f \in A^p_\alpha$ *the following inequality holds*

$$\int_{\mathbb{B}} |f(z)|^p dv_\alpha(z) \leqslant \frac{\Gamma(n)\Gamma(\alpha+n+1+\beta/2)}{\Gamma(\alpha+n+1)\Gamma(n+\beta/2)} \int_{\mathbb{B}} |f(z)|^p |z|^\beta dv_\alpha(z), \tag{42}$$

where the equality in (42) *holds if and only if* f *is constant.*

COROLLARY 8. *Assume that* $p > 1$, $\alpha > -1$ *and* $\beta > 0$. *Then for every* $u \in \mathscr{A}^p_\alpha$ *the following inequality holds*

$$\int_{\mathbb{B}_{\mathbb{R}}} |u(x)|^p dV_\alpha(x) \leqslant \frac{\Gamma(n/2)\Gamma(\alpha+1+(n+\beta)/2)}{\Gamma(\alpha+1+n/2)\Gamma((n+\beta)/2)} \int_{\mathbb{B}_{\mathbb{R}}} |u(x)|^p |x|^\beta dV_\alpha(x), \tag{43}$$

where equality in (43) *holds if and only if function* u *is constant.*

Proof. By using the polar coordinates we see that inequality (43) is equivalent to the following

$$\int_0^1 M_p^p(u,r)(1-r^2)^\alpha r^{n-1} dr \leqslant \frac{\Gamma(n/2)\Gamma(\alpha+1+(n+\beta)/2)}{\Gamma(\alpha+1+n/2)\Gamma((n+\beta)/2)} \times \int_0^1 M_p^p(u,r)(1-r^2)^\alpha r^{n-1+\beta} dr. \tag{44}$$

If we apply Theorem 4 with $p = q$ and $\omega(r) = (1-r^2)^\alpha r^{n-1}$, and note that function ω satisfies condition (10), we get

$$\int_0^1 M_p^p(u,r)(1-r^2)^\alpha r^{n-1} dr \leqslant \frac{\int_0^1 (1-r^2)^\alpha r^{n-1} dr}{\int_0^1 (1-r^2)^\alpha r^{n-1+\beta} dr} \times \int_0^1 M_p^p(u,r)(1-r^2)^\alpha r^{n-1+\beta} dr. \tag{45}$$

By using the change of variables $t = r^2$, we have that

$$\int_0^1 (1-r^2)^{\alpha+s} r^{n-1+\beta} dr = \frac{1}{2}\int_0^1 (1-t)^{\alpha} t^{(n+\beta)/2-1} dt = \frac{\Gamma(\alpha+1)\Gamma((n+\beta)/2)}{2\Gamma(\alpha+1+(n+\beta)/2)}, \quad (46)$$

and

$$\int_0^1 (1-r^2)^{\alpha} r^{n-1} dr = \frac{\Gamma(\alpha+1)\Gamma(n/2)}{2\Gamma(\alpha+1+n/2)}. \quad (47)$$

By using (46) and (47) into (45), we get (44), that is, (43). According to Theorem 4, the equality in (43) holds if and only if function u is constant. □

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