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ON AN APPROACH IN SERVICE OF MEAN–INEQUALITIES

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Abstract. In the present paper, an approach for constructing new means in two variables is investigated. Application of this approach for proving some mean-inequalities is also discussed.

1. Introduction

We understand by (bivariate) mean a binary map $m:(0,\infty)\times(0,\infty)\longrightarrow(0,\infty)$ satisfying the following statement.

$$\forall a,b>0 \quad \min(a,b)\leqslant m(a,b)\leqslant \max(a,b). \tag{1.1}$$

Two trivial means are $(a,b)\longmapsto \min(a,b)$ and $(a,b)\longmapsto \max(a,b)$ and will be denoted by $\min$ and $\max$, respectively. The standard examples of means are given in the following (see [1] for instance and the related references cited therein).

$$A:=A(a,b)=\frac{a+b}{2};\quad G:=G(a,b)=\sqrt{ab};\quad H:=H(a,b)=\frac{2ab}{a+b};$$

$$L:=L(a,b)=\frac{b-a}{\ln b-\ln a},\quad L(a,a)=a;$$

$$I:=I(a,b)=e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},\quad I(a,a)=a;$$

$$C:=C(a,b)=\frac{a^2+b^2}{a+b},\quad Q:=Q(a,b)=\sqrt{\frac{a^2+b^2}{2}};$$

$$P:=P(a,b)=\frac{b-a}{4\arctan\sqrt{b/a}-\pi}=\frac{b-a}{2\arcsin\frac{b-a}{b+a}},\quad P(a,a)=a;$$

$$T:=T(a,b)=\frac{b-a}{2\arctan\frac{b-a}{b+a}}=\frac{b-a}{2\arctan(b/a)-\pi/2},\quad T(a,a)=a;$$

$$M:=M(a,b)=\frac{b-a}{2\arcsin h\frac{b-a}{b+a}},\quad M(a,a)=a;$$

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and are known as the arithmetic mean, geometric mean, harmonic mean, logarithmic mean, identric mean, contra-harmonic mean, quadratic (or root-square) mean, first Seiffert mean [7], second Seiffert mean [8] and Neuman-Sándor mean [3], respectively.

A mean m is called symmetric if $m(a,b) = m(b,a)$ and m is homogeneous if $m(\alpha a, \alpha b) = \alpha m(a,b)$, for all $a,b,\alpha > 0$. Obviously, all the above means are homogeneous and symmetric.

For two means m_1 and m_2 we write $m_1 \leqslant m_2$ if and only if $m_1(a,b) \leqslant m_2(a,b)$ for every $a,b > 0$. We say that the mean-inequality $m_1 \leqslant m_2$ is strict, and we write $m_1 < m_2$, if and only if $m_1(a,b) < m_2(a,b)$ for all $a,b > 0$ with $a \neq b$. If $m_1 \neq m_2$, we say that m_1 and m_2 are comparable if $m_1 < m_2$ or $m_2 < m_1$. The above standard means are comparable to each. Precisely, the next chain of inequalities is well known in the literature, see [4] for instance:

$$\min < H < G < L < P < I < A < M < T < C < Q < \max. \tag{1.2}$$

For a given mean m, we set $m^*(a,b) = \Big(m\big(a^{-1}, b^{-1}\big)\Big)^{-1}$, and it is easy to see that m^* is also a mean, called the dual mean of m. If m is symmetric and homogeneous then so is m^* and in this case we have $m^*(a,b) = ab/m(a,b)$ which we can write $m^* = G^2/m$. Every mean m satisfies $m^{**} := (m^*)^* = m$ and, if m_1 and m_2 are two means such that $m_1 \leqslant m_2$ (resp. $m_1 < m_2$) then $m_1^* \geqslant m_2^*$ (resp. $m_1^* > m_2^*$). One can check that $\min^* = \max$ and $\max^* = \min$. Further, $A^* = H$, $H^* = A$ and $G^* = G$.

Let m be a homogeneous mean. Writing $m(a,b) = bm(a/b,1)$ we then associate to m a unique positive function ϕ, called the associate function to m, defined by $\phi(x) = m(x,1)$ for all $x > 0$. In this case, (1.1) is equivalent to $\min(x,1) \leqslant \phi(x) \leqslant \max(x,1)$ for every $x > 0$. For more details about these notions we refer the reader to [5].

We notice that the set of all means, denoted by $\mathscr{M}$, is power-convex, that is, for all $m_1, m_2 \in \mathscr{M}$ and every $\lambda \in [0,1]$ and $p \in \mathbb{R}$, $p \neq 0$, we have

$$\mathscr{B}_p^\lambda\big(m_1, m_2\big) := \Big((1-\lambda)m_1^p + \lambda m_2^p\Big)^{1/p} \in \mathscr{M}. \tag{1.3}$$

In particular, $\mathscr{M}$ is (linearly) convex and geometrically convex i.e. if $m_1, m_2 \in \mathscr{M}$ then

$$\mathscr{B}_1^\lambda(m_1, m_2) = (1-\lambda)m_1 + \lambda m_2 \in \mathscr{M}$$

and

$$\mathscr{B}_0^\lambda(m_1, m_2) := \lim_{p \to 0} \mathscr{B}_p^\lambda(m_1, m_2) = m_1^{1-\lambda} m_2^\lambda \in \mathscr{M}.$$

The reminder of this paper will be organized as follows: Section 2 is devoted to investigate an integral transform for means, denoted by $m \longmapsto m^\sigma$, and to apply it for computing m^σ when m belongs to the set of the above standard means. We obtain some good and simple relationships between such means, as $A^\sigma = H$, $G^\sigma = L$, $H^\sigma = T$. The mean $L^\sigma := R_1$ appears to be new and allows us to introduce more new means by analogy. In Section 3, we show under convenient assumption that the mean-map $m \longmapsto m^\sigma$ is one-to-one, with inverse map denoted by $m \longmapsto m^{-\sigma}$, whose transforms for some means among the above standard ones can be expressed only in terms of A and G. For example, we obtain

$$P^{-\sigma}=\left(\frac{2G^3}{A+G}\right)^{1/2},\quad M^{-\sigma}=\left(\frac{2G^4}{A^2+AG}\right)^{1/2},\quad T^{-\sigma}=H=G^2/A.$$

Section 4 displays more interesting properties for the mean-maps $m\longmapsto m^{\sigma}$ and $m\longmapsto m^{-\sigma}$ as well as some applications for obtaining/proving a lot of mean-inequalities via our present approach.

2. General approach

We start this section by stating the following result.

THEOREM 2.1. *Let $f:(0,\infty)\longrightarrow(0,\infty)$ be a continuous function such that*

$$\forall t>0\quad \min\left(1,\frac{1}{t^2}\right)\leqslant f(t)\leqslant \max\left(1,\frac{1}{t^2}\right). \tag{2.1}$$

Then the binary map $m_f:(0,\infty)\times(0,\infty)\longrightarrow(0,\infty)$ defined by $m_f(a,a)=a$ and

$$\Big(m_f(a,b)\Big)^{-1}=\frac{1}{b-a}\int_1^{b/a} f(t)dt \tag{2.2}$$

for all $a,b>0$ with $a\neq b$, is a continuous homogeneous mean. If further f satisfies

$$\forall t>0\quad f\big(1/t\big)=t^2 f(t) \tag{2.3}$$

then m_f is symmetric.

Proof. Assume that $a<b$. According to (2.1) we have

$$\int_1^{b/a}\frac{1}{t^2}dt=\int_1^{b/a}\min\left(1,\frac{1}{t^2}\right)dt\leqslant\int_1^{b/a} f(t)dt\leqslant\int_1^{b/a}\max\left(1,\frac{1}{t^2}\right)dt=\int_1^{b/a}dt.$$

This, with (2.2) and a simple computation, yields

$$\frac{1}{\max(a,b)}=\frac{1}{b}\leqslant\Big(m_f(a,b)\Big)^{-1}\leqslant\frac{1}{a}=\frac{1}{\min(a,b)}.$$

If $a>b$ we obtain in a similar manner

$$\frac{1}{\max(a,b)}=\frac{1}{a}\leqslant\Big(m_f(a,b)\Big)^{-1}\leqslant\frac{1}{b}=\frac{1}{\min(a,b)}.$$

From the above inequalities we deduce that m_f is a mean. The homogeneity of m_f is obvious from (2.2). For proving the continuity of m_f it is sufficient to show that

$x \longmapsto \left(m_f(x,1)\right)^{-1}$ is continuous for $x>0$. By (2.2) and the mean value theorem we can write, for all $x>0$ with $x \neq 1$,

$$\left(m_f(x,1)\right)^{-1} = \frac{1}{x-1}\int_1^x f(t)dt = f(c_x),$$

where c_x is between 1 and x. The continuity of $x \longmapsto m_f(1,x)$ on $(0,1)\cup(1,\infty)$ follows from the left side of the above equalities with the continuity of f. Now, by (2.1) one has $f(1)=1$, and again by the continuity of f we have

$$\lim_{x\to 1} f(c_x) = f\Big(\lim_{x\to 1} c_x\Big) = f(1) = 1,$$

since $1 \leqslant c_x \leqslant x$ or $x \leqslant c_x \leqslant 1$. The continuity of $x \longmapsto m_f(1,x)$ at $x=1$ follows, so completes the proof of continuity of m_f.

Now, assume that (2.3) is satisfied. Making the change of variable $t=1/u$ in (2.2) we obtain

$$\left(m_f(a,b)\right)^{-1} = \frac{1}{b-a}\int_1^{a/b} f\Big(\frac{1}{u}\Big)\Big(-\frac{du}{u^2}\Big),$$

which with (2.3) yields

$$\left(m_f(a,b)\right)^{-1} = \frac{1}{a-b}\int_1^{a/b} f(u)du = \left(m_f(b,a)\right)^{-1}.$$

The symmetry of m_f is proved, this completes the proof of the theorem. □

EXAMPLE 2.1. Let $f(t)=1/t$ for all $t>0$. It is easy to see that f satisfies (2.1) and (2.3) and a simple computation leads to $m_f(a,b)=L(a,b)$ the logarithmic mean.

The next result, deduced from the above theorem, is of interest in practical purposes.

COROLLARY 2.2. *Let m be a continuous homogeneous symmetric mean. Then the binary map m^σ defined by $m^\sigma(a,a)=a$ and*

$$\left(m^\sigma(a,b)\right)^{-1} = \frac{1}{b-a}\int_1^{b/a} m\Big(1,\frac{1}{t^2}\Big)dt \qquad (2.4)$$

for all $a,b>0$ with $a\neq b$, is also a continuous homogeneous symmetric mean, called the integral transform-mean of m.

Proof. Taking $f(t)=m\Big(1,\frac{1}{t^2}\Big)$, it is easy to see that f satisfies (2.1) and (2.3). The desired result follows by Theorem 2.1. □

The above corollary tells us that starting from a continuous homogenous symmetric mean m, which can be chosen among the above standard means, we obtain another mean m^σ whose expression can be more or less complicated than that of m. Let us observe a list of examples illustrating this latter point.

EXAMPLE 2.2. It is easy to see that $\min^\sigma = \max$ and $\max^\sigma = \min$. It is also simple to verify that $A^\sigma = H$. However, Q^σ is such that (for $a \neq b$)

$$\Big(Q^\sigma(a,b)\Big)^{-1} = \frac{1}{b-a}\int_1^{b/a} \frac{1}{t^2}\sqrt{\frac{t^4+1}{2}}dt,$$

which seems hard to be explicitly computed.

EXAMPLE 2.3. Let $m(a,b) = G(a,b)$ be the geometric mean. Thus $m(1,1/t^2) = 1/t$ and we are in the same situation as Example 2.1 i.e. $G^\sigma = L$ the logarithmic mean.

EXAMPLE 2.4. Let $m(a,b) = H(a,b)$ be the harmonic mean. Then $m(1,1/t^2) = \frac{2}{1+t^2}$ for all $t > 0$, and so we have for all $a,b > 0$ with $a \neq b$

$$\Big(H^\sigma(a,b)\Big)^{-1} = \frac{1}{b-a}\int_1^{b/a} \frac{2}{1+t^2}dt = \frac{2}{b-a}\Big(\arctan(b/a) - \pi/4\Big).$$

It follows that the mean H^σ is nothing other than the second Seiffert mean T i.e. $H^\sigma = T$.

EXAMPLE 2.5. Let $m(a,b) = L(a,b)$ be the logarithmic mean. Then

$$\forall t > 0,\ t \neq 1, \quad m(1,1/t^2) = \frac{t^2-1}{2t^2 \ln t}.$$

The mean L^σ is given by $L^\sigma(a,a) = a$ and

$$\forall a,b > 0,\ a \neq b, \quad \Big(L^\sigma(a,b)\Big)^{-1} = \frac{1}{2(b-a)}\int_1^{b/a} \frac{t^2-1}{t^2 \ln t}dt. \tag{2.5}$$

Explicit computation of $L^\sigma(a,b)$ in terms of elementary functions of a and b seems to be hard. Such mean L^σ appears to be new and can be expressed in terms of (non-elementary) special functions known in the literature. Precisely, the following result may be stated.

THEOREM 2.3. *The mean L^σ is given by*

$$L^\sigma(a,b) = \frac{b-a}{sinhi\Big(\ln(b/a)\Big)} \tag{2.6}$$

for all $a,b > 0$, $a \neq b$, with $L^\sigma(a,a) = a$, where the notation sinhi refers to the integral hyperbolic sine function defined for all real number x by

$$sinhi(x) = \int_0^x \frac{\sinh t}{t}dt.$$

Proof. Making the change of variable $u = \ln t$ in (2.5) we obtain after computation and reduction

$$\Big(L^{\sigma}(a,b)\Big)^{-1} = \frac{1}{b-a}\int_0^{\ln(b/a)} \frac{e^u - e^{-u}}{2u}du.$$

The desired result follows. □

Let us take a look at another example of interest presented below.

EXAMPLE 2.6. Let $m(a,b) = C(a,b)$ be the contra-harmonic mean. By (2.2), we have for $a,b > 0,\ a \neq b$

$$\Big(C^{\sigma}(a,b)\Big)^{-1} = \frac{1}{b-a}\int_1^{b/a} \frac{t^4+1}{t^2(t^2+1)}dt.$$

Since

$$\frac{t^4+1}{t^2(t^2+1)} = 1 + \frac{1}{t^2} - \frac{2}{t^2+1},$$

then a simple computation leads to

$$\Big(C^{\sigma}(a,b)\Big)^{-1} = 2\Big(H(a,b)\Big)^{-1} - \Big(T(a,b)\Big)^{-1}.$$

That is, C^{σ} is such that

$$\frac{2}{H} = \frac{1}{C^{\sigma}} + \frac{1}{T}. \tag{2.7}$$

REMARK 2.1. The following relationship

$$\forall a,b > 0 \quad G(a,b) = G\Big(A(a,b), H(a,b)\Big)$$

is well-known in the literature and is called the invariance property of G. We then say that G is (A,H)-invariant, see [2] for more detail. Relation (2.7) is equivalent to

$$\forall a,b > 0 \quad H(a,b) = H\Big(C^{\sigma}(a,b), T(a,b)\Big)$$

and tells us that H is (C^{σ}, T)-invariant. We omit the detail about this latter point which is out of our aim in this work.

Below, we need more notation. Let $f,g : (0,\infty) \longrightarrow (0,\infty)$ be two functions. We write $f \leqslant g$ for saying that $f(t) \leqslant g(t)$ for all $t > 0$ and $f \prec g$ for meaning that $f(t) < g(t)$ for all $t > 0$ with $t \neq 1$. If f and g both satisfy (2.3) then $f \prec g$ is satisfied if and only if $f(t) < g(t)$ for all $t \in (0,1)$. With this, the following result may be stated.

PROPOSITION 2.4. *With the above one has*

(i) Let $f,g : (0,\infty) \longrightarrow (0,\infty)$ be such that $f \prec g$. Then $m_f > m_g$.

(ii) The mean-map $m \longmapsto m^{\sigma}$ is point-wise strictly decreasing i.e. for all continuous homogeneous symmetric means m_1 and m_2 such that $m_1 < m_2$, we have $m_1^{\sigma} > m_2^{\sigma}$.

Proof. (i) and (ii) are immediate from (2.2) and (2.4), respectively. □

We end this section by stating the next result which ensures the point-wise injectivity of the mean-map $m \longmapsto m^\sigma$.

PROPOSITION 2.5. *Let m_1 and m_2 be continuous homogeneous symmetric means such that $m_1^\sigma = m_2^\sigma$. Then $m_1 = m_2$.*

Proof. Assume that $m_1^\sigma = m_2^\sigma$ then $m_1^\sigma(1,x) = m_2^\sigma(1,x)$ and by (2.4)

$$\int_1^x m_1\big(1,1/t^2\big)dt = \int_1^x m_2\big(1,1/t^2\big)dt,$$

for all $x > 0$. Since m_1 and m_2 are continuous we can take the derivatives, with respect to x, of the above sides for obtaining $m_1\big(1,1/x^2\big) = m_2\big(1,1/x^2\big)$ for each $x > 0$. By homogeneity of m_1 and m_2 (take $x^2 = b/a$) we deduce $m_1 = m_2$, so completes the proof. □

3. Inverse transform

In the above study, starting from a convenient function f we defined a mean m_f. Inversely, let m be a given continuous homogenous symmetric mean. Does exist a function f satisfying $m = m_f$? If yes, is it possible to explicit f_m when m is explicitly given? Before stating a positive answer for this latter question, we need the following definition for the sake of simplicity.

DEFINITION 3.1. A mean m is said to be:

(i) regular if m is continuous homogeneous and symmetric,

(ii) σ-regular if m is regular, the map $x \longmapsto m(x,1)$ is continuously differentiable on $(0,\infty)$ and the function f_m defined by

$$f_m(x) = \frac{d}{dx}\Big(\frac{x-1}{m(x,1)}\Big) \tag{3.1}$$

for all $x > 0$ with $f_m(1) = 1$, satisfies the double inequality (2.1).

In this case f_m is called the generated function of the mean m.

Now, we may state the following result answering the above question.

THEOREM 3.1. *Let m be a σ-regular mean with its generated function f_m defined through (3.1). Then f_m is continuous on $(0,\infty)$ and satisfies (2.3). Further, we have*

$$\Big(m(a,b)\Big)^{-1} = \frac{1}{b-a}\int_1^{b/a} f_m(t)dt. \tag{3.2}$$

Proof. The continuity of f_m on $(0,\infty)$ follows from (3.1) with the fact that $x \longmapsto m(x,1)$ is continuously differentiable on $(0,\infty)$. For fixed m, we set $g(x) = (x-1)\Big(m(x,1)\Big)^{-1}$ and we write $g'(x)$ instead of $\frac{dg}{dx}(x)$ for the sake of simplicity. Since m is homogeneous and symmetric then we have

$$g\Big(\frac{1}{x}\Big) = \Big(\frac{1}{x}-1\Big)\Big(m\Big(\frac{1}{x},1\Big)\Big)^{-1} = (1-x)\Big(m(1,x)\Big)^{-1} = -g(x).$$

We then have $g(1/x) = -g(x)$ which by derivation yields $\Big(g'(1/x)\Big)\big(\frac{-1}{x^2}\big) = -g'(x)$ or again, with (3.1), $f_m(1/x) = x^2 f_m(x)$, that is, f_m satisfies (2.3). Now, (3.1) is equivalent to

$$(x-1)\Big(m(x,1)\Big)^{-1} = \int_1^x f_m(t)dt$$

for all $x > 0$. Taking $x = b/a$ with the homogeneity and symmetry of m we deduce (3.2), so completes the proof of the theorem. □

A lot of examples explaining the above situation are presented below.

EXAMPLE 3.1. Let $m(a,b) = L(a,b)$ be the logarithmic mean. Applying (3.1) we immediately obtain

$$f_L(x) = \frac{d}{dx}\Big(\Big(x-1\Big)\Big(\frac{\ln x}{x-1}\Big)\Big) = \frac{1}{x},$$

which rejoins Example 2.3. We then can conclude that L is σ-regular.

We can also easily verify that H, G and A are σ-regular, with

$$f_H(x) = \frac{1}{2}\Big(1+\frac{1}{x^2}\Big), \quad f_G(x) = \frac{1}{2\sqrt{x}}\Big(1+\frac{1}{x}\Big), \quad f_A(x) = \frac{4}{(x+1)^2}.$$

EXAMPLE 3.2. Here we consider the means P, T and M. By (3.1) we have respectively (we omit the detail about computation).

(1) For the mean P:

$$f_P(x) = 2\frac{d}{dx}\Big(\arcsin\frac{x-1}{x+1}\Big) = \frac{2}{(x+1)\sqrt{x}}. \tag{3.3}$$

(2) For the mean T:

$$f_T(x) = 2\frac{d}{dx}\Big(\arctan\frac{x-1}{x+1}\Big) = \frac{2}{x^2+1}, \tag{3.4}$$

which rejoins Example 2.4.

(3) For the mean M:

$$f_M(x) = 2\frac{d}{dx}\Big(\operatorname{arcsinh}\frac{x-1}{x+1}\Big) = \frac{2\sqrt{2}}{(x+1)\sqrt{x^2+1}}. \tag{3.5}$$

It is not hard to verify that all the means P, T and M are σ-regular i.e. f_P, f_T and f_M satisfy (2.1). Now, for an example of no-σ-regular mean, let us observe the following.

EXAMPLE 3.3. Let $m(a,b) = C(a,b)$ be the contra-harmonic mean. The generated function f_C is

$$f_C(x) = \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{4x}{(x^2+1)^2}.$$

But f_C does not satisfy (2.1). In fact, assume that f_C satisfies (2.1) then it is necessary that

$$\frac{1}{x^2} \leqslant f_C(x) = \frac{4x}{(x^2+1)^2} \leqslant 1$$

for all $x \geqslant 1$. The left side of the above double inequality becomes $(x^2+1)^2 \leqslant 4x^3$ which is false for $x \geqslant 1$ enough large. It follows that C is not σ-regular.

The chain of inequalities (1.2), differently proved in the literature, can be here shown again in a simple way by using our present approach. For more clearness, we present some examples explaining this situation.

EXAMPLE 3.4. It is easy to see that $\frac{1}{t} > \frac{2}{(1+t)\sqrt{t}}$ for all $t > 0$ with $t \neq 1$, i.e $f_P \prec f_L$. According to Example 3.1 and Example 3.2,(1), with Proposition 2.4, (i), we infer that $L < P$.

EXAMPLE 3.5. Following Example 3.2 we have

$$\forall t > 0 \quad f_T(t) = \frac{2}{1+t^2}, \quad f_M(t) = \frac{2\sqrt{2}}{(1+t)\sqrt{1+t^2}}$$

It is very easy to verify that $f_T \prec f_M$ and by Proposition 2.4 we infer that $M < T$.

Now, a similar question as above can be put for Corollary 2.2: if m^σ is known, is m unique and how to find m such that (2.4) is satisfied? Below, an affirmative answer for this latter question will be discussed.

THEOREM 3.2. *Let m be a σ-regular mean with its generated function f_m. Then the binary map r_m defined by*

$$r_m(a,b) = bf_m\left(\sqrt{b/a}\right) \tag{3.6}$$

for all $a,b > 0$, is a regular mean with $r_m^\sigma = m$.

Proof. By Definition 3.1, f_m satisfies (2.1), that is, for all $t > 0$ one has

$$\min\left(1, \frac{1}{t^2}\right) \leqslant f_m(t) \leqslant \max\left(1, \frac{1}{t^2}\right).$$

Taking $t^2 = b/a$ we obtain

$$\min(a,b) = b\,\min\left(1,\frac{a}{b}\right) \leqslant b\,f_m\left(\sqrt{b/a}\right) \leqslant b\,\max\left(1,\frac{a}{b}\right) = \max(a,b).$$

Then r_m is a mean. By (3.6), r_m is obviously continuous and homogeneous. For proving the symmetry of r_m, we write that, by virtue of Theorem 3.1, the generated function f_m satisfies (2.3) and so

$$r_m(a,b) = bf_m\left(\sqrt{b/a}\right) = b\left(\sqrt{b/a}\right)^2 f_m\left(\sqrt{a/b}\right) = a\,f_m\left(\sqrt{a/b}\right) = r_m(b,a).$$

In summary, r_m is a regular mean. Now, by (2.4), (3.6) and the fact that f_m satisfies (2.3) again, we respectively obtain

$$\begin{aligned}\left(r_m^\sigma(a,b)\right)^{-1} &= \frac{1}{b-a}\int_1^{b/a} r_m\left(1,\frac{1}{t^2}\right)dt = \frac{1}{b-a}\int_1^{b/a}\frac{1}{t^2}f_m\left(\frac{1}{t}\right)dt\\ &= \frac{1}{b-a}\int_1^{b/a} f_m(t)dt,\end{aligned}$$

which, with (3.2), implies that $r_m^\sigma = m$, so completes the proof of the theorem. □

We pay attention to Theorem 3.2: before concluding, we should make sure that the function f_m satisfies (2.1), or equivalently to make sure that the binary map $(a,b)\longmapsto bf_m\left(\sqrt{b/a}\right)$ is really a mean. The following gives a counter-example for this latter situation.

EXAMPLE 3.6. Following Example 3.3, C is not σ-regular. That is, the associate binary map r_m defined through (3.6), and explicitly given by $r_m = G^3/A^2$, is not a mean.

By virtue of Theorem 3.2 with Proposition 2.5, for every convenient mean m there exists one and only one regular mean r_m such that $r_m^\sigma = m$. In another way, if we denote by $\mathscr{M}_r$ and $\mathscr{M}_\sigma$ the sets of all regular means and σ-regular means, respectively, then we have

$$\mathscr{M}_\sigma = \Big\{m\in\mathscr{M}_r,\ f_m \text{ satisfies } (2.1)\Big\} = \Big\{m\in\mathscr{M}_r,\ r_m \text{ is a regular mean}\Big\},$$

and the mean-map $m\longmapsto m^\sigma$ is a bijection from $\mathscr{M}_r$ into $\mathscr{M}_\sigma$.

We can then state the following.

DEFINITION 3.2. Let m be a σ-regular mean. The unique regular mean r_m such that $r_m^\sigma = m$ will be denoted by $r_m = m^{-\sigma}$ and called the σ-inverse transform of m.

It follows that $\left(m^\sigma\right)^{-\sigma} = m$ for all $m\in\mathscr{M}_r$ and $\left(m^{-\sigma}\right)^\sigma = m$ for all $m\in\mathscr{M}_\sigma$. That is,

$$\left(r_m^\sigma = m,\ r_m\in\mathscr{M}_r\right) \iff \left(r_m = m^{-\sigma},\ m\in\mathscr{M}_\sigma\right).$$

Now, we will discuss a list of examples illustrating the above.

EXAMPLE 3.7. According to Example 3.6, we can say that $C^{-\sigma}$ does not exist (as a regular mean). This, because C is not σ-regular.

EXAMPLE 3.8. With the above notation, Example 2.2, Example 2.3 and Example 2.4 imply that H, L and T are σ-regular, with

$$H^{-\sigma} = A, \quad L^{-\sigma} = G, \quad T^{-\sigma} = H,$$

while Example 2.5 with Theorem 2.3 yields that R_1 is also σ-regular, with $R_1^{-\sigma} = L$.

Let us take a look at the above example again: the means $L^{-\sigma}$ and $T^{-\sigma}$ have simple forms, simpler than those of L and T, respectively. We can then ask what is the explicit form of $m^{-\sigma}$ when the mean m is one of the other standard means previously mentioned. By virtue of its interest, the answer to this latter question will be presented in the form of a result as well.

THEOREM 3.3. *The following relationships hold*

$$A^{-\sigma} = \left(\frac{A+G}{2}\right)^* = \frac{2G^2}{A+G}, \qquad G^{-\sigma} = \left(\frac{AG+G^2}{2}\right)^{1/2},$$

$$P^{-\sigma} = \left(\left(\frac{A+G}{2}\right)^* G\right)^{1/2} = \left(\frac{2G^3}{A+G}\right)^{1/2},$$

$$M^{-\sigma} = \left(\left(\frac{A^2+AG}{2}\right)^{1/2}\right)^* = \left(\frac{2G^4}{A^2+AG}\right)^{1/2},$$

where the notation m^* refers to the dual mean of m.

Proof. We just show the formulae giving $P^{-\sigma}$ and we left to the reader the routine task for proving the other relationships in a similar manner. Recall that, see (3.3),

$$f_P(x) = \frac{2}{(x+1)\sqrt{x}}$$

and by (3.6) we obtain (after computation and reduction)

$$r_P(a,b) = b f_P\left(\sqrt{b/a}\right) = \frac{2a^{3/4}b^{3/4}}{\sqrt{a}+\sqrt{b}}.$$

Remarking that $a^{3/4}b^{3/4} = G^{3/2}(a,b)$ and $\frac{\sqrt{a}+\sqrt{b}}{2} = \left(\frac{A+G}{2}\right)^{1/2}(a,b)$, the desired relationship for $P^{-\sigma} := r_P$ follows. □

EXAMPLE 3.9. Contrary to the above, the mean $I^{-\sigma}$ can not be expressed only in terms of A and G. In fact, a long (but elementary) computation leads to (we omit the detail)

$$I^{-\sigma} = e\frac{\left(\frac{A+G}{2}\right)^{1/2} G^{3/2}}{L\, exp\left(\frac{A+G}{2L}\right)}.$$

It is not hard to check that I is a σ-regular mean and so $I^{-\sigma}$ is a regular mean. Otherwise, let us mention that, as pointed before for Q^{σ}, the means L^{σ}, I^{σ}, S^{σ}, P^{σ}, T^{σ} and M^{σ} are hard (and even impossible) to be explicitly computed.

Let S be the weighted geometric mean defined by $S := S(a,b) = \left(a^a b^b\right)^{1/(a+b)}$. We left the reader to verify if the mean S is σ-regular and discussing the existence of $S^{-\sigma}$ together with the computation, if possible, of $S^{-\sigma}$.

EXAMPLE 3.10. By analogy with the mean R_1 defined in Theorem 2.3, let us define the binary map R_2 by $R_2(a,a) = a$ and

$$\forall a,b>0,\ a\neq b \quad R_2(a,b) = \frac{b-a}{tanhi\Big(\ln(b/a)\Big)},$$

where *tanhi* denotes the integral hyperbolic tangent function defined for all real number x through

$$tanhi(x) = \int_0^x \frac{\tanh t}{t}dt.$$

We can ask if R_2 is a mean and if it is σ-regular. It is not hard to verify that the answer is positive with the following relationship

$$R_2^{-\sigma} = \frac{GL}{A}, \text{ or equivalently } R_2 = \left(\frac{GL}{A}\right)^{\sigma}.$$

We go back to the results of the above and specially to the relationships of Theorem 3.3, as well as that of Example 3.8: such results stem their importance in the fact that the means

$$G^{-\sigma},\ A^{-\sigma},\ Q^{-\sigma},\ L^{-\sigma},\ P^{-\sigma},\ T^{-\sigma},\ M^{-\sigma}$$

have algebraic expressions involving only the simplest (quasi-arithmetic) means A and G. We will write this in another way.

Let m be a σ-regular mean such that $m^{-\sigma}/G$ can be expressed only in term of A/G, as $m\in\{L,H,T,A,G,P,M\}$, we set

$$\frac{m^{-\sigma}}{G} := F_m(z), \text{ with } z = \frac{A}{G} \geqslant 1,$$

then the next corollary may be deduced from the above.

COROLLARY 3.4. *With the above notation, the following relationships are met:*

$$F_L(z) = 1, \quad F_H(z) = z, \quad F_T(z) = \frac{1}{z}, \quad F_A(z) = \frac{2}{z+1},$$
$$F_G(z) = \sqrt{\frac{z+1}{2}}, \quad F_P(z) = \sqrt{\frac{2}{z+1}}, \quad F_M(z) = \sqrt{\frac{2}{z^2+z}}. \tag{3.7}$$

Inversely, for some given function $g(z)$ we can ask if there is a σ-regular mean m such that $F_m(z) = g(z)$. It is necessary that $\Big(g(A/G)\Big)G$ is a mean. The next example explains more this latter situation.

EXAMPLE 3.11. (1) There is no mean m such that $F_m(z) = z^2$, this because $G(A/G)^2 = A^2/G$ is not a mean.

(2) There is no mean m such that $F_m(z) = \frac{1}{z^2}$ because G^3/A^2 is not a mean.

(3) The mean m such that $F_m(z) = \sqrt{z}$ is $m = \left((AG)^{1/2}\right)^{\sigma}$.

(4) The mean m such that $F_m(z) = \frac{z+1}{2}$ is $m = \left(\frac{A+G}{2}\right)^{\sigma}$.

(5) The mean m such that $F_m(z) = \sqrt{\frac{z^2+z}{2}}$ is $m = \left(\left(\frac{A^2+AG}{2}\right)^{1/2}\right)^{\sigma}$.

We can of course compute explicitly m for some of the above examples. For instance, it is not hard to verify that

$$\left(\frac{A+G}{2}\right)^{\sigma} = H(H,L) := \frac{2HL}{H+L} = \frac{2G^2L}{G^2+AL}.$$

4. Other interesting results

This section is focused to study other interesting properties of the mean-map $m \longmapsto m^{\sigma}$ as well as those of its inverse mean-map $m \longmapsto m^{-\sigma}$. We start with the following result, which is immediate from Definition 3.2 with the fact that the mean-map $m \longmapsto m^{\sigma}$ is point-wise decreasing.

PROPOSITION 4.1. *For all σ-regular means m_1 and m_2 such that $m_1^{-\sigma} > m_2^{-\sigma}$ we have $m_1 < m_2$.*

This proposition, with the relationships of Theorem 3.3, gives more simplification for proving some inequalities of (1.2). Let us observe this in the next example.

EXAMPLE 4.1. With Corollary 3.4, the following chain of inequalities

$$\frac{1}{z} = F_T(z) < F_M(z) < F_A(z) < F_P(z) < F_L(z) = 1 < F_G(z) < F_H(z) = z$$

is obviously satisfied for every $z = A/G \geqslant 1$. This with the fact that $F_m(z) = \frac{m^{-\sigma}}{G}$ and Proposition 4.1 gives (simultaneously and in a fast way)

$$H < G < L < P < A < M < T.$$

EXAMPLE 4.2. The mean $R_1 = L^{\sigma}$ defined by Theorem 2.3 interpolates the means H and G i.e. $H < R_1 < G$. In fact, it is sufficient to verify that

$$G^{-\sigma} = \left(\frac{AG+G^2}{2}\right)^{1/2} < R_1^{-\sigma} = L < H^{-\sigma} = A,$$

which are well-known mean-inequalities.

EXAMPLE 4.3. The means R_1 and R_2 satisfy $R_1 < G < L < R_2$. Indeed, it is sufficient to verify that

$$R_2^{-\sigma} = \frac{GL}{A} < L^{-\sigma} = G < G^{-\sigma} = \Big(\frac{AG+G^2}{2}\Big)^{1/2} < R_1^{-\sigma} = L.$$

All these inequalities are immediate.

To present more results here, we notice that the set of all regular means $\mathscr{M}_r$ is obviously (linearly) convex and geometrically convex, that is, for all $m_1, m_2 \in \mathscr{M}_r$ and every $\lambda \in (0,1)$ we have $(1-\lambda)m_1 + \lambda m_2 \in \mathscr{M}_r$ and $m_1^{1-\lambda} m_2^{\lambda} \in \mathscr{M}_r$. With this, we can now state the following result.

THEOREM 4.2. *The mean-map $m \longmapsto m^{\sigma}$ is point-wise convex. That is, for all real number $\lambda \in (0,1)$ and all regular means m_1 and m_2, one has*

$$\Big((1-\lambda)m_1 + \lambda m_2\Big)^{\sigma} \leqslant (1-\lambda)m_1^{\sigma} + \lambda m_2^{\sigma}. \tag{4.1}$$

If moreover $m_1 \neq m_2$ are comparable then the above mean-inequality is strict.

Proof. By virtue of (2.4), with the linearity of the integral, we can easily show that

$$\Big\{\Big((1-\lambda)m_1 + \lambda m_2\Big)^{\sigma}(a,b)\Big\}^{-1} = (1-\lambda)\Big(m_1^{\sigma}(a,b)\Big)^{-1} + \lambda\Big(m_2^{\sigma}(a,b)\Big)^{-1}.$$

Taking the inverse of the above sides, with the fact that the real-valued function $x \longmapsto 1/x$ is strictly convex on $(0,\infty)$, we obtain the desired result. □

Combining Theorem 4.2 with Proposition 2.4, (ii) we can state the next result.

COROLLARY 4.3. *Let $m_1 \neq m_2$ and m be three σ-regular means, with m_1 and m_2 comparable. Assume that there exists $\lambda \in (0,1)$ such that*

$$(1-\lambda)m_1^{-\sigma} + \lambda m_2^{-\sigma} \leqslant m^{-\sigma}. \tag{4.2}$$

Then there holds

$$m < (1-\lambda)m_1 + \lambda m_2. \tag{4.3}$$

REMARK 4.1. It is worth mentioning that, with $m_1 \neq m_2$ comparable, (4.3) is a strict mean-inequality even if (4.2) is an equality. This follows from the strict convexity of $m \longmapsto m^{\sigma}$.

The above theorem ensures the linear (arithmetic) convexity of the mean-map $m \longmapsto m^{\sigma}$, while the next one concerns the geometric concavity of this mean-map.

THEOREM 4.4. *The mean-map $m \longmapsto m^{\sigma}$ is point-wise geometrically strictly concave. That is, for all real number $\lambda \in (0,1)$ and all regular means m_1 and m_2, with $m_1 \neq m_1$, we have*

$$\Big(m_1^{1-\lambda} m_2^{\lambda}\Big)^{\sigma} > \big(m_1^{\sigma}\big)^{1-\lambda}\big(m_2^{\sigma}\big)^{\lambda}. \tag{4.4}$$

Proof. Without loss the generality, we can assume that $a < b$. By definition we have

$$\left\{\left(m_1^{1-\lambda} m_2^{\lambda}\right)^{\sigma}(a,b)\right\}^{-1} = \frac{1}{b-a}\int_1^{b/a} m_1^{1-\lambda}\left(1,\frac{1}{t^2}\right) m_2^{\lambda}\left(1,\frac{1}{t^2}\right) dt.$$

According to the Hölder inequality we obtain

$$\left\{\left(m_1^{1-\lambda} m_2^{\lambda}\right)^{\sigma}(a,b)\right\}^{-1} \leqslant \frac{1}{b-a}\left\{\int_1^{b/a}\left(m_1^{1-\lambda}\left(1,\frac{1}{t^2}\right)\right)^{\frac{1}{1-\lambda}} dt\right\}^{1-\lambda} \times\left\{\int_1^{b/a}\left(m_2^{\lambda}\left(1,\frac{1}{t^2}\right)\right)^{\frac{1}{\lambda}} dt\right\}^{\lambda}.$$

Since m_1 and m_2 are means with $m_1 \neq m_2$ then Hölder inequality is strict and so we have

$$\left\{\left(m_1^{1-\lambda} m_2^{\lambda}\right)^{\sigma}(a,b)\right\}^{-1} < \left\{\frac{1}{b-a}\int_1^{b/a} m_1\left(1,\frac{1}{t^2}\right) dt\right\}^{1-\lambda}\left\{\frac{1}{b-a}\int_1^{b/a} m_2\left(1,\frac{1}{t^2}\right) dt\right\}^{\lambda}.$$

It follows that

$$\left\{\left(m_1^{1-\lambda} m_2^{\lambda}\right)^{\sigma}(a,b)\right\}^{-1} < \left\{\left(m_1^{\sigma}(a,b)\right)^{-1}\right\}^{1-\lambda}\left\{\left(m_2^{\sigma}(a,b)\right)^{-1}\right\}^{\lambda}.$$

Taking the inverse of the sides of the above inequality we obtain (4.4), so completes the proof. $\square$

The next corollary is immediate from the above theorem when combined with Proposition 2.4, (ii).

COROLLARY 4.5. *Let $m_1 \neq m_2$ and m be three σ-regular means. Assume that there exists $\lambda \in (0,1)$ such that*

$$m^{-\sigma} \leqslant \left(m_1^{-\sigma}\right)^{1-\lambda}\left(m_2^{-\sigma}\right)^{\lambda}. \tag{4.5}$$

Then we have

$$m_1^{1-\lambda} m_2^{\lambda} < m. \tag{4.6}$$

REMARK 4.2. As pointed for Corollary 4.3, (4.6) is strict even if (4.5) is an equality, provided that $m_1 \neq m_2$. This because $m \longmapsto m^{\sigma}$ is strictly geometrically concave. Further, the above corollary is useful for proving mean-inequalities when the bounds are in a geometric form. See the examples below.

As before, using the notation of Corollary 3.4, (4.5) is equivalent to

$$F_m(z) \leqslant \left(F_{m_1}(z)\right)^{1-\lambda}\left(F_{m_2}(z)\right)^{\lambda}. \tag{4.7}$$

We can therefore state the following:

COROLLARY 4.6. *If for $m_1 \neq m_2$ and m as above, the inequality (4.7) is satisfied for some $\lambda \in (0,1)$, then (4.6) holds true.*

For some $\lambda \in (0,1)$ satisfying (4.7) and illustrating the previous corollary, see the three examples below.

We now state the following result which is also of interest.

THEOREM 4.7. *Let $p \geqslant 1$ and $\lambda \in (0,1)$. For all means m_1 and m_2, with $m_1 \neq m_2$ comparable, we have*

$$\Big(\mathscr{B}_p^{\lambda}(m_1,m_2)\Big)^{\sigma} < \mathscr{B}_p^{\lambda}\big(m_1^{\sigma},m_2^{\sigma}\big), \tag{4.8}$$

where $\mathscr{B}_p^{\lambda}(m_1,m_2)$ is defined by (1.3).

Proof. Since $p \geqslant 1$ then the real-valued function $x \longmapsto x^{1/p}$ is strictly concave on $(0,\infty)$. This with the definition of $\mathscr{B}_p^{\lambda}(m_1,m_2)$ gives

$$\mathscr{B}_p^{\lambda}(m_1,m_2) > (1-\lambda)m_1 + \lambda m_2,$$

and by Proposition 2.4, (ii) and Theorem 4.2 we obtain

$$\Big(\mathscr{B}_p^{\lambda}(m_1,m_2)\Big)^{\sigma} < \Big((1-\lambda)m_1+\lambda m_2\Big)^{\sigma} < (1-\lambda)m_1^{\sigma} + \lambda m_2^{\sigma}.$$

Again, by the strict concavity of $x \longmapsto x^{1/p}$ on $(0,\infty)$ we have

$$(1-\lambda)m_1^{\sigma}+\lambda m_2^{\sigma} < \Big((1-\lambda)\big(m_1^{\sigma}\big)^p + \lambda\big(m_2^{\sigma}\big)^p\Big)^{1/p} := \mathscr{B}_p^{\lambda}\big(m_1^{\sigma},m_2^{\sigma}\big).$$

The desired result follows by combining the above. $\square$

REMARK 4.3. For $p=1$, (4.8) coincides with (4.1). But (4.1) can not be considered as a consequence of (4.8) since in the proof of (4.8) we need to use (4.1). In another way, Theorem 4.2 can not be stated here as corollary of Theorem 4.7.

The above results are interesting in the practical purposes since, according to Corollary 3.4, the expansions of $F_m(z)$ for $m \in \{H,G,L,P,A,M,T\}$ are simple expressions in term of $z = A/G \geqslant 1$. The following examples explain more this latter situation.

EXAMPLE 4.4. (1) By Corollary 3.4 it is easy to see that $\Big(F_M(z)\Big)^2 = F_A(z)F_T(z)$, and by Corollary 4.6 with $m_1 = A$, $m_2 = T$, $m = M$ and $\lambda = 1/2$ we deduce $AT < M^2$ which was differently proved in [4].

(2) Similarly, we verify that $\Big(F_L(z)\Big)^2 = F_G(z)F_P(z)$ and $\Big(F_P(z)\Big)^2 = F_L(z)F_A(z)$. We then deduce $GP < L^2$ and $LA < P^2$, see [4].

EXAMPLE 4.5. By Corollary 3.4 we have, successively:

(1) $\left(F_A(z)\right)^2 \leqslant F_P(z)F_M(z)$, which after simple reduction is equivalent to $\left(\sqrt{z}-1\right)^2 \geqslant 0$. Corollary 4.6 gives $PM < A^2$, see [4].

(2) $\left(F_A(z)\right)^2 \leqslant F_L(z)F_T(z)$, equivalent to $\left(z-1\right)^2 \geqslant 0$. Then $LT < A^2$, see [4].

EXAMPLE 4.6. By Corollary 3.4 we easily see that $\left(F_P(z)\right)^3 = \left(F_A(z)\right)^2 F_G(z)$. This, with Corollary 4.6 for $m_1 = A$, $m_2 = G$, $m = P$ and $\lambda = 1/3$, yields $A^{2/3}G^{1/3} < P$. See [6] for comparison.

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