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# SOME INEQUALITIES FOR GENERAL $L_p$–HARMONIC BLASCHKE BODIES

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*Abstract.* Feng and Wang gave the extremum value of volume for the general $L_p$-harmonic Blaschke bodies. In this paper, associated with the general $L_p$-harmonic Blaschke bodies, we obtain the extremum value of dual quermassintegrals and the $L_p$-dual affine surface area, respectively. Further, two monotonic inequalities for the general $L_p$-harmonic Blaschke bodies are given.

## 1. Introduction

If $K$ is a compact star-shaped (about the origin) in Euclidean space $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \longrightarrow [0,+\infty)$, is defined by (see [5, 10])

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

If $\rho_K$ is positive and continuous, $K$ will be called a star body (about the origin). Let $\mathscr{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. For the set of origin-symmetric star bodies, we write $\mathscr{S}_{os}^n$. Two star bodies $K$ and $L$ are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$, where $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$.

Lutwak ([8]) introduced the notion of harmonic Blaschke combination for star bodies. For $K, L \in \mathscr{S}_o^n$, and $\lambda, \mu \geqslant 0$ (not both zero), the harmonic Blaschke combination, $\lambda K \hat{+} \mu L \in \mathscr{S}_o^n$, of $K$ and $L$ is defined by

$$\frac{\rho(\lambda K \hat{+} \mu L,\cdot)^{n+1}}{V(\lambda K \hat{+} \mu L)} = \lambda \frac{\rho(K,\cdot)^{n+1}}{V(K)} + \mu \frac{\rho(L,\cdot)^{n+1}}{V(L)}. \tag{1.1}$$

From definition (1.1), Lutwak ([8]) proved the Brunn-Minkowski inequality for the harmonic Blaschke combination as follows:

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THEOREM 1.A. *If* $K,L\in\mathscr{S}_o^n$, $\lambda,\mu\geqslant 0$ *(not both zero), then*

$$V(\lambda K\hat{+}\mu L)^{\frac{1}{n}}\geqslant\lambda V(K)^{\frac{1}{n}}+\mu V(L)^{\frac{1}{n}},\tag{1.2}$$

*with equality if and only if* $K$ *and* $L$ *are dilates.*

Based on definition (1.1) of harmonic Blaschke combination, Feng and Wang in [2] introduced the notion of $L_p$-harmonic Blaschke combination: For $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), the $L_p$-harmonic Blaschke combination, $\lambda\cdot K\hat{+}_p\mu\cdot L\in\mathscr{S}_o^n$, of $K$ and $L$ is given by

$$\frac{\rho(\lambda\cdot K\hat{+}_p\mu\cdot L,\cdot)^{n+p}}{V(\lambda\cdot K\hat{+}_p\mu\cdot L)}=\lambda\frac{\rho(K,\cdot)^{n+p}}{V(K)}+\mu\frac{\rho(L,\cdot)^{n+p}}{V(L)},\tag{1.3}$$

where the operation "$\hat{+}_p$" is called $L_p$-harmonic Blaschke addition. From (1.3), we easily know the harmonic Blaschke scalar multiplication and the usual scalar multiplication are related by $\lambda\cdot K=\lambda^{\frac{1}{p}}K$.

Let $\lambda=\mu=\frac{1}{2}$ and $L=-K$ in (1.3), the $L_p$-harmonic Blaschke body, $\widehat{\nabla}_pK$, of $K\in\mathscr{S}_o^n$ is given by (see [2])

$$\widehat{\nabla}_pK=\frac{1}{2}\cdot K\hat{+}_p\frac{1}{2}\cdot(-K).\tag{1.4}$$

From (1.3), Feng and Wang in [2] gave the following an extension of inequality (1.2).

THEOREM 1.B. *If* $K,L\in\mathscr{S}_o^n$, $p\geqslant 1$, $\lambda,\mu\geqslant 0$ *(not both zero), then*

$$V(\lambda\cdot K\hat{+}_p\mu\cdot L)^{\frac{p}{n}}\geqslant\lambda V(K)^{\frac{p}{n}}+\mu V(L)^{\frac{p}{n}},\tag{1.5}$$

*with equality if and only if* $K$ *and* $L$ *are dilates.*

Obviously, from (1.4) and (1.5), we have that if $K\in\mathscr{S}_o^n$, $p\geqslant 1$, then (see [2])

$$V(\widehat{\nabla}_pK)\geqslant V(K),\tag{1.6}$$

with equality if and only if $K$ is origin-symmetric.

Recently, Feng and Wang in [3] extended the notion of the $L_p$-harmonic Blaschke bodies and defined the general $L_p$-harmonic Blaschke bodies as follows: For $K\in\mathscr{S}_o^n$, $p\geqslant 1$ and $\tau\in[-1,1]$, the general $L_p$-harmonic Blaschke body, $\widehat{\nabla}_p^{\tau}K$, of $K$ is defined by

$$\frac{\rho(\widehat{\nabla}_p^{\tau}K,\cdot)^{n+p}}{V(\widehat{\nabla}_p^{\tau}K)}=f_1(\tau)\frac{\rho(K,\cdot)^{n+p}}{V(K)}+f_2(\tau)\frac{\rho(-K,\cdot)^{n+p}}{V(-K)}.\tag{1.7}$$

Associated with the definition of the $L_p$-harmonic Blaschke combination, it easily follows that

$$\widehat{\nabla}_p^{\tau}K=f_1(\tau)\cdot K\hat{+}_pf_2(\tau)\cdot(-K),\tag{1.8}$$

where

$$f_1(\tau)=\frac{(1+\tau)^p}{(1+\tau)^p+(1-\tau)^p},\qquad f_2(\tau)=\frac{(1-\tau)^p}{(1+\tau)^p+(1-\tau)^p}. \tag{1.9}$$

From (1.8) and (1.9), we easily get that if $\tau=0$ then $\widehat{\nabla}_p^0K=\widehat{\nabla}_pK$; if $\tau=\pm1$, then $\widehat{\nabla}_p^{+1}K=K$, $\widehat{\nabla}_p^{-1}K=-K$.

Further, Feng and Wang in [3] got the following extremum value of volume for the general $L_p$-harmonic Blaschke bodies.

THEOREM 1.C. *If $K\in\mathscr{S}_o^n$, $p\geqslant1$, $\tau\in[-1,1]$, then*

$$V(\widehat{\nabla}_pK)\geqslant V(\widehat{\nabla}_p^\tau K)\geqslant V(K). \tag{1.10}$$

*If $\tau\neq\pm1$, then equality holds in the right inequality of (1.10) if and only if $K$ is an origin-symmetric star body, and if $\tau\neq0$, then equality holds in the left inequality of (1.10) if and only if $K$ is also an origin-symmetric star body.*

In this article, from definition (1.7), we first give the extremum value of dual quermassintegrals for the general $L_p$-harmonic Blaschke bodies as follows.

THEOREM 1.1. *If $K\in\mathscr{S}_o^n$, $p\geqslant1$, $\tau\in[-1,1]$, $i$ is any real and $i\neq n$, then for $i>-p$,*

$$\frac{\widetilde{W}_i(\widehat{\nabla}_pK)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_pK)}\geqslant\frac{\widetilde{W}_i(\widehat{\nabla}_p^\tau K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p^\tau K)}\geqslant\frac{\widetilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)}. \tag{1.11}$$

*If $\tau\neq\pm1$, equality holds in the right inequality of (1.11) if and only if $K$ is an origin-symmetric star body, and if $\tau\neq0$, with equality in the left inequality of (1.11) if and only if $K$ is also an origin-symmetric star body. For $i<-p$, inequality (1.11) is reversed. For $i=-p$, (1.11) is identic.*

Here $\widetilde{W}_i(K)$ denotes the dual quermassintegrals of $K\in\mathscr{S}_o^n$ which be defined by (see [7])

$$\widetilde{W}_i(K)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i}du, \tag{1.12}$$

for any real $i$.

Obviously, let $i=0$ in inequality (1.11) and notice that $\widetilde{W}_0(K)=V(K)$, we immediately obtain Theorem 1.C.

Secondly, associated with the $L_p$-dual affine surface area (see (2.5)), we obtain its extremum value for the general $L_p$-harmonic Blaschke bodies.

THEOREM 1.2. *If $K\in\mathscr{S}_o^n$, $1\leqslant p<n$, $\tau\in[-1,1]$, then*

$$\widetilde{\Omega}_{-p}(\widehat{\nabla}_pK)\geqslant\widetilde{\Omega}_{-p}(\widehat{\nabla}_p^\tau K)\geqslant\widetilde{\Omega}_{-p}(K). \tag{1.13}$$

*If $\tau\neq\pm1$, equality holds in the right inequality of (1.13) if and only if $K$ is an origin-symmetric star body, and if $\tau\neq0$, equality holds in the left inequality of (1.13) if and only if $K$ is also an origin-symmetric star body.*

Recall that Feng and Wang in [2] proved that

*If $K \in \mathscr{S}_o^n$, $1 \leqslant p < n$, then*

$$\widetilde{\Omega}_{-p}(\widehat{\nabla}_p K) \geqslant \widetilde{\Omega}_{-p}(K), \tag{1.14}$$

*with equality if and only if $K$ is an origin-symmetric star body.*

Obviously, inequality (1.13) is an isolation of inequality (1.14).

In addition, we also give two monotonic inequalities for the general $L_p$-harmonic Blaschke bodies as follows:

THEOREM 1.3. *Let $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1,1]$. If $\widehat{\nabla}_p^\tau K \subseteq \widehat{\nabla}_p^\tau L$ and $L \in \mathscr{S}_{os}^n$, then*

$$V(\widehat{\nabla}_p^\tau K)V(K)^{\frac{p}{n}} \leqslant V(\widehat{\nabla}_p^\tau L)V(L)^{\frac{p}{n}}, \tag{1.15}$$

*with equality if and only if $K$ and $L$ are dilatant origin-symmetric star bodies.*

THEOREM 1.4. *Let $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\tau \in [-1,1]$. If $K \subseteq L$, then*

$$V(\widehat{\nabla}_p^\tau K)^{\frac{p}{n}}V(K) \leqslant V(\widehat{\nabla}_p^\tau L)^{\frac{p}{n}}V(L), \tag{1.16}$$

*with equality if and only if $K = L$.*

Actually, Theorem 1.3 may be regard as the Shephard problem of the general $L_p$-harmonic Blaschke bodies.

## 2. Preliminaries

### 2.1. $L_p$-dual mixed volume

For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$, $\lambda,\mu \geqslant 0$ (not both zero), the $L_p$-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L$, of $K$ and $L$ is defined by (see [4, 9])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K,\cdot)^{-p} + \mu \rho(L,\cdot)^{-p}.$$

The notion of $L_p$-dual mixed volume was introduced by Lutwak (see [9]). For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$ and $\varepsilon > 0$, the $L_p$-dual mixed volume, $\widetilde{V}_{-p}(K,L)$, of the $K$ and $L$ is defined by

$$\frac{n}{-p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

Further, Lutwak ([9]) proved that the $L_p$-dual mixed volume has the following integral representation:

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(L,u)^{-p}du. \tag{2.1}$$

From (2.1), we easily know

$$\widetilde{V}_{-p}(K,K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n}du = V(K). \tag{2.2}$$

The $L_p$-dual Minkowski inequality may be stated that (see [9]): For $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$, then

$$\widetilde{V}_{-p}(K,L) \geqslant V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}}, \tag{2.3}$$

with equality if and only if $K$ and $L$ are dilates.

### 2.2. $L_p$-dual affine surface

In 2008, Wang and He ([11]) gave the notion of $L_p$-dual affine surface area associated with the $L_p$-dual mixed volume. For $K \in S_o^n$ and $1 \leqslant p < n$, the $L_p$-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of $K$ is defined by

$$n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}} : Q \in \mathscr{K}_c^n\}. \tag{2.4}$$

For the $L_p$-dual affine surface area, except [11], Feng and Wang recently established some inequalities for $L_p$-dual affine surface area (see [1, 2]) . In particular, Feng and Wang in [2] improved definition (2.4) as follows: For $K \in S_o^n$ and $1 \leqslant p < n$, the $L_p$-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of $K$ is defined by

$$n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}} : Q \in \mathscr{S}_{os}^n\}. \tag{2.5}$$

From (2.5), we easily get that for $K \in S_o^n$ and $1 \leqslant p < n$,

$$\widetilde{\Omega}_{-p}(-K) = \widetilde{\Omega}_{-p}(K). \tag{2.6}$$

Associated with definition (2.5) and $L_p$-harmonic Blaschke addition (1.3), Feng and Wang ([2]) gave the following result:

THEOREM 2.A. *If $K, L \in \mathscr{S}_o^n$, $\lambda, \mu \geqslant 0$ (not both zero) and $1 \leqslant p < n$, then*

$$\frac{\widetilde{\Omega}_{-p}(\lambda \cdot K \hat{+}_p \mu \cdot L)^{\frac{n-p}{n}}}{V(\lambda \cdot K \hat{+}_p \mu \cdot L)} \geqslant \lambda \frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} + \mu \frac{\widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}}}{V(L)}, \tag{2.7}$$

*with equality if and only if $K$ and $L$ are dilates.*

### 2.3. Properties of the general $L_p$-harmonic Blaschke bodies

For the general $L_p$-harmonic Blaschke bodies. Feng and Wang in [3] proved the following properties.

THEOREM 2.B. *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$, $\tau \in [-1,1]$, then*

$$\widehat{\nabla}_p^{-\tau}K = \widehat{\nabla}_p^{\tau}(-K) = -\widehat{\nabla}_p^{\tau}K. \tag{2.8}$$

THEOREM 2.C. *Let $K \in \mathscr{S}_o^n$, $p \geqslant 1$, $\tau \in [-1,1]$. If $K$ is not an origin-symmetric star body, then $\widehat{\nabla}_p^{\tau}K = \widehat{\nabla}_p^{-\tau}K$ if and only if $\tau = 0$.*

THEOREM 2.D. *If $K \in \mathscr{S}_o^n$, $p \geqslant 1$, then for $\tau \in [-1,1]$,*

$$\widehat{\nabla}_p K = \frac{1}{2} \cdot \widehat{\nabla}_p^{\tau}K \hat{+}_p \frac{1}{2} \cdot \widehat{\nabla}_p^{-\tau}K. \tag{2.9}$$

## 3. Proofs of Theorems

In this section, we will complete the proofs of Theorems 1.1–1.4. In order to prove Theorem 1.1, we first give the following Brunn-Minkowski inequality for dual quermassintegrals of the $L_p$-harmonic Blaschke combinations.

THEOREM 3.1. *If $K,L \in \mathscr{S}_o^n$, $p \geqslant 1$, $\lambda,\mu \geqslant 0$ (not both zero), $i$ is any real and $i \neq n$, then for $i > -p$,*

$$\frac{\widetilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L)^{\frac{n+p}{n-i}}}{V(\lambda \cdot K \hat{+}_p \mu \cdot L)} \geqslant \lambda \frac{\widetilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)} + \mu \frac{\widetilde{W}_i(L)^{\frac{n+p}{n-i}}}{V(L)}; \tag{3.1}$$

*for $i < -p$,*

$$\frac{\widetilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L)^{\frac{n+p}{n-i}}}{V(\lambda \cdot K \hat{+}_p \mu \cdot L)} \leqslant \lambda \frac{\widetilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)} + \mu \frac{\widetilde{W}_i(L)^{\frac{n+p}{n-i}}}{V(L)}. \tag{3.2}$$

*In each case, equality holds if and only if $K$ and $L$ are dilates. For $i = -p$, (3.1) (or (3.2)) is identic.*

The proof of Theorem 3.1 requires the following Minkowski integral inequality (see [6]).

LEMMA 3.1. *Let $f$ and $g$ are nonnegative bounded Borel functions on measure space $X$. If $k > 1$, then*

$$\left(\int_X (f(x)+g(x))^k dx\right)^{1/k} \leqslant \left(\int_X f^k(x)dx\right)^{1/k} + \left(\int_X g^k(x)dx\right)^{1/k}; \tag{3.3}$$

*if $0 < k < 1$ or $k < 0$, then*

$$\left(\int_X (f(x)+g(x))^k dx\right)^{1/k} \geqslant \left(\int_X f^k(x)dx\right)^{1/k} + \left(\int_X g^k(x)dx\right)^{1/k}. \tag{3.4}$$

*Equality holds in every inequality if and only if $f(x)$ and $g(x)$ are effectively proportional or $f(x)g(x) = 0$ on $X$.*

*Proof of Theorem 3.1.* For $i > -p$, since $i \neq n$, thus $0 < (n-i)/(n+p) < 1$ when $-p < i < n$, or $(n-i)/(n+p) < 0$ when $i > n$. This together with (1.3), (1.12) and Minkowski integral inequality (3.4), it follows that

$$\begin{aligned}
\frac{\widetilde{W}_i(\lambda \cdot K \hat{+}_p \mu \cdot L)^{\frac{n+p}{n-i}}}{V(\lambda \cdot K \hat{+}_p \mu \cdot L)} &= \frac{1}{V(\lambda \cdot K \hat{+}_p \mu \cdot L)} \left[\frac{1}{n}\int_{S^{n-1}} \rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n-i} du\right]^{\frac{n+p}{n-i}} \\
&= \left\{\frac{1}{n}\int_{S^{n-1}} \left[\frac{\rho(\lambda \cdot K \hat{+}_p \mu \cdot L, u)^{n+p}}{V(\lambda \cdot K \hat{+}_p \mu \cdot L)}\right]^{\frac{n-i}{n+p}} du\right\}^{\frac{n+p}{n-i}} \\
&= \left\{\frac{1}{n}\int_{S^{n-1}} \left[\lambda \frac{\rho_K^{n+p}(u)}{V(K)} + \mu \frac{\rho_L^{n+p}(u)}{V(L)}\right]^{\frac{n-i}{n+p}} du\right\}^{\frac{n+p}{n-i}}
\end{aligned}$$

$$\geqslant \frac{\lambda}{V(K)}\left[\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)du\right]^{\frac{n+p}{n-i}}+\frac{\mu}{V(L)}\left[\frac{1}{n}\int_{S^{n-1}}\rho_L^{n-i}(u)du\right]^{\frac{n+p}{n-i}}$$
$$=\lambda\frac{\widetilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)}+\mu\frac{\widetilde{W}_i(L)^{\frac{n+p}{n-i}}}{V(L)}. \tag{3.5}$$

From this, we get inequality (3.1). According to the condition of equality holds in Minkowski integral inequality (3.4), we know that with equality in (3.5) if and only if $K$ and $L$ are dilates, this means that equality holds in (3.1) if and only if $K$ and $L$ are dilates.

For $i<-p$, because of $(n-i)/(n+p)>1$, thus inequality (3.5) is reversed by the Minkowski integral inequality (3.3). This shows that inequality (3.2) is true.

For $i=-p$, inequality (3.5) obviously is identic. This means that inequality (3.1) (or (3.2)) is identic. □

*Proof of Theorem 1.1.* For $i>-p$, we first prove the right inequality of (1.11). According to (1.8) and inequality (3.1), we have

$$\frac{\widetilde{W}_i(\widehat{\nabla}_p^{\tau}K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p^{\tau}K)}=\frac{\widetilde{W}_i(f_1(\tau)\cdot K\hat{+}_p f_2(\tau)\cdot(-K))^{\frac{n+p}{n-i}}}{V(f_1(\tau)\cdot K\hat{+}_p f_2(\tau)\cdot(-K))}$$
$$\geqslant f_1(\tau)\frac{\widetilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)}+f_2(\tau)\frac{\widetilde{W}_i(-K)^{\frac{n+p}{n-i}}}{V(-K)}. \tag{3.6}$$

From (1.9), we easily see

$$f_1(\tau)+f_2(\tau)=1. \tag{3.7}$$

Notice that for $K\in\mathscr{S}_o^n$ and any real $i$, $\widetilde{W}_i(-K)=\widetilde{W}_i(K)$ and $V(-K)=V(K)$. This together with (3.6) and (3.7), we obtain

$$\frac{\widetilde{W}_i(\widehat{\nabla}_p^{\tau}K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p^{\tau}K)}\geqslant\frac{\widetilde{W}_i(K)^{\frac{n+p}{n-i}}}{V(K)}. \tag{3.8}$$

This is just the right inequality of (1.11).

Clearly, if $\tau=\pm1$, then equality holds in (3.8). Besides, if $\tau\neq\pm1$, then by the condition of equality in (3.1), we see that equality holds in (3.8) if and only if $K$ and $-K$ are dilates, this yields $K=-K$, i.e., $K$ is an origin-symmetric star body. This means that if $\tau\neq\pm1$, then equality holds in the right inequality of (1.11) if and only if $K$ is an origin-symmetric star body.

Next, we give the proof of left inequality of (1.11). Using (2.9), (2.8) and inequality (3.1), we may get

$$
\begin{aligned}
\frac{\widetilde{W}_i(\widehat{\nabla}_p K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p K)} &= \frac{\widetilde{W}_i(\frac{1}{2}\cdot\widehat{\nabla}_p^{\tau}K \hat{+}_p \frac{1}{2}\cdot\widehat{\nabla}_p^{-\tau}K)^{\frac{n+p}{n-i}}}{V(\frac{1}{2}\cdot\widehat{\nabla}_p^{\tau}K \hat{+}_p \frac{1}{2}\cdot\widehat{\nabla}_p^{-\tau}K)} \\
&\geqslant \frac{1}{2}\frac{\widetilde{W}_i(\widehat{\nabla}_p^{\tau}K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p^{\tau}K)} + \frac{1}{2}\frac{\widetilde{W}_i(\widehat{\nabla}_p^{-\tau}K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p^{-\tau}K)} \\
&= \frac{1}{2}\frac{\widetilde{W}_i(\widehat{\nabla}_p^{\tau}K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p^{\tau}K)} + \frac{1}{2}\frac{\widetilde{W}_i(-\widehat{\nabla}_p^{\tau}K)^{\frac{n+p}{n-i}}}{V(-\widehat{\nabla}_p^{\tau}K)},
\end{aligned}
$$

i.e.,

$$
\frac{\widetilde{W}_i(\widehat{\nabla}_p K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p K)} \geqslant \frac{\widetilde{W}_i(\widehat{\nabla}_p^{\tau}K)^{\frac{n+p}{n-i}}}{V(\widehat{\nabla}_p^{\tau}K)}. \tag{3.9}
$$

This yields the left inequality of (1.11).

Obviously, if $\tau = 0$, then equality holds in (3.9). Hence, according to the equality condition of (3.1), we know that if $\tau \neq 0$, then equality holds in (3.9) if and only if $\widehat{\nabla}_p^{\tau}K$ and $\widehat{\nabla}_p^{-\tau}K$ are dilates, i.e., $\widehat{\nabla}_p^{\tau}K$ and $-\widehat{\nabla}_p^{\tau}K$ are dilates. This yields $\widehat{\nabla}_p^{\tau}K = -\widehat{\nabla}_p^{\tau}K$, thus $\widehat{\nabla}_p^{\tau}K = \widehat{\nabla}_p^{-\tau}K$. Therefore, using Theorem 2.C, we see that if $\tau \neq 0$, then equality holds in (3.9) if and only if $K$ is an origin-symmetric star body. This gives the equality condition in the left inequality of (1.11).

For $i < -p$, similar to above the proof of case $i > -p$ and combine with inequality (3.2), we may prove inequality (1.11) is reversed.

For $i = -p$, inequality (1.11) is identic by Theorem 3.1. □

*Proof of Theorem 1.2.* From (1.8), (2.7), (2.6) and (3.7), we have that for $n > p \geqslant 1$,

$$
\begin{aligned}
\frac{\widetilde{\Omega}_{-p}(\nabla_p^{\tau}K)^{\frac{n-p}{n}}}{V(\nabla_p^{\tau}K)} &= \frac{\widetilde{\Omega}_{-p}(f_1(\tau)\cdot K \hat{+}_p f_2(\tau)\cdot(-K))^{\frac{n-p}{n}}}{V(f_1(\tau)\cdot K \hat{+}_p f_2(\tau)\cdot(-K))} \\
&\geqslant f_1(\tau)\frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} + f_2(\tau)\frac{\widetilde{\Omega}_{-p}(-K)^{\frac{n-p}{n}}}{V(-K)} \\
&= f_1(\tau)\frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} + f_2(\tau)\frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)} \\
&= \frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)}.
\end{aligned}
$$

Thus

$$
\left(\frac{\widetilde{\Omega}_{-p}(\nabla_p^{\tau}K)}{\widetilde{\Omega}_{-p}(K)}\right)^{\frac{n-p}{n}} \geqslant \frac{V(\nabla_p^{\tau}K)}{V(K)},
$$

this combine with the right inequality of (1.10) and notice that $1 \leqslant p < n$, we easily get

$$\widetilde{\Omega}_{-p}(\nabla_p^{\tau} K) \geqslant \widetilde{\Omega}_{-p}(K).$$

Therefore, the right inequality of (1.13) is obtained.

According to the equality conditions in inequality (2.7) and the right inequality of (1.10), we see that if $\tau \neq \pm 1$, then equality holds in the right inequality of (1.13) if and only if $K$ is an origin-symmetric star body.

On the other hand, from inequality (2.7), equalities (2.9), (2.8) and (2.6), we obtain that for $n > p \geqslant 1$,

$$\begin{aligned}
\frac{\widetilde{\Omega}_{-p}(\nabla_p K)^{\frac{n-p}{n}}}{V(\nabla_p K)} &= \frac{\widetilde{\Omega}_{-p}\left(\frac{1}{2} \cdot \widehat{\nabla}_p^{\tau} K \hat{+}_p \frac{1}{2} \cdot \widehat{\nabla}_p^{-\tau} K\right)^{\frac{n-p}{n}}}{V\left(\frac{1}{2} \cdot \widehat{\nabla}_p^{\tau} K \hat{+}_p \frac{1}{2} \cdot \widehat{\nabla}_p^{-\tau} K\right)} \\
&\geqslant \frac{1}{2} \frac{\widetilde{\Omega}_{-p}(\widehat{\nabla}_p^{\tau} K)^{\frac{n-p}{n}}}{V(\widehat{\nabla}_p^{\tau} K)} + \frac{1}{2} \frac{\widetilde{\Omega}_{-p}(\widehat{\nabla}_p^{-\tau} K)^{\frac{n-p}{n}}}{V(\widehat{\nabla}_p^{-\tau} K)} \\
&= \frac{1}{2} \frac{\widetilde{\Omega}_{-p}(\widehat{\nabla}_p^{\tau} K)^{\frac{n-p}{n}}}{V(\widehat{\nabla}_p^{\tau} K)} + \frac{1}{2} \frac{\widetilde{\Omega}_{-p}(-\widehat{\nabla}_p^{\tau} K)^{\frac{n-p}{n}}}{V(-\widehat{\nabla}_p^{\tau} K)} \\
&= \frac{\widetilde{\Omega}_{-p}(\widehat{\nabla}_p^{\tau} K)^{\frac{n-p}{n}}}{V(\widehat{\nabla}_p^{\tau} K)}.
\end{aligned}$$

Hence,

$$\left(\frac{\widetilde{\Omega}_{-p}(\nabla_p K)}{\widetilde{\Omega}_{-p}(\widehat{\nabla}_p^{\tau} K)}\right)^{\frac{n-p}{n}} \geqslant \frac{V(\nabla_p K)}{V(\widehat{\nabla}_p^{\tau} K)}.$$

This together with the left inequality of (1.10) and notice that $1 \leqslant p < n$, we immediately get the left inequality of (1.13).

From the equality conditions in inequality (2.7) and the left inequality of (1.10), we see that if $\tau \neq 0$, then equality holds in the left inequality of (1.13) if and only if $K$ is an origin-symmetric star body. $\square$

Now we give the proofs of monotonic inequalities for the $L_p$-harmonic Blaschke bodies.

*Proof of Theorem 1.3.* From (1.8) and (2.1), we have that for any $M \in \mathscr{S}_o^n$,

$$\frac{\widetilde{V}_{-p}(\widehat{\nabla}_p^{\tau} K, M)}{V(\widehat{\nabla}_p^{\tau} K)} = f_1(\tau) \frac{\widetilde{V}_{-p}(K,M)}{V(K)} + f_2(\tau) \frac{\widetilde{V}_{-p}(-K,M)}{V(K)}. \tag{3.10}$$

Since $L \in \mathscr{S}_{os}^n$, i.e., $L = -L$, thus (2.1) yields

$$\widetilde{V}_{-p}(L,M) = \widetilde{V}_{-p}(-L,M), \tag{3.11}$$

for any $M \in \mathscr{S}_o^n$.

Hence, if $\widehat{\nabla}_p^\tau K \subseteq \widehat{\nabla}_p^\tau L$, then by (2.1) we get for any $M \in \mathscr{S}_o^n$,

$$\widetilde{V}_{-p}(\widehat{\nabla}_p^\tau K, M) \leqslant \widetilde{V}_{-p}(\widehat{\nabla}_p^\tau L, M),$$

this together with (3.10), (3.11) and (3.7), then

$$\begin{aligned}
&\frac{V(\widehat{\nabla}_p^\tau K)}{V(K)}[f_1(\tau)\widetilde{V}_{-p}(K,M)+f_2(\tau)\widetilde{V}_{-p}(-K,M)] \\
&\leqslant \frac{V(\widehat{\nabla}_p^\tau L)}{V(L)}[f_1(\tau)\widetilde{V}_{-p}(L,M)+f_2(\tau)\widetilde{V}_{-p}(-L,M)] \\
&= \frac{V(\widehat{\nabla}_p^\tau L)}{V(L)}\widetilde{V}_{-p}(L,M). \qquad (3.12)
\end{aligned}$$

Taking $M = L$ in (3.12), and using (2.2), inequality (2.3) and equality (3.7), we obtain

$$\begin{aligned}
V(\widehat{\nabla}_p^\tau L) &\geqslant \frac{V(\widehat{\nabla}_p^\tau K)}{V(K)}[f_1(\tau)\widetilde{V}_{-p}(K,L)+f_2(\tau)\widetilde{V}_{-p}(-K,L)] \\
&\geqslant \frac{V(\widehat{\nabla}_p^\tau K)}{V(K)}[f_1(\tau)V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}}+f_2(\tau)V(-K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}}] \\
&= \frac{V(\widehat{\nabla}_p^\tau K)}{V(K)}V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}},
\end{aligned}$$

this gives inequality (1.15).

According to the equality condition of (2.3), we see that equality holds in (1.15) if and only if $K$ and $L$, $-K$ and $L$ both are dilates. But $L \in \mathscr{S}_{os}^n$, i.e. $L$ is an origin-symmetric star body, this means $K$ also is an origin-symmetric star body. Therefore, equality holds in (1.15) if and only if $K$ and $L$ are dilates, and $K$ is an origin-symmetric star body. □

*Proof of Theorem 1.4.* Since $K \subseteq L$, thus $-K \subseteq -L$. So from (2.1), we know that for any $M \in \mathscr{S}_o^n$,

$$\widetilde{V}_{-p}(K,M) \leqslant \widetilde{V}_{-p}(L,M), \qquad \widetilde{V}_{-p}(-K,M) \leqslant \widetilde{V}_{-p}(-L,M), \qquad (3.13)$$

and with equality if and only if $K = L$. This together with (3.10), we have for any $M \in \mathscr{S}_o^n$,

$$\begin{aligned}
\frac{\widetilde{V}_{-p}(\widehat{\nabla}_p^\tau K, M)}{V(\widehat{\nabla}_p^\tau K)} &= f_1(\tau)\frac{\widetilde{V}_{-p}(K,M)}{V(K)} + f_2(\tau)\frac{\widetilde{V}_{-p}(-K,M)}{V(K)} \\
&\leqslant \frac{V(L)}{V(K)}\left[f_1(\tau)\frac{\widetilde{V}_{-p}(L,M)}{V(L)} + f_2(\tau)\frac{\widetilde{V}_{-p}(-L,M)}{V(L)}\right] \\
&= \frac{V(L)\widetilde{V}_{-p}(\widehat{\nabla}_p^\tau L, M)}{V(K)V(\widehat{\nabla}_p^\tau L)}. \qquad (3.14)
\end{aligned}$$

Taking $M = \widehat{\nabla}_p^\tau L$ in (3.14), and using (2.2) and inequality (2.3), we obtain

$$\frac{V(L)}{V(K)} \geqslant \frac{\widetilde{V}_{-p}(\widehat{\nabla}_p^\tau K, \widehat{\nabla}_p^\tau L)}{V(\widehat{\nabla}_p^\tau K)} \geqslant \frac{V(\widehat{\nabla}_p^\tau K)^{\frac{n+p}{n}} V(\widehat{\nabla}_p^\tau L)^{-\frac{p}{n}}}{V(\widehat{\nabla}_p^\tau K)}, \tag{3.15}$$

with equality if and only if $\widehat{\nabla}_p^\tau K$ and $\widehat{\nabla}_p^\tau L$ are dilates. From this, inequality (1.16) is obtained.

Because $K = L$ implies that $\widehat{\nabla}_p^\tau K$ and $\widehat{\nabla}_p^\tau L$ are dilates, thus according to the equality conditions of (3.13) and (3.15), we see that equality holds in (1.16) if and only if $K = L$. □

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