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CHARACTERIZATIONS FOR VECTORIAL PREQUASI–INVEX TYPE FUNCTIONS VIA JENSEN TYPE INEQUALITIES

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Abstract. The purpose of this paper is to derive some criteria for vectorial prequasi-invex type functions via Jensen type inequalities. It is shown that a Jensen type inequality is sufficient and necessary for a vector function to be prequasi-invex under the condition of lower level-closedness, cone lower semicontinuity, cone upper semicontinuity and semistrict prequasi-invexity, respectively. Analogous results are established for vectorial semistrictly prequasi-invex functions and vectorial strictly prequasi-invex functions.

1. Introduction

A classical result for numerical convex functions is that a numerical function is convex if and only if it is midconvex and continuous. In [1], Yang relaxed continuity by lower semicontinuity and further replaced the midconvex by the following Jensen's condition: there exists some constant $\alpha \in (0,1)$ such that

$$f(\alpha x+(1-\alpha)y) \leqslant \alpha f(x)+(1-\alpha)f(y), \quad \forall x,y \in K,$$

where $K \subset R^n$ is a convex set and $f: K \to R$ is a numerical function. In [2], by using Yang's ideas, Mukherjee and Reddy derived some criteria of numerical prequasi-convex functions under lower semicontinuity and upper semicontinuity conditions via Jensen type inequalities. In [3], Yang et al introduced two new types of generalized convex functions, which are called semistrictly prequasi-invex functions and strictly prequasi-invex functions. Under certain conditions, Yang et al [3] propose some Jensen type inequalities to derive criteria for prequasi-invex, semistrictly prequasi-invex, and strictly prequasi-invex functions, respectively. Luo and Xu [4] further improved partial results of Yang et al [3] under weaker conditions. Motivated and inspired by the above works, in this paper, we attempt to use Jensen type inequalities to derive the criteria for vectorial prequasi-invex, semistrictly prequasi-invex, and strictly prequasi-invex functions. It is worth noting that partial results presented is more general than existing ones even when the functions are numerical.

The rest of this paper is organized as follows: In Section 2, we give some preliminary notations and definitions. In Section 3, we give the characterization of vectorial prequasi-invex functions. Section 4 is devoted to the study of criteria for vectorial semistrictly prequasi-invex functions. In Section 5, some characterizations of vectorial strictly prequasi-invex functions are established.

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2. Preliminaries

Throughout this paper, without otherwise specified, we always suppose that X and Y are two vector topological spaces, $K \subset X$ is a nonempty set, $C \subset Y$ is a pointed, closed and convex cone with $int\,C \neq \emptyset$, where $int\,C$ denotes the interior of C. For a given cone P in Y, define the following relations: for given $a,b \in Y$,

$$a \leqslant_P b \quad \Leftrightarrow \quad b-a \in P \quad \text{and} \quad a \nleqslant_P b \quad \Leftrightarrow \quad b-a \notin P.$$

We always suppose that Y is a vector lattice with respect to the order $\leqslant_C$ induced by C, $\eta : X \times X \to X$ is a vectorial function, $F : K \to Y$ is a vectorial function, and $f : K \to R$ is a numerical function. For any $a,b \in Y$, $a \vee b$ denotes the supremum of a and b with respect to $\leqslant_C$. Now we first give some definitions.

DEFINITION 2.1. See [5, 6, 7]. For a given set $K \subset X$ and a given function η : $X \times X \to X$, K is said to be invex with respect to η iff

$$\forall x,y \in K, \forall \lambda \in [0,1] \quad \Rightarrow \quad y + \lambda \eta(x,y) \in K.$$

Note that a convex set is invex with respect to η with $\eta(x,y) = x - y$.

DEFINITION 2.2. Let K be invex with respect to η. A vectorial function $F : K \to Y$ is said to be preinvex iff

$$F(y + \lambda \eta(x,y)) \leqslant_C \lambda F(x) + (1-\lambda)F(y), \quad \forall x,y \in K, \lambda \in [0,1].$$

REMARK 2.1. Definition 2.2 is a vectorial generalization of numerical invex functions [6, 7].

Pini [8] introduced the concept of numerical prequasi-invex functions. Now we generalize it to vectorial functions as follows.

DEFINITION 2.3. Let K be invex with respect to η. A vectorial function $F : K \to Y$ is said to be

(i) prequasi-invex of type (I) iff

$$F(y + \lambda \eta(x,y)) \leqslant_C F(x) \vee F(y), \quad \forall x,y \in K, \lambda \in [0,1].$$

(ii) prequasi-invex of type (II) iff, for any $x,y \in K$,

$$F(x) \leqslant_C F(y) \quad \Rightarrow \quad F(y + \lambda \eta(x,y)) \leqslant_C F(y), \quad \forall \lambda \in [0,1].$$

REMARK 2.2. (1) A prequasi-invex of type (I) vectorial function is also prequasi-invex of type (II), but the converse is not true in general;

(2) If $Y = R$ and $C = R_+$, then (i) and (ii) of Definition 2.3 are equivalent and both reduce to the definition of numerical prequasi-invex functions [8].

The following proposition gives a characterization of vectorial prequasi-invex of type (I).

PROPOSITION 2.1. *Let K be invex with respect to η and $F : K \to Y$ be a vectorial function. Then F is prequasi-invex of type (I) if and only if for each $a \in Y$, the set $\{z \in K : F(z) \leqslant_C a\}$ is invex with respect to η*

Proof. Suppose that F is prequasi-invex of type (I). Let $a \in Y$ and $x, y \in K$ such that

$$F(x) \leqslant_C a, \quad F(y) \leqslant_C a.$$

Since F is prequasi-invex of type (I),

$$F(y + \lambda\eta(x,y)) \leqslant_C F(x) \vee F(y) \leqslant_C a, \forall \lambda \in [0,1].$$

This implies that $y + \lambda\eta(x,y) \in \{z \in K : F(z) \leqslant_C a\}$ for all $\lambda \in [0,1]$ and so $\{z \in K : F(z) \leqslant_C a\}$ is invex with respect to η for all $a \in Y$.

Conversely, suppose that $\{z \in K : F(z) \leqslant_C a\}$ is invex with respect to η for all $a \in Y$. Let $x, y \in K$ and $a = F(x) \vee F(y)$. It follows that

$$x, y \in \{z \in K : F(z) \leqslant_C a\}.$$

The invexity of $\{z \in K : F(z) \leqslant_C a\}$ implies that

$$F(y + \lambda\eta(x,y)) \leqslant_C a = F(x) \vee F(y), \quad \forall \lambda \in [0,1].$$

Thus F is prequasi-invex of type (I). □

REMARK 2.3. If for each $a \in Y$, the set $\{z \in K : F(z) \leqslant_C a\}$ is invex with respect to η with $\eta(x,y) = x - y$, then F is said to be C-quasiconvex in the sense of Luc [9].

DEFINITION 2.4. Let K be invex with respect to η. A vectorial function $F : K \to Y$ is said to be

(i) strictly prequasi-invex of type (I) iff, $\forall x, y \in K, x \neq y, \forall \lambda \in (0,1)$,

$$F(y + \lambda\eta(x,y)) \leqslant_{C\setminus\{0\}} F(x) \vee F(y).$$

(ii) strictly prequasi-invex of type (II) iff, for any $x, y \in K, x \neq y$,

$$F(x) \leqslant_C F(y) \quad \Rightarrow \quad F(y + \lambda\eta(x,y)) \leqslant_{C\setminus\{0\}} F(y), \quad \forall \lambda \in (0,1).$$

REMARK 2.4. (1) Definition 2.4 generalizes Definition 1.4 of [3] to vectorial functions.

(2) A strictly prequasi-invex of type (I) vectorial function is also strictly prequasi-invex of type (II), but the converse is not true in general;

(3) If $Y = R$ and $C = R_+$, then (i) and (ii) of Definition 2.4 are equivalent and both reduce to the definition of numerical strictly prequasi-invex functions [3, 10].

DEFINITION 2.5. Let K be invex with respect to η. A vectorial function $F : K \to Y$ is said to be

(i) semistrictly prequasi-invex of type (I) iff, $\forall x, y \in K$ with $F(x) \neq F(y)$,

$$F(y + \lambda\eta(x,y)) \leqslant_{C\setminus\{0\}} F(x) \vee F(y), \quad \forall \lambda \in (0,1).$$

(ii) semistrictly prequasi-invex of type (II) iff, for any $x, y \in K$ with $F(x) \leqslant_{C\setminus\{0\}} F(y)$,

$$F(y + \lambda\eta(x,y)) \leqslant_{C\setminus\{0\}} F(y), \quad \forall \lambda \in (0,1).$$

REMARK 2.5. (1) Definition 2.5 is vectorial generalizations of numerical semistrictly prequasi-invex functions (see [3, 4, 10, 11]).

(2) A semistrictly prequasi-invex of type (I) vectorial function is also semistrictly prequasi-invex of type (II), but the converse is not true in general.

(3) If $Y = R$ and $C = R_+$, then (i)-(ii) of Definition 2.5 are equivalent and both reduce to the definition of numerical semistrictly prequasi-invex functions.

In the study of generalized invex functions, the following condition is important.

Condition C See [13]. Let $K \subset R^n$ be invex with respect to $\eta : X \times X \to X$. We say that η satisfies Condition C iff, for any $x, y \in X, \lambda \in [0,1]$,

$$\eta(y, y + \lambda\eta(x,y)) = -\lambda\eta(x,y), \quad \eta(x, y + \lambda\eta(x,y)) = (1-\lambda)\eta(x,y).$$

PROPOSITION 2.2. *Let K be invex with respect to η and η satisfy Condition C. Then for any $x, y \in K, \lambda, \lambda_1, \lambda_2 \in [0,1]$,*

$$\eta(y + \lambda_1\eta(x,y), y + \lambda_2\eta(x,y)) = (\lambda_1 - \lambda_2)\eta(x,y),$$
$$\eta(y + \lambda\eta(x,y), y) = \lambda\eta(x,y).$$

Proof. The conclusions have been shown in the proofs of Theorem 3.1 in [10] and Theorem 2.4 in [3]. □

Condition D Let $F : K \subset X \to Y$ be a vectorial function. We say that F satisfies Condition D iff

$$F(y + \eta(x,y)) \leqslant_C F(x), \quad \forall x, y \in K.$$

REMARK 2.6. Condition D is a vectorial generalization of Condition D in the numerical sense (see [3, 4]).

REMARK 2.7. Strict prequasi-invexity of type (I) (res. (II)) implies that prequasi-invexity of type (I) (res. (II)) as well as semistrict prequasi-invexity of type (I) (res. (II)). However, prequasi-invexities do not imply semistrict prequasi-invexities, and semistrict prequasi-invexities do not imply prequasi-invexities. For details, we refer to Examples 1.1-1.4 of [3].

The following cone semicontinuity was introduced in [14].

DEFINITION 2.6. Let $F : K \to Y$ be a vectorial function. F is said to be cone lower semicontinuous iff for each $x_0 \in K$ and any $d \in int\, C$, there exists a neighborhood U of x_0 such that $F(x) \in F(x_0) - d + int\, C$ for all $x \in U$. We say F is cone upper semicontinuous if and only if $-F$ is cone lower semicontinuous.

The lower level-closedness was introduced in [9].

DEFINITION 2.7. Let $F : K \to Y$ be a vectorial function. F is said to be lower level-closed iff, for all $a \in Y$, the set $\{x \in K : F(x) \leqslant_C a\}$ is closed.

REMARK 2.8. If $Y = R$ and $C = R_+$, both cone lower semicontinuity and lower level-closedness reduce to the ordinary lower semicontinuity.

3. Characterizations of Vectorial Prequasi-invex Functions

In what follows, unless otherwise specified, we always assume that:

(i) $K \subset X$ is an invex subset with respect to $\eta : X \times X \to X$.

(i) η satisfies Condition C, $F : K \to Y$ is a vectorial function and $f : K \to R$ is a numerical function.

First we recall a criteria for convex set.

LEMMA 3.1. *See* [2, 12]. *Let S be a nonempty closed set in R^n. Then S is convex if and only if for each $x, y \in S$, there exists $\beta \in (0,1)$ such that*

$$\beta x + (1-\beta) y \in S.$$

Now we generalize Lemma 3.1 as follows:

LEMMA 3.2. *Let S be a nonempty closed set in X and $\eta : X \times X \to X$ be a vectorial function such that $y + \eta(x,y) \in S$ for all $x, y \in S$. Then S is invex with respect to η if and only if for each $x, y \in S$, there exists $\beta \in (0,1)$ such that*

$$y + \beta \eta(x,y) \in S. \tag{1}$$

Proof. The necessity is obvious from Definition 2.1. We prove the sufficiency. Suppose on the contrary that there exist $x, y \in S$ and $\lambda \in (0,1)$ such that

$$y + \lambda\eta(x,y) \notin S.$$

Let

$$u = \sup\{t \in [0,\lambda) : y + t\eta(x,y) \in S\} \text{ and } v = \inf\{t \in (\lambda,1] : y + t\eta(x,y) \in S\}.$$

Obviously,

$$0 \leqslant u < \lambda < v \leqslant 1, \quad y_t := y + t\eta(x,y) \notin S, \quad \forall t \in (u,v).$$

Moreover, $y_u, y_v \in S$ since S is closed. Therefore, for any $\beta \in (0,1)$,

$$y_u + \beta\eta(y_v, y_u) = y + [u + \beta(v-u)]\eta(x,y) \notin S,$$

which contradicts (1) since $y_u, y_v \in S$. Thus S is invex with respect to η. $\square$

The following result gives a characterization of vectorial prequasi-invex of type (I) functions under lower level-closedness condition.

THEOREM 3.1. *Let $F : K \to Y$ be lower level-closed and satisfy Condition D. Then F is prequasi-invex of type (I) if and only if for each $x, y \in K$, there exists $\alpha \in (0,1)$ such that*

$$F(y + \alpha\eta(x,y)) \leqslant_C F(x) \vee F(y). \tag{2}$$

Proof. Since F is lower level-closed,

$$S_a = \{z \in K : F(z) \leqslant_C a\}$$

is closed for all $a \in Y$. We know that $y + \eta(x,y) \in S_a$ for all $x, y \in S_a$ since F satisfies Condition D. Now we show S_a is invex with respect to η. Let $x, y \in S_a$. It follows from (2) that there exists $\alpha \in (0,1)$ such that

$$F(y + \alpha\eta(x,y)) \leqslant_C F(x) \vee F(y) \leqslant_C a.$$

This implies that for each $x, y \in S_a$, there exists $\alpha \in (0,1)$ such that

$$y + \alpha\eta(x,y) \in S_a.$$

By Lemma 3.2, S_a is invex with respect to η for all $a \in Y$. Thus F is prequasi-invex of type (I) from Proposition 2.1. $\square$

To establish the criteria for vectorial prequasi-invex functions of type (II), we first introduce the following condition.

Condition P Let $F : K \to Y$ be a vectorial function. We say that F satisfies Condition P iff, for all $x, y \in K$ with $F(x) \leqslant_C F(y)$, and for all $\lambda_1, \lambda_2 \in [0,1]$, one has

$$\text{either} \quad F(y + \lambda_1\eta(x,y)) \geqslant_C F(y + \lambda_2\eta(x,y))$$

$$\text{or} \quad F(y + \lambda_1\eta(x,y)) \leqslant_C F(y + \lambda_2\eta(x,y)).$$

THEOREM 3.2. *Let F be cone lower semicontinuous (or lower level-closed) and satisfy Condition D and Condition P. Then F is prequasi-invex of type (II) if and only if for any given $x,y \in K$ with $F(x) \leqslant_C F(y)$, there exists $\alpha \in (0,1)$ such that*

$$F(y+\alpha\eta(x,y)) \leqslant_C F(y). \tag{3}$$

.

Proof. The necessity is obvious. To prove the sufficiency, we assume on the contrary that there exist $x,y \in$ with $F(x) \leqslant_C F(y)$ and $\bar{\lambda} \in (0,1)$ such that

$$F(y+\bar{\lambda}\eta(x,y)) \nleqslant_C F(y).$$

Let

$$A_1(x,y) = \{\lambda \in [0,1] : F(y+\lambda\eta(x,y)) \leqslant_C F(y)\}.$$

It is easy to verify that

$$\{\lambda \in A_1(x,y) : \lambda < \bar{\lambda}\} \neq \emptyset$$

and

$$\{\lambda \in A_1(x,y) : \lambda > \bar{\lambda}\} \neq \emptyset \text{ (from Condition D)}.$$

Define

$$\lambda_1 = \sup\{\lambda \in A_1(x,y) : \lambda < \bar{\lambda}\}$$

and

$$\lambda_2 = \inf\{\lambda \in A_1(x,y) : \lambda > \bar{\lambda}\}.$$

It is easy to see that

$$0 \leqslant \lambda_1 < \bar{\lambda} < \lambda_2 \leqslant 1 \quad \text{and} \quad \lambda \notin A_1(x,y), \quad \forall \lambda \in (\lambda_1,\lambda_2). \tag{4}$$

Now we show $\lambda_1, \lambda_2 \in A_1(x,y)$. From the definitions of λ_1 and λ_2, there exist $\lambda_1^n, \lambda_2^n \in A_1(x,y)$ satisfying $\lambda_1^n \to \lambda_1$ and $\lambda_2^n \to \lambda_2$ as $n \to \infty$.

Case I: F is cone lower semicontinuous. Since F is cone lower semicontinuous, for any $d \in int\, C$, there exists an integer N such that

$$F(y+\lambda_1\eta(x,y)) \leqslant_{int\, C} F(y+\lambda_1^n\eta(x,y)) + d \leqslant_C F(y) + d$$

and

$$F(y+\lambda_2\eta(x,y)) \leqslant_{int\, C} F(y+\lambda_2^n\eta(x,y)) + d \leqslant_C F(y) + d$$

for all $n > N$. Since $d \in int\, C$ is arbitrary, it follows that

$$F(y+\lambda_1\eta(x,y)) \leqslant_C F(y) \text{ and } F(y+\lambda_2\eta(x,y)) \leqslant_C F(y).$$

Case II: F is lower level-closed. Since $\lambda_1^n, \lambda_2^n \in A_1(x,y)$ and $\lambda_1^n \to \lambda_1$ and $\lambda_2^n \to \lambda_2$, we have

$$y+\lambda_1^n\eta(x,y), y+\lambda_2^n\eta(x,y) \in \{z \in K : F(z) \leqslant_C F(y)\} \tag{5}$$

and

$$y+\lambda_1^n\eta(x,y)\to y+\lambda_1\eta(x,y), y+\lambda_2^n\eta(x,y)\to y+\lambda_2\eta(x,y). \tag{6}$$

Since F is lower level-closed, the set $\{z\in K : F(z)\leqslant_C F(y)\}$ is closed. It follows from (5) and (6) that

$$F(y+\lambda_1\eta(x,y))\leqslant_C F(y), \quad F(y+\lambda_2\eta(x,y))\leqslant_C F(y).$$

Thus $\lambda_1,\lambda_2\in A_1(x,y)$. Since Condition P is satisfied,

$$F(y+\lambda_1\eta(x,y))\leqslant_C F(y+\lambda_2\eta(x,y)) \text{ or } F(y+\lambda_1\eta(x,y))\geqslant_C F(y+\lambda_2\eta(x,y)).$$

By the above inequality and (3), there exists $\bar{\alpha}\in(0,1)$ such that

$$F(y+\lambda_1\eta(x,y)+\bar{\alpha}\eta(y+\lambda_2\eta(x,y),y+\lambda_1\eta(x,y)))\leqslant_C F(y+\lambda_1\eta(x,y))\vee F(y+\lambda_1\eta(x,y)). \tag{7}$$

Let

$$\lambda_0=\lambda_1+\bar{\alpha}(\lambda_2-\lambda_1).$$

It is east to see that $\lambda_1<\lambda_0<\lambda_2$. By Condition C and Proposition 2.2,

$$y+\lambda_0\eta(x,y)=y+\lambda_1\eta(x,y)+\bar{\alpha}\eta(y+\lambda_2\eta(x,y),y+\lambda_1\eta(x,y)).$$

Since $\lambda_1,\lambda_2\in A_1(x,y)$, it follows from (7) that

$$F(y+\lambda_0\eta(x,y))\leqslant_C F(y),$$

which contradicts (4). This completes the proof. □

COROLLARY 3.1. *Let $f: K\to R$ be lower semicontinuous and satisfy Condition D. Then f is prequasi-invex if and only if, for every $x,y\in K$, there exists some $\alpha\in(0,1)$ such that*

$$f(y+\alpha\eta(x,y))\leqslant\max\{f(x),f(y)\}$$

.

REMARK 3.1. Corollary 3.1 was first established in the setting of finite-dimension spaces by Yang et al. (see Theorem 2.3 of [3]). It is worth mentioning that their proof is based on Lemma 3.1 of [3] where the following stronger condition should be needed: there exists some $\alpha\in(0,1)$ such that

$$f(y+\alpha\eta(x,y))\leqslant\max\{f(x),f(y)\}, \quad \forall x,y\in K.$$

REMARK 3.2. Theorems 3.1 and 3.2 generalize and improve Theorem 2.3 of [3].

In order to establish criteria for vectorial prequasi-invex functions under cone upper semicontinuity condition, we need the following lemmas.

LEMMA 3.3. *In addition to Condition D, assume that the following Jensen type condition is satisfied: there exists* $\alpha \in (0,1)$ *such that*

$$F(y+\alpha\eta(x,y)) \leqslant_C F(x) \vee F(y), \quad \forall x,y \in K. \tag{8}$$

Then the set

$$A_2 = \{\lambda \in [0,1] : F(y+\lambda\eta(y,x)) \leqslant_C F(x) \vee F(y), \forall x,y \in K\}$$

is dense in $[0,1]$.

Proof. Suppose that A_2 is not dense in $[0,1]$. Then there exist a $\lambda_0 \in (0,1)$ and a neighborhood $N(\lambda_0)$ of λ_0 such that

$$N(\lambda_0) \cap A_2 = \emptyset. \tag{9}$$

Then

$$\{\lambda \in A_2 : \lambda \geqslant \lambda_0\} \neq \emptyset \quad (\text{ from Condition D})$$

and

$$\{\lambda \in A_2 : \lambda \leqslant \lambda_0\} \neq \emptyset.$$

Define

$$\lambda_1 = \inf\{\lambda \in A_2 : \lambda \geqslant \lambda_0\}, \quad \lambda_2 = \sup\{\lambda \in A_2 : \lambda \geqslant \lambda_0\}. \tag{10}$$

It follows from (9) that

$$0 \leqslant \lambda_2 < \lambda_1 \leqslant 1.$$

Since $\alpha \in (0,1)$, we can choose $u_1, u_2 \in A_2$ such that

$$u_1 \geqslant \lambda_1, u_2 \leqslant \lambda_2 \text{ and } \max\{\alpha, 1-\alpha\}(u_1-u_2) < \lambda_1 - \lambda_2. \tag{11}$$

It follows from (8) that

$$\begin{aligned} &F(y+u_2\eta(x,y)+\alpha\eta(y+u_1\eta(x,y), y+u_2\eta(x,y))) \\ \leqslant_C\ &F(y+u_1\eta(x,y)) \vee F(y+u_2\eta(x,y)). \end{aligned}$$

Let $\bar{\lambda} = \alpha u_1 + (1-\alpha)u_2$. It follows from Proposition 2.2 that

$$\begin{aligned} &y+\bar{\lambda}\eta(x,y) \\ =\ &y+[u_2+\alpha(u_1-u_2)]\eta(x,y) \\ =\ &y+u_2\eta(x,y)+\alpha\eta(y+u_1\eta(x,y), y+u_2\eta(x,y)). \end{aligned}$$

Since $u_1, u_2 \in A_2$,

$$\begin{aligned} &F(y+\bar{\lambda}\eta(x,y)) \\ =\ &F(y+u_2\eta(x,y)+\alpha\eta(y+u_1\eta(x,y), y+u_2\eta(x,y)))) \\ \leqslant_C\ &F(y+u_1\eta(x,y)) \vee F(y+u_2\eta(x,y)) \\ \leqslant_C\ &F(x) \vee F(y). \end{aligned}$$

This implies $\bar{\lambda} \in A_2$.

If $\bar{\lambda} \geqslant \lambda_0$, from (11), we have

$$\bar{\lambda} - u_2 = \bar{\alpha}(u_1 - u_2) < \lambda_1 - \lambda_2.$$

Again from (11), we have $\bar{\lambda} < \lambda_1$, which contradicts (10).

Similarly, $\bar{\lambda} < \lambda_0$ provides a contradiction to (10). Therefore, A_2 is dense in $[0,1]$. □

In addition to Condition P, by using similar ideas as in Lemma 3.3, we can prove the following result.

LEMMA 3.4. *In addition to Condition D and Condition P, assume that the following condition is satisfied: there exists* $\alpha \in (0,1)$ *such that for all* $x, y \in K$ *with* $F(x) \leqslant_C F(y)$,

$$F(y + \alpha\eta(x,y)) \leqslant_C F(y). \tag{12}$$

Then, the set

$$A_3 = \{\lambda \in [0,1] : F(y + \lambda\eta(y,x)) \leqslant_C F(y), \forall x, y \in K \text{ with } F(x) \leqslant_C F(y)\}$$

is dense in $[0,1]$.

THEOREM 3.3. *Let* F *be cone upper semicontinuous and satisfy Condition D and Condition P. Then* F *is prequasi-invex of type (II) if and only if* F *satisfies inequality (12).*

Proof. The necessity is obvious. We prove the sufficiency. By Lemma 3.4, the set

$$A_3 = \{\lambda \in [0,1] : F(y + \lambda\eta(y,x)) \leqslant_C F(y), \forall x, y \in K \text{ with } F(x) \leqslant_C F(y)\}$$

is dense in $[0,1]$. Then, $\forall \bar{\alpha} \in (0,1), \exists \{\alpha_n\} \subset (0,1) \cap A_3$ such that $\alpha_n < \bar{\alpha}$ for all n and $\alpha_n \to \bar{\alpha}$ as $n \to \infty$.

Let $x, y \in K$ such that $F(x) \leqslant_C F(y)$. Define

$$z = y + \bar{\alpha}\eta(x,y), \quad y_n = y + \frac{\bar{\alpha} - \alpha_n}{1 - \alpha_n}\eta(x,y).$$

It is easy to see that

$$0 < \frac{\bar{\alpha} - \alpha_n}{1 - \alpha_n} < 1 \text{ and } y_n \to y \text{ as } n \to \infty.$$

Since K is invex with respect to η, $y_n \in K$ for all n. It follows from Proposition 2.2 that

$$y_n + \alpha_n\eta(x, y_n) = y + \bar{\alpha}\eta(x,y) = z.$$

The cone upper semicontinuity of F implies that for any $d \in int\,C$, there exists an N such that

$$F(y_n) \leqslant_{int\,C} F(y) + d, \quad \forall n > N.$$

Since $\alpha_n \in A_3$ and F satisfies Condition P,

$$\begin{aligned} F(z) &= F(y_n + \alpha_n \eta(x, y_n)) \\ &\leqslant_C F(x) \vee F(y_n) \\ &\leqslant_C F(x) \vee (F(y) + d), \quad \forall n > N. \end{aligned}$$

Since $d \in int\, C$ is arbitrary,

$$F(y + \bar{\alpha}\eta(x,y)) \leqslant_C F(x) \vee F(y) = F(y).$$

Thus F is prequasi-invex of type (II). $\square$

By using similar method as in Theorem 3.3, we can prove the following result.

THEOREM 3.4. *Let F be cone upper semicontinuous and satisfy Condition D. Then F is prequasi-invex of type (I) if and only if F satisfies inequality (8).*

REMARK 3.3. Theorems 3.3 and 3.4 generalize Theorem 2.1 of [3] and Theorem 2.4 of [4].

The following result gives a characterization of vectorial prequasi-invex of type (II) functions under semistrict prequasi-invexity condition.

THEOREM 3.5. *Let $F : K \to Y$ be semistrictly prequasi-invex of type (II) and satisfy Condition P. Then F is prequasi-invex of type (II) if and only if F satisfies inequality (12).*

Proof. The necessity is obvious. To prove the sufficiency, by contradiction, suppose that there exist $x, y \in K$ with $F(x) \leqslant_C F(y)$ and $\lambda \in (0,1)$ such that

$$F(y + \lambda\eta(x,y)) \not\leqslant_C F(y).$$

Let

$$z = y + \lambda\eta(x,y).$$

It follows that

$$F(z) \not\leqslant_C F(y) \geqslant_C F(x).$$

Since F satisfies Condition P, the above inequality implies that

$$F(z) \geqslant_{C\setminus\{0\}} F(y) \geqslant_C F(x). \tag{13}$$

If $F(x) \leqslant_{C\setminus\{0\}} F(y)$, then we have

$$F(z) \leqslant_{C\setminus\{0\}} F(y)$$

since F is semistrictly prequasi-invex of type (II). This contradicts (13).

If $F(x) = F(y)$, it follows from (13) that

$$F(z) \geqslant_{C\setminus\{0\}} F(y) = F(x). \tag{14}$$

Consider the following two cases:

(i) $0 < \lambda < \alpha < 1$. Define

$$z_1 = y + \frac{\lambda}{\alpha}\eta(x,y).$$

Since Condition C is satisfied, from Proposition 2.2, we have

$$z = y + \alpha\eta(z_1,y) = y + \lambda\eta(x,y). \tag{15}$$

Further, from Condition P, we have

$$F(y) \leqslant_C F(z_1) \quad \text{or} \quad F(y) \geqslant_C F(z_1). \tag{16}$$

It follows from (12), (15) and (16) that

$$F(z) \leqslant_C F(z_1) \vee F(y). \tag{17}$$

Again from Condition P, we have

$$F(z_1) \leqslant_C F(z) \quad \text{or} \quad F(z_1) \geqslant_C F(z).$$

By (14), (17) and the above inequality,

$$F(z) \leqslant_C F(z_1). \tag{18}$$

Define

$$b = \frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}.$$

Then $0 < b < 1$ since $0 < \lambda < \alpha < 1$. By Proposition 2.2,

$$z + b\eta(x,z) = y + \frac{\lambda}{\alpha}\eta(x,y) = z_1.$$

The semistrict prequasi-invexity of type (II) of F implies that

$$F(z_1) \leqslant_{C\setminus\{0\}} F(z),$$

which contradicts (18).

(ii) $0 < \alpha < \lambda < 1$. In this case, we can get a contradiction by exchanging the roles of α and $1-\alpha$ and λ and $\lambda-\alpha$ in case (i). □

REMARK 3.4. Theorem 3.5 generalizes Theorem 2.3 of [3].

4. Characterizations of Vectorial Semistrictly Prequasi-invex Functions

This section is devoted to criteria for vectorial semistrictly prequasi-invex functions via Jensen type inequalities.

THEOREM 4.1. *Let $F : K \to Y$ be prequasi-invex of type (II) and satisfy Condition P. Then F is semistictly prequasi-invex of type (II) if and only if there exists $\alpha \in (0,1)$ such that for every $x, y \in$ with $F(x) \leqslant_{C\setminus\{0\}} F(y)$,*

$$F(y+\alpha\eta(x,y)) \leqslant_{C\setminus\{0\}} F(y). \tag{19}$$

Proof. The necessity is obvious. To prove the sufficiency, suppose on the contrary that there exist $x, y \in K$ with $F(x) \leqslant_{C\setminus\{0\}} F(y)$ and $\lambda \in (0,1)$ such that

$$F(y+\lambda\eta(x,y)) \nleqslant_{C\setminus\{0\}} F(y). \tag{20}$$

Let

$$z = y+\lambda\eta(x,y).$$

By Condition P,

$$F(y+\lambda\eta(x,y)) \leqslant_C F(y) \quad \text{or} \quad F(y+\lambda\eta(x,y)) \geqslant_C F(y). \tag{21}$$

It follows from (20), (21) and the fact $F(x) \leqslant_{C\setminus\{0\}} F(y)$ that

$$F(z) \geqslant_C F(y) \geqslant_{C\setminus\{0\}} F(x). \tag{22}$$

The prequasi-invexity of type (II) of F further implies that

$$F(z) \leqslant_C F(y).$$

The above inequality together with (22) leads to

$$F(z) = F(y) \geqslant_{C\setminus\{0\}} F(x). \tag{23}$$

Define a sequence $\{z_k\} \subset K$ by

$$z_k = z+\alpha\eta(z_{k-1},z), \quad k = 1,2,3,\cdots.$$

where $z_0 = z$. It follows from (19) that

$$F(z_k) \leqslant_{C\setminus\{0\}} F(z), \quad k = 1,2,3,\cdots. \tag{24}$$

By Proposition 2.2,

$$z_k = z+\alpha^k\eta(x,z) = y+[\lambda+\alpha^k(1-\lambda)]\eta(x,y).$$

Choose an integer k_1 such that

$$\lambda > (1-\alpha)\alpha^{k_1-1}(1-\lambda).$$

Let

$$\begin{aligned}\beta_1 &= \lambda + \alpha^{k_1}(1-\lambda),\\ \beta_2 &= \lambda - (1-\alpha)\alpha^{k_1-1}(1-\lambda),\\ \bar{y} &= y + \beta_1\eta(x,y),\\ \bar{x} &= y + \beta_2\eta(x,y).\end{aligned}$$

It is easy to see that

$$0 < \beta_2 < \lambda < \beta_1 < 1.$$

Inequality (24) implies that

$$F(\bar{y}) = F(z_{k_1}) \leqslant_{C\setminus\{0\}} F(z). \tag{25}$$

Furthermore, from Condition P, we have

$$F(\bar{x}) \leqslant_C F(\bar{y}) \quad \text{or} \quad F(\bar{x}) \geqslant_{C\setminus\{0\}} F(\bar{y}).$$

Consider the following two cases:

(i) $F(\bar{x}) \leqslant_C F(\bar{y})$. By Condition C and Proposition 2.2,

$$\bar{y} + \alpha\eta(\bar{x},\bar{y}) = y + \beta_1\eta(x,y) + \alpha(\beta_2 - \beta_1)\eta(x,y) = y + \lambda\eta(x,y) = z.$$

Since F is prequasi-invex of type (II),

$$F(z) \leqslant_C F(\bar{y}),$$

which contradicts (25).

(ii) $F(\bar{x}) \geqslant_{C\setminus\{0\}} F(\bar{y})$. Since $z = \bar{y} + \alpha\eta(\bar{x},\bar{y})$, it follows from (19) that

$$F(z) \leqslant_{C\setminus\{0\}} F(\bar{x}). \tag{26}$$

On the other hand, we have

$$F(\bar{x}) \leqslant_C F(y) \tag{27}$$

since F is prequasi-invex of type (II). Combining (26) and (27), we have

$$F(z) \leqslant_{C\setminus\{0\}} F(y),$$

which contradicts (23). The proof is complete. □

The following result is a direct consequence of Theorem 4.1.

COROLLARY 4.1. *Let $f: K \to R$ be a numerical prequasi-invex function. Then f is semistictly prequasi-invex if and only if there exists $\alpha \in (0,1)$ such that for every $x,y \in$ with $f(x) \neq f(y)$,*

$$f(y + \alpha\eta(x,y)) < \max\{f(x), f(y)\}.$$

REMARK 4.1. Corollary 4.1 was first proved by Yang et al (see Theorem 3.1 of [3]) with additional condition: for every $x,y \in K$ with $f(x) \neq f(y)$,

$$f(y+(1-\alpha)\eta(x,y)) < \max\{f(x), f(y)\}$$

in the finite-dimension setting.

THEOREM 4.2. *Let $F: K \to Y$ be lower level-closed and satisfy Condition D and Condition P. If there exits $\alpha \in (0,1)$ such that for every $x,y \in K$ with $F(x) \neq F(y)$,*

$$F(y+\alpha\eta(x,y)) \leqslant_{C\setminus\{0\}} F(x) \vee F(y), \tag{28}$$

then F is prequasi-invex of type (I).

Proof. , By Theorem 3.1, it is sufficient to show that, for every $x,y \in K$, there exists $\lambda \in (0,1)$ such that

$$F(y+\lambda\eta(x,y)) \leqslant_C F(x) \vee F(y).$$

Suppose on the contrary that there exist $x,y \in K$ such that

$$F(y+\lambda\eta(x,y)) \nleqslant_C F(x) \vee F(y), \quad \forall\lambda \in (0,1). \tag{29}$$

If $F(x) \neq F(y)$, it follows from (28) that

$$F(y+\alpha\eta(x,y)) \leqslant_{C\setminus\{0\}} F(x) \vee F(y),$$

which contradicts (29).

If $F(x) = F(y)$, then (29) implies that

$$F(y+\lambda\eta(x,y)) \nleqslant_C F(x) = F(y), \quad \forall\lambda \in (0,1),$$

which together Condition P leads to

$$F(y+\lambda\eta(x,y)) \geqslant_{C\setminus\{0\}} F(x) = F(y), \quad \forall\lambda \in (0,1). \tag{30}$$

It follows from (30) and Proposition 2.2 that

$$\begin{aligned} &F(y+\lambda\eta(x,y)+\alpha\eta(x,y+\lambda\eta(x,y)) \\ &= F(y+\gamma\eta(x,y)) \geqslant_{C\setminus\{0\}} F(x) = F(y), \quad \forall\lambda \in (0,1), \end{aligned} \tag{31}$$

where $\gamma = \lambda + \alpha(1-\lambda) \in (0,1)$.

Inequalities (28) and (30) imply that

$$\begin{aligned} &F(y+\lambda\eta(x,y)+\alpha\eta(x,y+\lambda\eta(x,y)) \\ &\leqslant_{C\setminus\{0\}} F(y+\lambda\eta(x,y)) \vee F(x) = F(y+\lambda\eta(x,y)), \forall\lambda \in (0,1). \end{aligned} \tag{32}$$

It follows from (28), (31) and (32), and Proposition 2.2 that

$$\begin{aligned}
&F(y+(1-\alpha)\gamma\eta(x,y))\\
&=F(y+\gamma\eta(x,y)+\alpha\eta(y,y+\gamma\eta(x,y)))\\
&\leqslant_{C\setminus\{0\}} F(y)\vee F(y+\gamma\eta(x,y))\\
&=F(y+\gamma\eta(x,y))\\
&\leqslant_{C\setminus\{0\}} F(y+\lambda\eta(x,y)),\quad \forall\lambda\in(0,1).
\end{aligned}$$

Letting

$$\lambda=\frac{1-\alpha}{2-\alpha}$$

in the above inequality, we get a contradiction. □

By using Theorem 3.2 and similar method as in Theorem 4.2, we can prove the following result.

THEOREM 4.3. *Let $F:K\to Y$ be cone lower semi-continuous (or lower level-closed) and satisfy Condition D and Condition P. Suppose that inequality (19) holds. Then F is prequasi-invex of type (II).*

The following result is a direct consequence of Theorem 4.2 and 4.3.

COROLLARY 4.2. *Let $f:K\to R$ be lower semi-continuous and satisfy Conditions D. If there exits $\alpha\in(0,1)$ such that for every $x,y\in K$ with $f(x)\neq f(y)$,*

$$f(y+\alpha\eta(x,y))<\max\{f(x),f(y)\},$$

then f is prequasi-invex.

REMARK 4.2. Corollary 4.2 was first proved by Yang et al (see Theorem 3.3 of [3]) in the finite-dimension setting. However, our result improves Theorem 3.3 of [3] since their proof is based on Theorem 2.3 of [3] (see Remark 3.1).

By Theorems 4.1 and 4.3, we have the following result.

THEOREM 4.4. *Let $F:K\to Y$ be cone lower semi-continuous (or lower level-closed) and satisfy Condition D and Condition P. Then F is semitrictly prequasi-invex of type (II) if and only if F satisfies inequality (19).*

REMARK 4.3. Theorem 4.4 generalizes Corollary 3.1 of [3].

5. Characterizations of Vectorial Strictly Prequasi-invex Functions

In this section, we give some criteria for vectorial strictly prequasi-invex functions via Jensen type inequalities.

THEOREM 5.1. *Assume that $F : K \to Y$ satisfies the following conditions:*

(i) F is prequasi-invex of type (I).

(ii) $\eta(x,y) = 0$ if and only if $x = y$.

Then F is strictly prequasi-invex of type (I) if and only if for every $x, y \in K$ with $x \neq y$, there exists $\alpha \in (0,1)$ such that

$$F(y + \alpha\eta(x,y)) \leqslant_{C\setminus\{0\}} F(x) \vee F(y). \tag{33}$$

Proof. The necessity is obvious. To prove the sufficiency, we suppose on the contrary that there exist $x, y \in$ with $x \neq y$ and $\lambda \in (0,1)$ such that

$$F(y + \lambda\eta(x,y)) \not\leqslant_{C\setminus\{0\}} F(x) \vee F(y).$$

Let

$$z = y + \lambda\eta(x,y).$$

Since F is prequasi-invex of type (I),

$$F(z) \leqslant_C F(x) \vee F(y).$$

The above two inequalities imply that

$$F(z) = F(x) \vee F(y). \tag{34}$$

From Condition C and assumption (ii), we have

$$x \neq z \quad \text{and} \quad y \neq z.$$

By (33), there exist $\beta_1, \beta_2 \in (0,1)$ such that

$$F(z + \beta_1\eta(x,z)) \leqslant_{C\setminus\{0\}} F(x) \vee F(z) \text{ and } F(y + \beta_2\eta(z,y)) \leqslant_{C\setminus\{0\}} F(y) \vee F(z). \tag{35}$$

Let

$$\bar{x} = z + \beta_1\eta(x,z) \quad \text{and} \quad \bar{y} = y + \beta_2\eta(z,y).$$

Combining (34) and (35), we have

$$F(\bar{x}) \leqslant_{C\setminus\{0\}} F(z) \text{ and } F(\bar{y}) \leqslant_{C\setminus\{0\}} F(z). \tag{36}$$

By Proposition 2.2,

$$\bar{x} = y + [\lambda + \beta_1(1-\lambda)]\eta(x,y) \text{ and } \bar{y} = y + \beta_2\lambda\eta(x,y).$$

Let

$$u_1 = \lambda + \beta_1(1-\lambda), u_2 = \beta_2\lambda, u = \frac{\lambda - u_2}{u_1 - u_2}.$$

It is easy to see that $u_1, u_2, u \in (0,1)$. Again from Proposition 2.2, we have

$$\bar{y} + u\eta(\bar{x}, \bar{y}) = y + \lambda\eta(x,y) = z.$$

Since F is prequasi-invex of type (I),

$$F(z) \leqslant_C F(\bar{x}) \vee F(\bar{y}). \quad \square$$

In addition to Condition P, by using similar method as in Theorem 5.1, we can prove the following result.

THEOREM 5.2. *Assume that $F : K \to Y$ satisfies the following conditions:*

(i) F is prequasi-invex of type (II) and satisfies Condition P.

(ii) $\eta(x,y) = 0$ if and only if $x = y$.

Then F is strictly prequasi-invex of type (II) if and only if for every $x, y \in K$ with $x \neq y$ and $F(x) \leqslant_C F(y)$, there exists $\alpha \in (0,1)$ such that

$$F(y + \alpha\eta(x,y)) \leqslant_{C\setminus\{0\}} F(y). \tag{37}$$

REMARK 5.1. Some analogous results for numerical functions were proved by Yang et al. (see Theorem 4.1 of [3]) and Luo and Xu (see Theorem 2.4 of [4].

THEOREM 5.3. *Let $F : K \to Y$ satisfy the following conditions:*

(i) F is lower level-closed and satisfies Conditions D and P.

(ii) $\eta(x,y) = 0$ if and only if $x = y$.

Then F is strictly prequasi-invex of type (I) if and only if there exists $\alpha \in (0,1)$ such that for every $x, y \in K$ with $x \neq y$,

$$F(y + \alpha\eta(x,y)) \leqslant_{C\setminus\{0\}} F(x) \vee F(y). \tag{38}$$

Proof. The conclusion follows from Theorems 4.2 and 5.1. $\square$

THEOREM 5.4. *Let $F : K \to Y$ satisfy the following conditions:*

(i) F is cone lower semi-continuous (or lower level-closed) and satisfies Condition D and Condition P.

(ii) $\eta(x,y) = 0$ if and only if $x = y$.

Then F is strictly prequasi-invex of type (II) if and only if there exists $\alpha \in (0,1)$ such that for every $x, y \in K$ with $x \neq y$ and $F(x) \leqslant_C F(y)$,

$$F(y + \alpha\eta(x,y)) \leqslant_{C\setminus\{0\}} F(y). \tag{39}$$

Proof. The conclusion follows from Theorems 4.3 and 5.2. □

THEOREM 5.5. *Let $F : K \to Y$ satisfy the following conditions:*

(i) F is cone upper semi-continuous and satisfies Condition D.

(ii) $\eta(x,y) = 0$ if and only if $x = y$.

Then F is strictly prequasi-invex of type (I) if and only if F satisfies inequality (38).

Proof. The conclusion follows from Theorems 3.4 and 5.1. □

THEOREM 5.6. *Let $F : K \to Y$ satisfy the following conditions:*

(i) F is cone upper semi-continuous and satisfies Conditions D and P.

(ii) $\eta(x,y) = 0$ if and only if $x = y$.

Then F is strictly prequasi-invex of type (II) if and only if F satisfies inequality (39).

Proof. , The conclusion follows from Theorems 3.3 and 5.2. □
In [3], Yang et al proved the following result.

THEOREM 5.7. *See Theorem 4.3 of* [3]. *A numerical function $f : K \to R$ is strictly prequasi-invex function if and only if f is semistrictly prequasi-invex and the following condition holds: there exists $\alpha \in (0,1)$ such that for every $x,y \in K$ with $x \neq y$,*

$$f(y+\alpha\eta(x,y)) < \max\{f(x), f(y)\}.$$

Now we improve the above theorem as follows:

THEOREM 5.8. *Let $F : K \to Y$ be a vectorial function. Then F is strictly prequasi-invex of type (I) if and only if F is semistrictly prequasi-invex of type (I) and satisfies inequality (33).*

Proof. The necessity is obvious. To prove the sufficiency, it is is sufficient to show that $x \neq y, F(x) = F(y)$ imply that

$$F(y+\lambda\eta(x,y)) \leqslant_{C\setminus\{0\}} F(x) \vee F(y), \quad \forall\lambda \in (0,1). \tag{40}$$

For every $x,y \in K$ with $x \neq y$ and $F(x) = F(y)$, it follows from (33) that there exists $\beta \in (0,1)$ such that

$$F(y+\beta\eta(x,y)) \leqslant_{C\setminus\{0\}} F(x) = F(y). \tag{41}$$

Set

$$z = y+\beta\eta(x,y).$$

Let $\lambda \in (0,1)$. If $\lambda < \beta$, then

$$u = \frac{\beta - \lambda}{\beta} \in (0,1).$$

By Proposition 2.2,

$$z + u\eta(y,z) = y + \lambda\eta(x,y).$$

Since $F(z) \neq F(y)$, the semistrict prequasi-invexity of type (I) of F implies that

$$F(y+\lambda\eta(x,y)) = F(z+u\eta(y,z)) \leqslant_{C\setminus\{0\}} F(y) \vee F(z) = F(x) \vee F(y). \tag{42}$$

If $\lambda > \beta$, then

$$v = \frac{\lambda - \beta}{1-\beta} \in (0,1).$$

Again from Proposition 2.2, we have

$$z + v\eta(x,z) = y + \lambda\eta(x,y).$$

Since $F(z) \neq F(x)$ and F is semistrictly prequasi-invex of type (I),

$$F(y+\lambda\eta(x,y)) = F(z+v\eta(x,z)) \leqslant_{C\setminus\{0\}} F(x) \vee F(z) = F(x) \vee F(y). \tag{43}$$

Inequalities (41), (42) and (43) show that (40) holds. This completes the proof. □

By using similar method as in Theorem 5.8, we can obtain the following result.

THEOREM 5.9. *Let $F : K \to Y$ be a vectorial function. Then F is strictly prequasi-invex of type (II) if and only if F is semistrictly prequasi-invex of type (II) and satisfies inequality (37).*

REMARK 5.2. A uniform $\alpha \in (0,1)$ in Theorem 5.7 is needed, while conditions in Theorems 5.8 and 5.9 have been weakened to a great extent.

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