SOME INEQUALITIES FOR GENERAL Lp–HARMONIC BLASCHKE BODIES

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Abstract. Feng and Wang gave the extremum value of volume for the general Lp-harmonic Blaschke bodies. In this paper, associated with the general Lp-harmonic Blaschke bodies, we obtain the extremum value of dual quermassintegrals and the Lp-dual affine surface area, respectively. Further, two monotonic inequalities for the general Lp-harmonic Blaschke bodies are given.

1. Introduction

If \( K \) is a compact star-shaped (about the origin) in Euclidean space \( \mathbb{R}^n \), its radial function, \( \rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, +\infty) \), is defined by (see [5, 10])

\[
\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\} , \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

If \( \rho_K \) is positive and continuous, \( K \) will be called a star body (about the origin). Let \( \mathcal{S}_o^n \) denote the set of star bodies (about the origin) in \( \mathbb{R}^n \). For the set of origin-symmetric star bodies, we write \( \mathcal{S}_{os}^n \). Two star bodies \( K \) and \( L \) are said to be dilates (of one another) if \( \rho_K(u)/\rho_L(u) \) is independent of \( u \in S^{n-1} \), where \( S^{n-1} \) denotes the unit sphere in \( \mathbb{R}^n \).

Lutwak ([8]) introduced the notion of harmonic Blaschke combination for star bodies. For \( K, L \in \mathcal{S}_o^n \), and \( \lambda, \mu \geq 0 \) (not both zero), the harmonic Blaschke combination, \( \lambda K + \mu L \in \mathcal{S}_o^n \), of \( K \) and \( L \) is defined by

\[
\frac{\rho(\lambda K + \mu L, \cdot)^{n+1}}{V(\lambda K + \mu L)} = \lambda \frac{\rho(K, \cdot)^{n+1}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+1}}{V(L)}. \tag{1.1}
\]

From definition (1.1), Lutwak ([8]) proved the Brunn-Minkowski inequality for the harmonic Blaschke combination as follows:

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THEOREM 1.A. If \( K, L \in \mathcal{S}^n_o \), \( \lambda, \mu \geq 0 \) (not both zero), then
\[
V(\lambda K +_p \mu L)^{\frac{1}{n}} \geq \lambda V(K)^{\frac{1}{n}} + \mu V(L)^{\frac{1}{n}},
\]
with equality if and only if \( K \) and \( L \) are dilates.

Based on definition (1.1) of harmonic Blaschke combination, Feng and Wang in [2] introduced the notion of \( L_p \)-harmonic Blaschke combination: For \( K, L \in \mathcal{S}^n_o \), \( p \geq 1 \) and \( \lambda, \mu \geq 0 \) (not both zero), the \( L_p \)-harmonic Blaschke combination, \( \lambda \cdot K +_p \mu \cdot L \in \mathcal{S}^n_o \), of \( K \) and \( L \) is given by
\[
\frac{\rho(\lambda \cdot K +_p \mu \cdot L, \cdot)^{n+p}}{V(\lambda \cdot K +_p \mu \cdot L)} = \lambda \frac{\rho(K, \cdot)^{n+p}}{V(K)} + \mu \frac{\rho(L, \cdot)^{n+p}}{V(L)},
\]
where the operation \( +_p \) is called \( L_p \)-harmonic Blaschke addition. From (1.3), we easily know the harmonic Blaschke scalar multiplication and the usual scalar multiplication are related by \( \lambda \cdot K = \lambda^\frac{1}{p} K \).

Let \( \lambda = \mu = \frac{1}{2} \) and \( L = -K \) in (1.3), the \( L_p \)-harmonic Blaschke body, \( \nabla_p K \), of \( K \in \mathcal{S}^n_o \) is given by (see [2])
\[
\nabla_p K = \frac{1}{2} \cdot K +_p \frac{1}{2} \cdot (-K).
\]

From (1.3), Feng and Wang in [2] gave the following an extension of inequality (1.2).

THEOREM 1.B. If \( K, L \in \mathcal{S}^n_o \), \( p \geq 1 \), \( \lambda, \mu \geq 0 \) (not both zero), then
\[
V(\lambda K +_p \mu L)^{\frac{1}{n}} \geq \lambda V(K)^{\frac{1}{n}} + \mu V(L)^{\frac{1}{n}},
\]
with equality if and only if \( K \) and \( L \) are dilates.

Obviously, from (1.4) and (1.5), we have that if \( K \in \mathcal{S}^n_o \), \( p \geq 1 \), then (see [2])
\[
V(\nabla_p K) \geq V(K),
\]
with equality if and only if \( K \) is origin-symmetric.

Recently, Feng and Wang in [3] extended the notion of the \( L_p \)-harmonic Blaschke bodies and defined the general \( L_p \)-harmonic Blaschke bodies as follows: For \( K \in \mathcal{S}^n_o \), \( p \geq 1 \) and \( \tau \in [-1, 1] \), the general \( L_p \)-harmonic Blaschke body, \( \nabla_p^\tau K \), of \( K \) is defined by
\[
\frac{\rho(\nabla_p^\tau K, \cdot)^{n+p}}{V(\nabla_p^\tau K)} = f_1(\tau) \frac{\rho(K, \cdot)^{n+p}}{V(K)} + f_2(\tau) \frac{\rho(-K, \cdot)^{n+p}}{V(-K)}.
\]

Associated with the definition of the \( L_p \)-harmonic Blaschke combination, it easily follows that
\[
\nabla_p^\tau K = f_1(\tau) \cdot K +_p f_2(\tau) \cdot (-K),
\]
where

\[
\begin{align*}
  f_1(\tau) &= \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \\
  f_2(\tau) &= \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}.
\end{align*}
\]

From (1.8) and (1.9), we easily get that if \( \tau = 0 \) then \( \hat{V}_p^0 K = \hat{V}_p K \); if \( \tau = \pm 1 \), then \( \hat{V}_p^{\pm 1} K = K \), \( \hat{V}_p^{-1} K = -K \).

Further, Feng and Wang in [3] got the following extremum value of volume for the general \( L_p \)-harmonic Blaschke bodies.

**Theorem 1.C.** If \( K \in \mathcal{S}^n_o \), \( p \geq 1 \), \( \tau \in [-1,1] \), then

\[
V(\hat{V}_p K) \geq V(\hat{V}_p^\tau K) \geq V(K).
\]

If \( \tau \neq \pm 1 \), then equality holds in the right inequality of (1.10) if and only if \( K \) is an origin-symmetric star body, and if \( \tau \neq 0 \), then equality holds in the left inequality of (1.10) if and only if \( K \) is also an origin-symmetric star body.

In this article, from definition (1.7), we first give the extremum value of dual quermassintegrals for the general \( L_p \)-harmonic Blaschke bodies as follows.

**Theorem 1.1.** If \( K \in \mathcal{S}^n_o \), \( p \geq 1 \), \( \tau \in [-1,1] \), \( i \) is any real and \( i \neq n \), then for \( i > -p \),

\[
\frac{\tilde{W}_i(\hat{V}_p K)^{n+i}}{V(\hat{V}_p K)} \geq \frac{\tilde{W}_i(\hat{V}_p^\tau K)^{n+i}}{V(\hat{V}_p^\tau K)} \geq \frac{\tilde{W}_i(K)^{n+i}}{V(K)}. \tag{1.11}
\]

If \( \tau \neq \pm 1 \), equality holds in the right inequality of (1.11) if and only if \( K \) is an origin-symmetric star body, and if \( \tau \neq 0 \), with equality in the left inequality of (1.11) if and only if \( K \) is also an origin-symmetric star body. For \( i < -p \), inequality (1.11) is reversed. For \( i = -p \), (1.11) is identical.

Here \( \tilde{W}_i(K) \) denotes the dual quermassintegrals of \( K \in \mathcal{S}^n_o \) which be defined by (see [7])

\[
\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-i} du, \tag{1.12}
\]

for any real \( i \).

Obviously, let \( i = 0 \) in inequality (1.11) and notice that \( \tilde{W}_0(K) = V(K) \), we immediately obtain Theorem 1.C.

Secondly, associated with the \( L_p \)-dual affine surface area (see (2.5)), we obtain its extremum value for the general \( L_p \)-harmonic Blaschke bodies.

**Theorem 1.2.** If \( K \in \mathcal{S}^n_o \), \( 1 \leq p < n \), \( \tau \in [-1,1] \), then

\[
\tilde{\Omega}_{-p}(\hat{V}_p K) \geq \tilde{\Omega}_{-p}(\hat{V}_p^\tau K) \geq \tilde{\Omega}_{-p}(K). \tag{1.13}
\]

If \( \tau \neq \pm 1 \), equality holds in the right inequality of (1.13) if and only if \( K \) is an origin-symmetric star body, and if \( \tau \neq 0 \), equality holds in the left inequality of (1.13) if and only if \( K \) is also an origin-symmetric star body.

Recall that Feng and Wang in [2] proved that
If \( K \in \mathscr{S}_o^n \), \( 1 \leq p < n \), then

\[
\tilde{\Omega}_{-p}(\hat{\Omega}_p K) \geq \tilde{\Omega}_{-p}(K),
\]

with equality if and only if \( K \) is an origin-symmetric star body.

Obviously, inequality (1.13) is an isolation of inequality (1.14).

In addition, we also give two monotonic inequalities for the general \( L_p \)-harmonic Blaschke bodies as follows:

**Theorem 1.3.** Let \( K, L \in \mathscr{S}_o^n \), \( p \geq 1 \) and \( \tau \in [-1, 1] \). If \( \hat{\Omega}_p^\tau K \subseteq \hat{\Omega}_p^\tau L \) and \( L \in \mathscr{S}_{os}^n \), then

\[
V(\hat{\Omega}_p^\tau K)V(K)^{\frac{p}{n}} \leq V(\hat{\Omega}_p^\tau L)V(L)^{\frac{p}{n}},
\]

with equality if and only if \( K \) and \( L \) are dilatant origin-symmetric star bodies.

**Theorem 1.4.** Let \( K, L \in \mathscr{S}_o^n \), \( p \geq 1 \) and \( \tau \in [-1, 1] \). If \( K \subseteq L \), then

\[
V(\hat{\Omega}_p^\tau K)^{\frac{p}{n}}V(K) \leq V(\hat{\Omega}_p^\tau L)^{\frac{p}{n}}V(L),
\]

with equality if and only if \( K = L \).

Actually, Theorem 1.3 may be regard as the Shephard problem of the general \( L_p \)-harmonic Blaschke bodies.

### 2. Preliminaries

#### 2.1. \( L_p \)-dual mixed volume

For \( K, L \in \mathscr{S}_o^n \), \( p \geq 1 \), \( \lambda, \mu \geq 0 \) (not both zero), the \( L_p \)-harmonic radial combination, \( \lambda*K_{-p} \mu*L \), of \( K \) and \( L \) is defined by (see [4, 9])

\[
\rho(\lambda*K_{-p} \mu*L, \cdot)^{-p} = \lambda\rho(K, \cdot)^{-p} + \mu\rho(L, \cdot)^{-p}.
\]

The notion of \( L_p \)-dual mixed volume was introduced by Lutwak (see [9]). For \( K, L \in \mathscr{S}_o^n \), \( p \geq 1 \) and \( \varepsilon > 0 \), the \( L_p \)-dual mixed volume, \( \tilde{V}_{-p}(K, L) \), of the \( K \) and \( L \) is defined by

\[
\frac{n}{-p}\tilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K_{-p} \varepsilon*L) - V(K)}{\varepsilon}.
\]

Further, Lutwak ([9]) proved that the \( L_p \)-dual mixed volume has the following integral representation:

\[
\tilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} du.
\]

From (2.1), we easily know

\[
\tilde{V}_{-p}(K, K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n du = V(K).
\]

The \( L_p \)-dual Minkowski inequality may be stated that (see [9]): For \( K, L \in \mathscr{S}_o^n \), \( p \geq 1 \), then

\[
\tilde{V}_{-p}(K, L) \geq V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}},
\]

with equality if and only if \( K \) and \( L \) are dilates.
2.2. $L_p$-dual affine surface

In 2008, Wang and He ([11]) gave the notion of $L_p$-dual affine surface area associated with the $L_p$-dual mixed volume. For $K \in S^n_o$ and $1 \leq p < n$, the $L_p$-dual affine surface area, $\tilde{\Omega}_{-p}(K)$, of $K$ is defined by

$$n^\frac{p}{n} \tilde{\Omega}_{-p}(K) \frac{n-p}{n} = \inf \{ n\tilde{V}_{-p}(K, Q^*) V(Q) \frac{-p}{n} : Q \in \mathcal{K}_{c}^n \}. \quad (2.4)$$

For the $L_p$-dual affine surface area, except [11], Feng and Wang recently established some inequalities for $L_p$-dual affine surface area (see [1, 2]). In particular, Feng and Wang in [2] improved definition (2.4) as follows: For $K \in S^n_o$ and $1 \leq p < n$, the $L_p$-dual affine surface area, $\tilde{\Omega}_{-p}(K)$, of $K$ is defined by

$$n^\frac{p}{n} \tilde{\Omega}_{-p}(K) \frac{n-p}{n} = \inf \{ n\tilde{V}_{-p}(K, Q^*) V(Q) \frac{-p}{n} : Q \in \mathcal{K}_{os}^n \}. \quad (2.5)$$

From (2.5), we easily get that for $K \in S^n_o$ and $1 \leq p < n$,

$$\tilde{\Omega}_{-p}(-K) = \tilde{\Omega}_{-p}(K). \quad (2.6)$$

Associated with definition (2.5) and $L_p$-harmonic Blaschke addition (1.3), Feng and Wang ([2]) gave the following result:

**THEOREM 2.A.** If $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero) and $1 \leq p < n$, then

$$\frac{\tilde{\Omega}_{-p}(\lambda \cdot K + \mu \cdot L)}{V(\lambda \cdot K + \mu \cdot L)} \geq \frac{n^\frac{p}{n} \tilde{\Omega}_{-p}(K) \frac{n-p}{n}}{V(K)} + \frac{n^\frac{p}{n} \tilde{\Omega}_{-p}(L) \frac{n-p}{n}}{V(L)}, \quad (2.7)$$

with equality if and only if $K$ and $L$ are dilates.

2.3. Properties of the general $L_p$-harmonic Blaschke bodies

For the general $L_p$-harmonic Blaschke bodies. Feng and Wang in [3] proved the following properties.

**THEOREM 2.B.** If $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$, then

$$\hat{\nabla}_p^{-\tau} K = \hat{\nabla}_p^{\tau} (-K) = -\hat{\nabla}_p^{\tau} K. \quad (2.8)$$

**THEOREM 2.C.** Let $K \in \mathcal{S}_o^n$, $p \geq 1$, $\tau \in [-1, 1]$. If $K$ is not an origin-symmetric star body, then $\hat{\nabla}_p^{\tau} K = \hat{\nabla}_p^{-\tau} K$ if and only if $\tau = 0$.

**THEOREM 2.D.** If $K \in \mathcal{S}_o^n$, $p \geq 1$, then for $\tau \in [-1, 1]$,

$$\hat{\nabla}_p K = \frac{1}{2} \cdot \hat{\nabla}_p^{\tau} K + \frac{1}{p} \cdot \hat{\nabla}_p^{-\tau} K. \quad (2.9)$$
3. Proofs of Theorems

In this section, we will complete the proofs of Theorems 1.1–1.4. In order to prove Theorem 1.1, we first give the following Brunn-Minkowski inequality for dual quermassintegrals of the $L_p$-harmonic Blaschke combinations.

**Theorem 3.1.** If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $\lambda, \mu \geq 0$ (not both zero), $i$ is any real and $i \neq n$, then for $i > -p$,

$$\frac{\tilde{W}_i(\lambda \cdot K^+_p \mu \cdot L)}{V(\lambda \cdot K^+_p \mu \cdot L)^{\frac{n+p}{n-i}}} \geq \lambda \frac{\tilde{W}_i(K)}{V(K)^{\frac{n+p}{n-i}}} + \mu \frac{\tilde{W}_i(L)}{V(L)^{\frac{n+p}{n-i}}};$$

(3.1)

for $i < -p$,

$$\frac{\tilde{W}_i(\lambda \cdot K^+_p \mu \cdot L)}{V(\lambda \cdot K^+_p \mu \cdot L)^{\frac{n+p}{n-i}}} \leq \lambda \frac{\tilde{W}_i(K)}{V(K)^{\frac{n+p}{n-i}}} + \mu \frac{\tilde{W}_i(L)}{V(L)^{\frac{n+p}{n-i}}}.$$

(3.2)

In each case, equality holds if and only if $K$ and $L$ are dilates. For $i = -p$, (3.1) (or (3.2)) is identic.

The proof of Theorem 3.1 requires the following Minkowski integral inequality (see [6]).

**Lemma 3.1.** Let $f$ and $g$ are nonnegative bounded Borel functions on measure space $X$. If $k > 1$, then

$$\left(\int_X (f(x) + g(x))^k dx\right)^{\frac{1}{k}} \leq \left(\int_X f^k(x) dx\right)^{\frac{1}{k}} + \left(\int_X g^k(x) dx\right)^{\frac{1}{k}};$$

(3.3)

if $0 < k < 1$ or $k < 0$, then

$$\left(\int_X (f(x) + g(x))^k dx\right)^{\frac{1}{k}} \geq \left(\int_X f^k(x) dx\right)^{\frac{1}{k}} + \left(\int_X g^k(x) dx\right)^{\frac{1}{k}}.$$

(3.4)

Equality holds in every inequality if and only if $f(x)$ and $g(x)$ are effectively proportional or $f(x)g(x) = 0$ on $X$.

**Proof of Theorem 3.1.** For $i > -p$, since $i \neq n$, thus $0 < (n-i)/(n+p) < 1$ when $-p < i < n$, or $(n-i)/(n+p) < 0$ when $i > n$. This together with (1.3), (1.12) and Minkowski integral inequality (3.4), it follows that

$$\frac{\tilde{W}_i(\lambda \cdot K^+_p \mu \cdot L)^{\frac{n+p}{n-i}}}{V(\lambda \cdot K^+_p \mu \cdot L)} = \frac{1}{V(\lambda \cdot K^+_p \mu \cdot L)} \left[ \frac{1}{n} \int_{S^{n-1}} \rho(\lambda \cdot K^+_p \mu \cdot L, u)^{n-i} du \right]^{\frac{n+p}{n-i}}$$

$$= \left\{ \frac{1}{n} \int_{S^{n-1}} \left[ \frac{\rho(\lambda \cdot K^+_p \mu \cdot L, u)^{n+p}}{V(\lambda \cdot K^+_p \mu \cdot L)} \right]^{\frac{n-i}{n+p}} du \right\}^{\frac{n+p}{n-i}}$$

$$= \left\{ \frac{1}{n} \int_{S^{n-1}} \left[ \lambda \frac{\rho^{n+p}(u)}{V(K)} + \mu \frac{\rho^{n+p}(u)}{V(L)} \right]^{\frac{n-i}{n+p}} du \right\}^{\frac{n+p}{n-i}}$$
\[
\geq \frac{\lambda}{V(K)} \left[ \frac{1}{n} \int_{S^{n-1}} \rho^n_{K}(u) du \right]^{\frac{n+p}{n-i}} + \frac{\mu}{V(L)} \left[ \frac{1}{n} \int_{S^{n-1}} \rho^n_{L}(u) du \right]^{\frac{n+p}{n-i}}
\]

\[
= \lambda \frac{\tilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)} + \mu \frac{\tilde{W}_i(L)^{\frac{n+p}{n-i}}}{V(L)}. \tag{3.5}
\]

From this, we get inequality (3.1). According to the condition of equality holds in Minkowski integral inequality (3.4), we know that with equality in (3.5) if and only if \( K \) and \( L \) are dilates, this means that equality holds in (3.1) if and only if \( K \) and \( L \) are dilates.

For \( i < -p \), because of \( (n-i)/(n+p) > 1 \), thus inequality (3.5) is reversed by the Minkowski integral inequality (3.3). This shows that inequality (3.2) is true.

For \( i = -p \), inequality (3.5) obviously is identic. This means that inequality (3.1) (or (3.2)) is identic. \( \square \)

**Proof of Theorem 1.1.** For \( i > -p \), we first prove the right inequality of (1.11). According to (1.8) and inequality (3.1), we have

\[
\frac{\tilde{W}_i(\tilde{\nabla}^\tau_pK)^{\frac{n+p}{n-i}}}{V(\tilde{\nabla}^\tau_pK)} = \frac{\tilde{W}_i(f_1(\tau) \cdot K)_{\tau} f_2(\tau) \cdot (-K))^{\frac{n+p}{n-i}}}{V(f_1(\tau) \cdot K)_{\tau} f_2(\tau) \cdot (-K))}
\]

\[
\geq f_1(\tau) \frac{\tilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)} + f_2(\tau) \frac{\tilde{W}_i(-K)^{\frac{n+p}{n-i}}}{V(-K)}. \tag{3.6}
\]

From (1.9), we easily see

\[
f_1(\tau) + f_2(\tau) = 1. \tag{3.7}
\]

Notice that for \( K \in \mathcal{S}_o^n \) and any real \( i \), \( \tilde{W}_i(-K) = \tilde{W}_i(K) \) and \( V(-K) = V(K) \). This together with (3.6) and (3.7), we obtain

\[
\frac{\tilde{W}_i(\tilde{\nabla}^\tau_pK)^{\frac{n+p}{n-i}}}{V(\tilde{\nabla}^\tau_pK)} \geq \frac{\tilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)}. \tag{3.8}
\]

This is just the right inequality of (1.11).

Clearly, if \( \tau = \pm 1 \), then equality holds in (3.8). Besides, if \( \tau \neq \pm 1 \), then by the condition of equality in (3.1), we see that equality holds in (3.8) if and only if \( K \) and \(-K\) are dilates, this yields \( K = -K \), i.e., \( K \) is an origin-symmetric star body. This means that if \( \tau \neq \pm 1 \), then equality holds in the right inequality of (1.11) if and only if \( K \) is an origin-symmetric star body.

Next, we give the proof of left inequality of (1.11). Using (2.9), (2.8) and inequal-
ity (3.1), we may get
\[
\frac{\tilde{W}_i(\tilde{\nabla}_p K)_{n+p}}{V(\tilde{\nabla}_p K)} = \frac{\tilde{W}_i(\frac{1}{2} \cdot \tilde{\nabla}_p K + \frac{1}{2} \cdot \tilde{\nabla}_p K)_{n+p}}{V(\frac{1}{2} \cdot \tilde{\nabla}_p K + \frac{1}{2} \cdot \tilde{\nabla}_p K)} \\
\geq \frac{1}{2} \frac{\tilde{W}_i(\tilde{\nabla}_p K)_{n+p}}{V(\tilde{\nabla}_p K)} + \frac{1}{2} \frac{\tilde{W}_i(\tilde{\nabla}_p K)_{n+p}}{V(\tilde{\nabla}_p K)} \\
= \frac{\tilde{W}_i(\tilde{\nabla}_p K)_{n+p}}{2 V(\tilde{\nabla}_p K)} + \frac{1}{2} \frac{\tilde{W}_i(\tilde{\nabla}_p K)_{n+p}}{V(\tilde{\nabla}_p K)}.
\]

i.e.,
\[
\frac{\tilde{W}_i(\tilde{\nabla}_p K)_{n+p}}{V(\tilde{\nabla}_p K)} \geq \frac{\tilde{W}_i(\tilde{\nabla}_p K)_{n+p}}{V(\tilde{\nabla}_p K)}. \tag{3.9}
\]

This yields the left inequality of (1.11).

Obviously, if $\tau = 0$, then equality holds in (3.9). Hence, according to the equality condition of (3.1), we know that if $\tau \neq 0$, then equality holds in (3.9) if and only if $\tilde{\nabla}_p K$ and $\tilde{\nabla}_p K$ are dilates, i.e., $\tilde{\nabla}_p K$ and $-\tilde{\nabla}_p K$ are dilates. This yields $\tilde{\nabla}_p K = -\tilde{\nabla}_p K$, thus $\tilde{\nabla}_p K = \tilde{\nabla}_p K$. Therefore, using Theorem 2.C, we see that if $\tau \neq 0$, then equality holds in (3.9) if and only if $K$ is an origin-symmetric star body. This gives the equality condition in the left inequality of (1.11).

For $i < -p$, similar to above the proof of case $i > -p$ and combine with inequality (3.2), we may prove inequality (1.11) is reversed.

For $i = -p$, inequality (1.11) is identical by Theorem 3.1. \qed

**Proof of Theorem 1.2.** From (1.8), (2.7), (2.6) and (3.7), we have that for $n > p \geq 1$,
\[
\frac{\tilde{\Omega}_{-p}(\tilde{\nabla}_p K)_{n+p}}{V(\tilde{\nabla}_p K)} = \frac{\tilde{\Omega}_{-p}(f_1(\tau) \cdot K + f_2(\tau) \cdot (-K))_{n+p}}{V(f_1(\tau) \cdot K + f_2(\tau) \cdot (-K))} \\
\geq f_1(\tau) \frac{\tilde{\Omega}_{-p}(K)_{n+p}}{V(K)} + f_2(\tau) \frac{\tilde{\Omega}_{-p}(-K)_{n+p}}{V(-K)} \\
= f_1(\tau) \frac{\tilde{\Omega}_{-p}(K)_{n+p}}{V(K)} + f_2(\tau) \frac{\tilde{\Omega}_{-p}(K)_{n+p}}{V(K)} \\
= \frac{\tilde{\Omega}_{-p}(K)_{n+p}}{V(K)}.
\]

Thus
\[
\left( \frac{\tilde{\Omega}_{-p}(\tilde{\nabla}_p K)_{n+p}}{\tilde{\Omega}_{-p}(K)_{n+p}} \right) \geq \frac{V(\tilde{\nabla}_p K)}{V(K)},
\]
this combine with the right inequality of (1.10) and notice that $1 \leq p < n$, we easily get

$$\tilde{\Omega}_{-p}(\nabla_{p}^{T}K) \geq \tilde{\Omega}_{-p}(K).$$

Therefore, the right inequality of (1.13) is obtained.

According to the equality conditions in inequality (2.7) and the right inequality of (1.10), we see that if $\tau \neq \pm 1$, then equality holds in the right inequality of (1.13) if and only if $K$ is an origin-symmetric star body.

On the other hand, from inequality (2.7), equalities (2.9), (2.8) and (2.6), we obtain that for $n > p \geq 1$,

$$\frac{\tilde{\Omega}_{-p}(\nabla_{p}K)^{n-p}}{V(\nabla_{p}K)} = \frac{\tilde{\Omega}_{-p} \left( \left. \frac{1}{2} \lightning \nabla_{p}^{T}K \land \frac{1}{2} \hat{\nabla}_{p}^{T}K \right)^{n-p} \right. }{V \left( \frac{1}{2} \hat{\nabla}_{p}^{T}K \land \frac{1}{2} \hat{\nabla}_{p}^{T}K \right)} \geq \frac{1}{2} \frac{\tilde{\Omega}_{-p}(\hat{\nabla}_{p}^{T}K)^{n-p}}{V(\hat{\nabla}_{p}^{T}K)} + \frac{1}{2} \frac{\tilde{\Omega}_{-p}(\hat{\nabla}_{p}^{T}K)^{n-p}}{V(\hat{\nabla}_{p}^{T}K)}.$$

Hence,

$$\frac{\tilde{\Omega}_{-p}(\nabla_{p}K)^{n-p}}{\tilde{\Omega}_{-p}(\nabla_{p}K)^{n-p}} = \frac{\tilde{\Omega}_{-p}(\hat{\nabla}_{p}^{T}K)^{n-p}}{V(\hat{\nabla}_{p}^{T}K)}.$$

This together with the left inequality of (1.10) and notice that $1 \leq p < n$, we immediately get the left inequality of (1.13).

From the equality conditions in inequality (2.7) and the left inequality of (1.10), we see that if $\tau \neq 0$, then equality holds in the left inequality of (1.13) if and only if $K$ is an origin-symmetric star body. □

Now we give the proofs of monotonic inequalities for the $L_p$-harmonic Blaschke bodies.

**Proof of Theorem 1.3.** From (1.8) and (2.1), we have that for any $M \in \mathcal{S}_o^n$,

$$\frac{\tilde{V}_{-p}(\nabla_{p}^{T}K,M)}{V(\nabla_{p}^{T}K)} = f_1(\tau) \frac{\tilde{V}_{-p}(K,M)}{V(K)} + f_2(\tau) \frac{\tilde{V}_{-p}(-K,M)}{V(K)}. \quad (3.10)$$

Since $L \in \mathcal{S}_{os}^n$, i.e., $L = -L$, thus (2.1) yields

$$\tilde{V}_{-p}(L,M) = \tilde{V}_{-p}(-L,M), \quad (3.11)$$

This concludes the proof of Theorem 1.3.
for any $M \in \mathcal{S}_n$. 

Hence, if $\hat{\nabla}_p^\tau K \subseteq \hat{\nabla}_p^\tau L$, then by (2.1) we get for any $M \in \mathcal{S}_n$,

$$\tilde{V}_{-p}(\hat{\nabla}_p^\tau K, M) \leq \tilde{V}_{-p}(\hat{\nabla}_p^\tau L, M),$$

this together with (3.10), (3.11) and (3.7), then

$$\frac{V(\hat{\nabla}_p^\tau K)}{V(K)} [f_1(\tau)\tilde{V}_{-p}(K, M) + f_2(\tau)\tilde{V}_{-p}(-K, M)] \\
\leq \frac{V(\hat{\nabla}_p^\tau L)}{V(L)} [f_1(\tau)\tilde{V}_{-p}(L, M) + f_2(\tau)\tilde{V}_{-p}(-L, M)] \\
= \frac{V(\hat{\nabla}_p^\tau L)}{V(L)} \tilde{V}_{-p}(L, M).$$

(3.12)

Taking $M = L$ in (3.12), and using (2.2), inequality (2.3) and equality (3.7), we obtain

$$V(\hat{\nabla}_p^\tau L) \geq \frac{V(\hat{\nabla}_p^\tau K)}{V(K)} [f_1(\tau)V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}} + f_2(\tau)V(-K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}}] \\
= \frac{V(\hat{\nabla}_p^\tau K)}{V(K)} V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}},$$

this gives inequality (1.15).

According to the equality condition of (2.3), we see that equality holds in (1.15) if and only if $K$ and $L$, $-K$ and $L$ both are dilates. But $L \in \mathcal{S}_n^{\#}$, i.e. $L$ is an origin-symmetric star body, this means $K$ also is an origin-symmetric star body. Therefore, equality holds in (1.15) if and only if $K$ and $L$ are dilates, and $K$ is an origin-symmetric star body. □

**Proof of Theorem 1.4.** Since $K \subseteq L$, thus $-K \subseteq -L$. So from (2.1), we know that for any $M \in \mathcal{S}_n$,

$$\tilde{V}_{-p}(K, M) \leq \tilde{V}_{-p}(L, M), \quad \tilde{V}_{-p}(-K, M) \leq \tilde{V}_{-p}(-L, M),$$

(3.13)

and with equality if and only if $K = L$. This together with (3.10), we have for any $M \in \mathcal{S}_n$

$$\frac{\tilde{V}_{-p}(\hat{\nabla}_p^\tau K, M)}{V(\hat{\nabla}_p^\tau K)} = f_1(\tau)\frac{\tilde{V}_{-p}(K, M)}{V(K)} + f_2(\tau)\frac{\tilde{V}_{-p}(-K, M)}{V(K)} \\
\leq \frac{V(L)}{V(K)} \left[ f_1(\tau)\frac{\tilde{V}_{-p}(L, M)}{V(L)} + f_2(\tau)\frac{\tilde{V}_{-p}(-L, M)}{V(L)} \right] \\
= \frac{V(L)\tilde{V}_{-p}(\hat{\nabla}_p^\tau L, M)}{V(K)V(\hat{\nabla}_p^\tau L)}.$$  

(3.14)
Taking $M = \hat{V}_p^\tau L$ in (3.14), and using (2.2) and inequality (2.3), we obtain

$$\frac{V(L)}{V(K)} \geq \frac{\hat{V}_p^\tau (\hat{V}_p^\tau K, \hat{V}_p^\tau L)}{V(\hat{V}_p^\tau K)} \geq \frac{V(\hat{V}_p^\tau K)^{\frac{n-p}{n}} V(\hat{V}_p^\tau L)^{-\frac{p}{n}}}{V(\hat{V}_p^\tau K)},$$

(3.15)

with equality if and only if $\hat{V}_p^\tau K$ and $\hat{V}_p^\tau L$ are dilates. From this, inequality (1.16) is obtained.

Because $K = L$ implies that $\hat{V}_p^\tau K$ and $\hat{V}_p^\tau L$ are dilates, thus according to the equality conditions of (3.13) and (3.15), we see that equality holds in (1.16) if and only if $K = L$. □

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