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NON–LINEAR OPERATOR AND THE SUFFICIENT CONDITIONS OF UNIVALENCE WITH APPLICATIONS

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Abstract. In the present paper, by using a nonlinear operator, we obtain a general theorem of univalence which refines and generalizes many results. Some applications of the main results are also considered.

1. Introduction

Let P denote the class of analytic functions $p(z)$ in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ which satisfies $p(0) = 0$. Also, let A denote the subclass of P of all functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ (z \in \mathbb{U}). \tag{1.1}$$

Moreover, let S denote the subclass of A consisting of functions which are univalent functions in $\mathbb{U}$. Furthermore, we use Ω to denote the class of bounded analytic functions $w(z)$ in $\mathbb{U}$, satisfying the conditions $w(0) = 0$ and for $|w| \leqslant 1$.

For $f(z) \in A$ and $z \in \mathbb{U}$, let the integral operators $I(f), L(f), L_\gamma(f)$ and $G_\alpha(f)$ be defined as

$$I(f)(z) = \int_0^z \frac{f(t)}{t}\, dt, \tag{1.2}$$

$$L(f)(z) = \frac{2}{z} \int_0^z f(t)\, dt \tag{1.3}$$

$$L_\gamma(f)(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t)\, t^{\gamma-1}\, dt \qquad (\gamma \in \mathbb{C};\ \mathrm{Re}(\gamma) > -1). \tag{1.4}$$

and

$$G_\alpha(f)(z) = \left[\alpha \int_0^z u^{-1}\, (f(u))^\alpha\, du\right]^{\frac{1}{\alpha}} \qquad (\mathrm{Re}(\alpha) > 0). \tag{1.5}$$

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The operators $I(f)$ and $L(f)$ are Alexander operator and Libera operator which were introduced earlier by Alexander [1] and Libera [8]. Also, $L_\gamma(f)$ is the general form of Bernardi operator, the operator $L_\gamma(f)$ when $\gamma \in \mathbb{N} = \{1,2,...\}$ was introduced by Bernardi [3]. Furthermore, the operator $G_\alpha(f)$ was introduced by Miller and Mocanu [9].

2. Preliminaries

Firstly, we denote by

$$F_\alpha(f) : P \longrightarrow A$$

the nonlinear operator defined by:

$$F_\alpha(f)(z) := \left[\alpha \int_0^z u^{\alpha-1} \exp\left(\int_0^u \frac{f(t)}{t} dt\right) du\right]^{\frac{1}{\alpha}} \qquad (z \in \mathbb{U};\ f \in P;\ \mathrm{Re}(\alpha) > 0)\,, \tag{2.1}$$

where is the exponential function in the above relation is the principal value.

In this paper we need the following lemmas.

LEMMA 2.1. [4], (see [12]) *Let α be a complex number,* $\mathrm{Re}(\alpha) > 0$ *and* $f(z) \in A$. *If*

$$\frac{1-|z|^{2\,\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left(\frac{zf''(z)}{f'(z)}\right) \in \Omega \quad (z \in \mathbb{U})\,, \tag{2.2}$$

then the integral operator

$$H_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u)\, du\right]^{\frac{1}{\alpha}} \tag{2.3}$$

is in the class S.

Also, we consider the general Schwarz Lemma, see e.g.[16]:

LEMMA 2.2. *Let the function $f(z)$ be regular in $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| \leqslant M$, M fixed. If $f(z)$ has one zero with multiply $\geqslant m$. Then*

$$|f(z)| \leqslant \frac{M}{R^m}|z|^m \quad (z \in \mathbb{U}_R), \tag{2.4}$$

the equality (2.4) can hold true only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m \ \ (z \neq 0), \tag{2.5}$$

where θ is constant.

3. Main results

THEOREM 3.1. *Let* α *be a complex number,* $\operatorname{Re}(\alpha) > 0$ *and* $p(z) \in P$. *Suppose also that*

$$|p(z)| \leqslant M \qquad (M \ \text{constant}). \tag{3.1}$$

Then

$$F_\alpha(p)(z) \in S, \tag{3.2}$$

where

$$M \leqslant \frac{1}{2}\left(1+2\operatorname{Re}(\alpha)\right)^{1+\frac{1}{2\operatorname{Re}(\alpha)}}. \tag{3.3}$$

Moreover, if $f(z) \in A$ *satisfies* $|f(z)| \leqslant L_1$ *and* $\left|\frac{z^2 f'(z)}{f(z)^2} - L_2\right| < L_3$, *then*

$$G_{\alpha,\beta}(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(\frac{f(u)}{u}\right)^\beta du\right]^{\frac{1}{\alpha}} \in S, \tag{3.4}$$

where

$$L_1(L_2+L_3) \leqslant \frac{1}{2|\beta|}\left(1+2\operatorname{Re}(\alpha)\right)^{1+\frac{1}{2\operatorname{Re}(\alpha)}} - 1 \tag{3.5}$$

and $\alpha,\ \beta \in \{w \in \mathbb{C} : \operatorname{Re}(w) > 0\}$.

Proof. Let $|p(z)| \leqslant M \quad (z \in \mathbb{U})$, By using Lemma 2.2, we have

$$|p(z)| \leqslant M\ |z| \qquad (|z| < 1). \tag{3.6}$$

Define the function $g(z)$ by

$$g(z) = \int_0^z \exp\left(\int_0^u \frac{p(t)}{t} dt\right) du, \tag{3.7}$$

then, we have $g(0) = g'(0) - 1 = 0$, therefore,

$$\begin{aligned} \frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\left|\frac{zg''(z)}{g'(z)}\right| &= \frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}|p(z)| \\ &\leqslant \frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} M|z|. \end{aligned} \tag{3.8}$$

Since the maximum value of the function

$$h(r) = \frac{1}{a} r\left(1-r^{2a}\right) \qquad (a > 0;\ 0 \leqslant r < 1)$$

occurs when $r=(1+\ 2\,a)^{-\frac{1}{2a}}$, therefore,

$$\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\,|z|<\frac{2}{(1+2\operatorname{Re}(\alpha))^{1+\frac{1}{2\operatorname{Re}(\alpha)}}}. \tag{3.9}$$

Using (3.6), we have

$$\frac{1-|z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)}\left|\frac{zg''(z)}{g'(z)}\right|\leqslant\frac{2}{(1+2\operatorname{Re}(\alpha))^{1+\frac{1}{2\operatorname{Re}(\alpha)}}}M\ . \tag{3.10}$$

Then by using Lemma 2.1, $F_\alpha(z)\in S$, when $M\leqslant\frac{1}{2}(1+2\operatorname{Re}(\alpha))^{1+\frac{1}{2\operatorname{Re}(\alpha)}}$, which gives (3.2).

Also, if $f(z)\in A$, then we have

$$\begin{aligned}\left|\frac{z\,f'(z)}{f(z)}-1\right|&\leqslant\left|\frac{zf'(z)}{f(z)}\right|+1\\&=\left|\frac{z^2f'(z)}{f(z)^2}\right|\frac{|f(z)|}{|z|}+1\\&\leqslant\left|\frac{z^2f'(z)}{f(z)^2}\right|L_1+1\\&\leqslant\left|\frac{z^2f'(z)}{f(z)^2}-L_2\right|L_1+L_1L_2+1\\&\leqslant L_1(L_2+L_3)+1\ .\end{aligned}$$

By putting $p(z)=\beta\left(\frac{zf'(z)}{f(z)}-1\right)$, $M=|\beta|\,(L_1(L_2+L_3)+1)$ in (3.1) and using (3.2), which gives (3.4). Therefore, the proof of the theorem is completed. □

Putting $p(z)=\beta\left(\frac{zf'\ (z)}{f(z)}-1\right)$, $M=|\beta|M_1$ and using Theorem 3.1, we have the following corollary.

COROLLARY 3.1. *Let* $f(z)\ \in A$ *satisfy the condition* $\left|\frac{zf'(z)}{f(z)}-1\right|<M_1$, *then*

$$G_{\alpha,\beta}(z)=\left[\alpha\int_0^z u^{\alpha-1}\left(\frac{f(u)}{u}\right)^\beta du\right]^{\frac{1}{\alpha}}\in S, \tag{3.11}$$

where

$$M_1\leqslant\frac{1}{2\,|\beta|}(1+2\operatorname{Re}(\alpha))^{1+\frac{1}{2\operatorname{Re}(\alpha)}} \tag{3.12}$$

and $\alpha,\ \beta\in\{w\in\mathbb{C}:\operatorname{Re}(w)>0\}$.

Putting $\alpha=1,\ \beta=1$ and using Theorem 3.1, we have the following property of the Alexander operator.

COROLLARY 3.2. *Let* $f(z) \in A$ *satisfy* $|f(z)| \leqslant L_1$ *and* $\left|\frac{z^2 f'(z)}{f(z)^2} - L_2\right| < L_3$, *then*

$$I(f)\ (z) = \int_0^z \frac{f(u)}{u} du \in S, \tag{3.13}$$

where

$$L_1(L_2 + L_3) \leqslant \frac{3\sqrt{3}}{2} - 1\ . \tag{3.14}$$

Putting $p(z) = \gamma z f'(z)$, $M = \beta\,|\gamma|$ and using Theorem 3.1, we have the following Corollary.

COROLLARY 3.3. *Let* $f(z) \in A$ *satisfy* $\left|zf^{'}(z)\right| \leqslant \beta$ *(*β *constant) and* $\alpha,\ \gamma \in \{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

Then

$$G_{\alpha,\gamma}(z) = \left[\alpha \int_0^z u^{\alpha-1} (e^{f(u)})^{\gamma} du\right]^{\frac{1}{\alpha}} \in S\ , \tag{3.15}$$

where

$$\beta \leqslant \frac{1}{2\,|\gamma|}\,(1 + 2\,\mathrm{Re}(\alpha))^{1+\frac{1}{2\,\mathrm{Re}(\alpha)}}\ .$$

Putting $p(z) = \sum_{i=1}^{n} \beta_i (\frac{z f_{i'}'(z)}{f_i(z)} - 1)$ and using the same technique in Theorem 3.1, we have the following corollary.

COROLLARY 3.4. *Let* $f_i(z) \in A$ *satisfy* $|f_i(z)| \leqslant L_{1i}$ *and* $\left|\frac{z^2 f_i'(z)}{f_i(z)^2} - L_{2i}\right| < L_{3i}$ *for all* $i = 1, 2, 3, ..., n$.

Then

$$G_{\alpha,\beta_1,\ldots,\beta_n}(z) = \left[\alpha \int_0^z u^{\alpha-1} \prod_{i=1}^{n} \left(\frac{f_i(u)}{u}\right)^{\beta_i} du\right]^{\frac{1}{\alpha}} \in S\ , \tag{3.16}$$

where

$$\sum_{i=1}^{n} |\beta_i|\,(L_{1i}(L_{2i} + L_{3i}) + 1) \leqslant \frac{1}{2}\,(1 + 2\,\mathrm{Re}(\alpha))^{1+\frac{1}{2\ \mathrm{Re}(\alpha)}} \tag{3.17}$$

$$(\alpha,\ \beta_1, \beta_2, ..., \beta_n\ \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0)\ .$$

In the following remarks we give some special cases of our results which generalize and improve some recent results.

REMARKS.

i) Putting $M = 1,\ \alpha = 1$ and $p(z) = \sum_{i=1}^{n} \gamma_i \left(\frac{z\,(L(a,c)\ f_i(z))'}{L(a,c)\ f_i(z)} - 1\right)$; $(\gamma_i \in \mathbb{C}\ )$.

If $\sum_{i=1}^{n} |\gamma_i| \leqslant 1$ and $\left|\frac{z(L(a,c)\, f_i(z))'}{L(a,c)\, f_i(z)} - 1\right| < 1$, where $L(a,c)\,(f)$ is the Carlson-Shafer linear operator depends on generalized hypergeometric function (see [7]). Applying Theorem 3.1, we have the main result due to Selvaraj and Karthikeyan [15].

ii) Putting $M = 1$, $\alpha = 1$ and $p(z) = \sum_{i=1}^{n}\gamma_i \left(\frac{z(D_n\, f_i(z))'}{D_n\, f_i(z)} - 1\right)$; $(\gamma_i \in \mathbb{C}\,)$.

If $\sum_{i=1}^{n} |\gamma_i| \leqslant 1$ and $\left|\frac{z(D_n\, f_i(z))'}{D_n\, f_i(z)} - 1\right| < 1$, where $D_n\,(f)$ is the Ruscheweyh differential operator. Applying Theorem 3.1, we have the main result due to Oros et. al. [10].

iii) Putting $M = 1$, $\alpha = 1$ and $p(z) = \sum_{i=1}^{n}\gamma_i \left(\frac{z(f_i(z))'}{f_i(z)} - 1\right)$; $(\gamma_i \in \mathbb{C}\,)$.

If $\sum_{i=1}^{n} |\gamma_i| \leqslant 1$ and $\left|\frac{z(f_i(z))'}{f_i(z)} - 1\right| < 1$. Applying Theorem 3.1, we have the main result due to Breaz and Breaz [5].

iv) Applying Theorem 3.1, by putting $L_1 = L_2 = L_3 = 1$ and $\beta = \frac{1}{\alpha}$, we improve the main results due to Pescar and Breaz [13].

v) Putting $\gamma = \alpha$ in Corollary 3.3, we improve the main result due to Pescar [11].

vi) Putting $p(z) = \sum_{i=1}^{n} \beta \left(\frac{z(g_{v_i}(z))'}{g_{v_i}(z)} - 1\right)$; $(\beta \in \mathbb{C}$, g_{v_i} is the normalized Bessel function of the first kind) and $\alpha = n\beta + 1$ in Theorem 3.1, we improve Theorem 2 due to Baricz and Frasin [2].

vii) Putting $\gamma = \alpha$ and $f(z) = g_v$ (g_v is the normalized Bessel function of the first kind) in Corollary 3.3, we improve Theorem 3 due to Baricz and Frasin [2].

viii) Putting $L_{1i} = M_i$, $L_{2i} = 1$ and $L_{3i} = \gamma_i$ in Corollary 3.4, we improve the main result due to Ravichandran [14].

ix) Putting $p(z) = \sum_{j=1}^{n} \frac{1}{\gamma_j}\left(\frac{z(f_j(z))'}{f_j(z)} - 1\right)$ and $M = \sum_{j=1}^{n} \frac{M_j}{\gamma_j}$; $(\gamma_j \in \mathbb{C}$ and $\left|\frac{z(f_j(z))'}{f_j(z)} - 1\right| < M_j$). Applying Theorem 3.1, we improve the main result due to Pescar [12].

x) Putting $p(z) = \sum_{j=1}^{[|\mathrm{Re}(\eta)|]} \frac{1}{\gamma_j}\left(\frac{z(f_j(z))'}{f_j(z)} - 1\right)$, $M = \sum_{j=1}^{[|\mathrm{Re}(\eta)|]} \frac{M_j}{\gamma_j}$ and $\alpha = \eta\beta$ ($\gamma_j, \eta, \beta \in \mathbb{C}$ and $\left|\frac{z(f_j(z))'}{f_j(z)} - 1\right| < M_j = \frac{(2a+1)^{\frac{2a+1}{a}}}{2\,\mathrm{Re}(\eta)}|\gamma_j|$, $a \leqslant \mathrm{Re}(\alpha)$) in the first part of Theorem 3.1, we have the main result due to Breaz et al. [6].

4. Applications

THEOREM 4.1. *Let* $f(z) \in A$ *and*

$$F_{\alpha}(z) = \left[\alpha \int_0^z u^{\alpha-2} f(u)\, du\right]^{\frac{1}{\alpha}}. \tag{4.1}$$

i) If $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1,$ *then* $F_{\alpha}(z) \in S,$ *for all complex values of* α *satisfying* $\mathrm{Re}(\alpha) > 0.$

ii) If $|f(z)| \leqslant 1$ *and* $\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| < 1,$ *then* $F_{\alpha}(z) \in S,$ *for all complex values of* α *satisfying* $\mathrm{Re}(\alpha) \geqslant 1.3470133\ldots.$

Proof. By putting $p(z) = \left(\frac{zf'(z)}{f(z)} - 1\right)$ and $M = 1$ in Theorem 3.1, then we have $F_{\alpha}(z) \in S$ for all α which satisfy

$$1 \leqslant \frac{1}{2}\left(1 + 2\ \mathrm{Re}(\alpha)\right)^{1+\frac{1}{2\ \mathrm{Re}(\alpha)}}, \tag{4.2}$$

since the function $h(a) = \frac{1}{2}(1+2a)^{1+\frac{1}{2a}}$ is an increasing function of $a > 0$ which satisfies

$$\lim_{a \to 0} h(a) = \frac{e}{2} > 1,$$

therefore, the condition (4.2) is satisfied.

Also, by using Theorem 3.1, for $L_1 = L_2 = L_3 = M = \beta = 1$ we have

$$6 \leqslant \left(1 + 2\ \mathrm{Re}(\alpha)\right)^{1+\frac{1}{2\ \mathrm{Re}(\alpha)}},$$

since the function $h(a) = (1 + 2\,a)^{1+\frac{1}{2\,a}}$ is an increasing function of $a > 0,$ therefore $\mathrm{Re}(\alpha)$ constrained by (4.2) must satisfy the condition $\mathrm{Re}(\alpha) \geqslant 1.3470133\ldots.$ $\square$

Putting $\alpha = 2$ in Theorem 4.1, we have

COROLLARY 4.1. *Let* $f(z) \in A$ *satisfy either* $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1,$ *or* $|f(z)| \leqslant 1$ *and* $\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| < 1.$ *Then*

$$\sqrt{z\,L(f)(z)} \in S, \tag{4.3}$$

where $L(f)$ *is the Libera operator.*

Putting $\alpha = \gamma + 1$ in Theorem 4.1, we have

COROLLARY 4.2. *Let* $f(z) \in A$.
i) If $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$, *then*

$$z\left(\frac{L_\gamma(f)(z)}{z}\right)^{\frac{1}{1+\gamma}} \in S,$$

for all complex values of γ *satisfying* $\mathrm{Re}(\gamma) > -1$.
ii) If $|f(z)| \leqslant 1$ *and* $\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| < 1$, *then*

$$z\left(\frac{L_\gamma(f)(z)}{z}\right)^{\frac{1}{1+\gamma}} \in S,$$

for all complex values of γ *satisfying* $\mathrm{Re}(\gamma) \geqslant 0.3470133...$.

Putting $p(z) = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)$ and $M = |\alpha| M_1$ in Theorem 3.1, we have

COROLLARY 4.3. *Let* $f(z) \in A$.
i) If $\left|\frac{zf'(z)}{f(z)} - 1\right| < M_1$, *then* $G_\alpha(f)(z) \in S$, *where*

$$M_1 \leqslant \frac{1}{2|\alpha|}(1 + 2\mathrm{Re}(\alpha))^{1+\frac{1}{2\,\mathrm{Re}(\alpha)}} \tag{4.4}$$

for all complex values of α *satisfying* $\mathrm{Re}(\alpha) \geqslant 0$.
ii) If $|f(z)| \leqslant L_1$ *and* $\left|\frac{z^2 f'(z)}{f(z)^2} - L_2\right| < L_3$ *then* $G_\alpha(f)(z) \in S$, *where*

$$L_1(L_2 + L_3) \leqslant \frac{1}{2|\alpha|}(1 + 2\mathrm{Re}(\alpha))^{1+\frac{1}{2\,\mathrm{Re}(\alpha)}} - 1 \tag{4.5}$$

for all complex values of α *satisfying* $\mathrm{Re}(\alpha) \geqslant 0$.

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