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CONTINUOUS MONOTONE MAPS ON MATRICES FOR ORDERS INDUCED BY THE GROUP INVERSE

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Abstract. We characterize continuous injective maps on the set of complex matrices which are monotone with respect to $\overset{\sharp}{\leqslant}$-order or $\overset{\mathrm{cn}}{\leqslant}$-order. In particular, we prove that all such maps must be automatically $\mathbb{R}$-linear and surjective. We also present several examples of monotone maps showing that our assumptions are indispensable.

1. Introduction

Let $M_n(\mathbb{F})$ denote the space of square matrices of order n with coefficients from the field $\mathbb{F}$, and let $GL_n(\mathbb{F})$ denote the group of invertible matrices. By $\mathscr{D}_n(\mathbb{F}) \subseteq M_n(\mathbb{F})$ we denote the set of all *diagonalizable matrices*, where $A \in M_n(\mathbb{F})$ is diagonalizable if there exists $P \in GL_n(\mathbb{F})$ such that $P^{-1}AP$ is diagonal. It is said that a matrix $A \in M_n(\mathbb{F})$ has the *index* l ($\mathrm{Ind}A = l$) if $\mathrm{rk}A^l = \mathrm{rk}A^{l+1}$ and l is the smallest positive number with this property. Note that any diagonalizable matrix A has index 1. A pair of matrices $A, B \in M_n(\mathbb{F})$ is called *orthogonal*, see [19], if $AB = BA = 0$, it is denoted by $A \perp B$.

DEFINITION 1. [15] Let $A \in M_n(\mathbb{F})$. The system of matrix equations $AXA = A, \quad XAX = X, \quad AX = XA$ has a solution X if and only if $\mathrm{Ind}A = 1$. This solution is unique. It is called the *group inverse* of A, and is denoted by $A^\sharp$.

Group inverse is one of the matrix generalized inverses which has many useful properties and applications. A more detailed information about group inverse can be found for example in [1, 18]. An interesting application of the group inverse is the fact that it can be utilized to introduce an order relation on matrices:

DEFINITION 2. [15] Let $A, B \in M_n(\mathbb{F})$. Then $A \overset{\sharp}{\leqslant} B$ if and only if $A = B$ or $\mathrm{Ind}A = 1$ and $AA^\sharp = BA^\sharp = A^\sharp B$. Moreover, if $A \overset{\sharp}{\leqslant} B$ and $A \neq B$, then $A \overset{\sharp}{<} B$.

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A *core-nilpotent decomposition* of a matrix $A \in M_n(\mathbb{F})$ is the sum $A = C_A + N_A$, such that $C_A \perp N_A$, $\operatorname{Ind} C_A = 1$, N_A is a nilpotent matrix. This decomposition exists and is unique for all $A \in M_n(\mathbb{F})$, see [1, Chapter 4.8].

Below we provide the definitions of the matrix partial orders which will be useful for our further considerations.

DEFINITION 3. [11, 16] We say that $A \overline{\leqslant} B$ for an arbitrary pair of matrices A and B if and only if $\operatorname{rk}(B-A) = \operatorname{rk} B - \operatorname{rk} A$.

DEFINITION 4. [12] Let $A, B \in M_n(\mathbb{F})$. Then $A \overset{\mathrm{cn}}{\leqslant} B$ if and only if $C_A \overset{\sharp}{\leqslant} C_B$ and $N_A \overline{\leqslant} N_B$.

Let $\leqslant$ be a certain partial order relation on $M_n(\mathbb{F})$. For $M \subseteq M_n(\mathbb{F})$, the map $T: M \to M$ is called *monotone* with respect to $\leqslant$-order if for arbitrary two matrices $A, B \in M$ the condition $A \leqslant B$ implies $T(A) \leqslant T(B)$.

There are many results related to the investigations of monotone transformations on matrices and operators. See for example [4, 5, 6, 13, 14, 17, 19, 20, 21] and references therein. The investigation of monotone transformations for orders related to the group inverse was started in [2]. In that paper the characterizations of linear bijective maps for matrices over an arbitrary field which are monotone with respect to $\overset{\sharp}{<}$- and $\overset{\mathrm{cn}}{<}$-orders were obtained. In the paper [7] the approach which enabled to remove the bijectivity assumption was discovered. After that in the paper [8] additive monotone maps were characterized. Also, the characterization of injective maps preserving $\overset{\sharp}{\leqslant}$-order on the set of diagonalizable matrices was obtained in [10]. The main goal of the present paper is to characterize continuous injective maps on the set of complex matrices that are monotone with respect to $\overset{\sharp}{\leqslant}$-order or $\overset{\mathrm{cn}}{\leqslant}$-order. As a corollary we show that all such transformations are automatically surjective and $\mathbb{R}$-linear. After that we present some examples showing that our assumptions are indispensable.

We note that this paper is devoted only to finite dimensional matrix spaces. The results for infinite dimension spaces will appear elsewhere.

In Section 2 of this paper we give the necessary definitions and facts about spectrally orthogonal matrix decompositions. Such decompositions are convenient tools for describing the monotone maps. In Section 3 we formulate and prove the main characterization result for such maps. Section 4 contains some corollaries and examples.

2. Preliminaries

The notion of spectrally orthogonal matrix decompositions are introduced and investigated in our paper [9]. These decompositions are essentially used in the present paper, so we recall below basic definitions and properties. Firstly we need the following counting functions:

$k_A : \overline{\mathbb{F}} \times \mathbb{N} \to \mathbb{Z}_+$ defined by the rule: for $\lambda \in \overline{\mathbb{F}}$ and $r \in \mathbb{N}$ the value $k_A(\lambda, r)$ is equal to the number of Jordan blocks of A of the size r, corresponding to the eigenvalue λ (if there are no Jordan blocks of A with λ of the size r then $k_A(\lambda, r) = 0$);

$K_A : \overline{\mathbb{F}} \to \mathbb{Z}_+$ determines the total number of Jordan blocks of A, corresponding to the eigenvalue λ, i.e., $K_A(\lambda) = \sum\limits_{r=1}^{\infty} k_A(\lambda, r)$.

Observe that the set of eigenvalues $\operatorname{Spec} A = \{\lambda \in \overline{\mathbb{F}} \mid K_A(\lambda) > 0\}$.

Now we are ready to define spectrally orthogonal matrix decompositions.

DEFINITION 5. Let $\mathbb{F}$ be a field and let $A \in M_n(\mathbb{F})$ with $A = C_A + N_A$ be the core-nilpotent decomposition of A. The maps $S^i : \overline{\mathbb{F}} \times M_n(\mathbb{F}) \to M_n(\overline{\mathbb{F}})$, $i = 1,2,3$ are called *spectrally orthogonal decompositions* of A if $S^1_A(0) = N_A$ and for any $\lambda \neq 0$ the matrix $S^1_A(\lambda) = X_\lambda$ is such that $X_\lambda \overset{\sharp}{\leqslant} A$, $K_{X_\lambda}(\lambda) = K_A(\lambda)$ and $K_{X_\lambda}(\mu) = 0$ for all $\mu \in \overline{\mathbb{F}} \setminus \{0, \lambda\}$.

$$S^2_A(\lambda) = S^1_{A+I}(\lambda + 1) - S^1_A(\lambda) \text{ for all } \lambda \in \overline{\mathbb{F}};$$

$$S^3_A(\lambda) = S^1_A(\lambda) - \lambda S^2_A(\lambda) \text{ for all } \lambda \in \overline{\mathbb{F}}.$$

The correctness of this definition is proved in [9, Lemma 2.14]. Below we list the most important properties of these maps.

THEOREM 1. [9, Theorems 2.18, 2.20] *Let* $A \in M_n(\mathbb{F})$.

1. If $\lambda \notin \operatorname{Spec} A \subseteq \overline{\mathbb{F}}$ *then* $S^i_A(\lambda) = 0$ *for* $i = 1,2,3$.

2. $\operatorname{rk}(S^2_A(\lambda)) = \deg_{\chi_A}(z - \lambda)$ *is the multiplicity of* λ *in the characteristic polynomial* χ_A.

3. $S^i_A(\lambda) \perp S^j_A(\mu)$ *for all* $\lambda \neq \mu$, $i, j = 1,2,3$.

4. $S^i_A(\lambda) S^2_A(\lambda) = S^2_A(\lambda) S^i_A(\lambda) = S^i_A(\lambda)$ *for all* $\lambda \in \overline{\mathbb{F}}$, $i = 1,2,3$.

5. The matrix $S^2_A(\lambda)$ *is idempotent for all* $\lambda \in \overline{\mathbb{F}}$.

6. The matrix $S^3_A(\lambda)$ *is nilpotent for all* $\lambda \in \overline{\mathbb{F}}$.

7. $A = \sum\limits_{\lambda \in \overline{\mathbb{F}}} S^1_A(\lambda) = \sum\limits_{\lambda \in \overline{\mathbb{F}}} (\lambda S^2_A(\lambda) + S^3_A(\lambda))$, $I = \sum\limits_{\lambda \in \overline{\mathbb{F}}} S^2_A(\lambda)$.

8. For any polynomial $f \in \overline{\mathbb{F}}[t]$ *it holds that*

$$f(A) = \sum_{\lambda \in \overline{\mathbb{F}}} (f(\lambda) S^2_A(\lambda) + \frac{f'(\lambda)}{1!} S^3_A(\lambda) + \ldots + \frac{f^{(n-1)}(\lambda)}{(n-1)!} (S^3_A(\lambda))^{n-1}).$$

9. $\overline{\mathbb{F}}[A] = \{f(A)\}_{f \in \overline{\mathbb{F}}[t]} = \langle \{S^2_A(\lambda), S^3_A(\lambda), \ldots, (S^3_A(\lambda))^{n-1}\}_{\lambda \in \overline{\mathbb{F}}} \rangle$, *and nonzero matrices from the system* $\{S^2_A(\lambda), S^3_A(\lambda), \ldots, (S^3_A(\lambda))^{n-1}\}_{\lambda \in \overline{\mathbb{F}}}$ *are linearly independent.*

10. If $\lambda \in \mathbb{F}$ *then* $S^i_A(\lambda) \in M_n(\mathbb{F})$, $i = 1,2,3$.

11. If A *commutes with some* $B \in M_n(\mathbb{F})$, *then* $S^i_A(\lambda)$ *commute with* B *for all* $\lambda \in \overline{\mathbb{F}}$ *and* $i = 1,2,3$.

12. If $\operatorname{Ind} A = 1$ *and* A *is orthogonal to some* $B \in M_n(\mathbb{F})$ *then*

a) for $\lambda \neq 0$ *all matrices* $S^i_A(\lambda)$ *are orthogonal to* B,

b) $S^i_{A+B}(\lambda) = S^i_A(\lambda) + S^i_B(\lambda)$ *for* $\lambda \neq 0$ *and* $i = 1,2,3$.

c) $S^i_A(\lambda) \perp S^j_B(\mu)$ *for all* $\lambda, \mu \in \mathbb{F} \setminus \{0\}$, $i, j = 1,2,3$.

13. If $A \overset{\sharp}{\leqslant} C$ *for some* $C \in M_n(\mathbb{F})$, *then for all* $\Lambda \subseteq \overline{\mathbb{F}} \setminus \{0\}$ *we have* $\sum\limits_{\lambda \in \Lambda} S_A^i(\lambda) \overset{\sharp}{\leqslant} \sum\limits_{\lambda \in \Lambda} S_C^i(\lambda)$, $i = 1,2$. *In particular,* $S_A^i(\lambda) \overset{\sharp}{\leqslant} S_C^i(\lambda)$ *for* $\lambda \neq 0$ *and* $i = 1,2$.

The following property of matrix maps, which is closely related with the monotonicity, was introduced in [9]:

DEFINITION 6. The map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is *0-additive*, if for any matrices $A,B \in M_n(\mathbb{C})$ with $A \perp B$ we have:
(i) $T(A) \perp T(B)$; (ii) $T(A+B) = T(A)+T(B)$.

The theorem below is the important tool to prove our main result.

THEOREM 2. [10, Theorem 1.12] *Let* $\mathbb{F}$ *be an arbitrary algebraically closed field. Assume* $n \geqslant 3$ *and consider injective map* $T\colon \mathscr{D}_n(\mathbb{F}) \to \mathscr{D}_n(\mathbb{F})$ *which is monotone with respect to* $\overset{\sharp}{\leqslant}$*-order. Then there exist a matrix* $P \in GL_n(\mathbb{F})$, *a nonzero endomorphism* $f\colon \mathbb{F} \to \mathbb{F}$, *and an injective map* $\sigma\colon \mathbb{F} \to \mathbb{F}$ *satisfying the condition* $\sigma(0) = 0$ *such that*

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1} (S_A^2(\lambda))^f P \text{ for all } A \in \mathscr{D}_n(\mathbb{F})$$

or

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1} [(S_A^2(\lambda))^f]^t P \text{ for all } A \in \mathscr{D}_n(\mathbb{F}),$$

here spectrally orthogonal matrix decomposition $S_A^i(\lambda) \in M_n(\overline{\mathbb{F}})$, $i = 1,2,3$ *is defined above, see Definition 5.*

3. Main result

We will further assume that $\mathbb{F} = \mathbb{C}$.

THEOREM 3. *Let* $n \geqslant 3$ *and assume that the map* $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *is injective and continuous. Assume that at least one of the following conditions is true:*

a) T *is monotone with respect to* $\overset{\sharp}{\leqslant}$*-order;*
b) T *is monotone with respect to* $\underset{cn}{\leqslant}$*-order;*
c) T *is 0-additive map.*
Then there are $P \in GL_n(\mathbb{C})$, $\alpha \in \mathbb{C} \setminus \{0\}$ *such that*

$$\begin{aligned} T(X) &= \alpha P^{-1} X P && \text{for all } X \in M_n(\mathbb{C}) \text{ or} \\ T(X) &= \alpha P^{-1} X^t P && \text{for all } X \in M_n(\mathbb{C}) \text{ or} \\ T(X) &= \alpha P^{-1} \overline{X} P && \text{for all } X \in M_n(\mathbb{C}) \text{ or} \\ T(X) &= \alpha P^{-1} \overline{X}^t P && \text{for all } X \in M_n(\mathbb{C}), \end{aligned}$$

here $\overline{X}$ *is the matrix obtained from* X *by the elementwise complex conjugation.*

Proof. We start by assuming the condition a).

1. Let $A \in \mathscr{D}_n(\mathbb{C})$ be an arbitrary diagonalizable matrix, $\mathrm{rk}A = r$. It can be easily checked that there exist diagonalizable matrices $A_1, \dots, A_n$, such that $A_r = A$ and

$$0 \overset{\sharp}{<} A_1 \overset{\sharp}{<} \cdots \overset{\sharp}{<} A_n.$$

Thus

$$0 \overset{\sharp}{<} T(A_1) \overset{\sharp}{<} \cdots \overset{\sharp}{<} T(A_n).$$

Therefore by [2, Lemma 3.4] matrices $T(A_1), \dots, T(A_n)$ are also diagonalizable. In this case $T(A) = T(A_r)$ is diagonalizable and $T(\mathscr{D}_n(\mathbb{C})) \subseteq \mathscr{D}_n(\mathbb{C})$.

2. By Theorem 1.12 from [10] (see Theorem 2 in this text) we can assume that there are $P \in GL_n(\mathbb{C})$, nonzero endomorphism $f\colon \mathbb{C} \to \mathbb{C}$ and injective map $\sigma_0\colon \mathbb{C} \to \mathbb{C}$, $\sigma_0(0) = 0$ such that

$$T(A) = \sum_{\lambda \in \mathbb{C}} \sigma_0(\lambda) P^{-1} (S_A^2(\lambda))^f P \text{ for all } A \in \mathscr{D}_n(\mathbb{C})$$

or

$$T(A) = \sum_{\lambda \in \mathbb{C}} \sigma_0(\lambda) P^{-1} [(S_A^2(\lambda))^f]^t P \text{ for all } A \in \mathscr{D}_n(\mathbb{C}).$$

Composing, if necessary, the map T with the similarity by P^{-1} and transposition, we obtain an injective continuous map $T_1\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$, satisfying the condition

$$T_1(A) = \sum_{\lambda \in \mathbb{C}} \sigma_0(\lambda) (S_A^2(\lambda))^f \text{ for all } A \in \mathscr{D}_n(\mathbb{C}).$$

Now we define a map $d\colon \mathbb{C} \to M_n(\mathbb{C})$ by the following rule:

$$d(\lambda) = T_1(E_{11} + \lambda E_{12}) = \sigma_0(1)(E_{11} + f(\lambda) E_{12}).$$

Since $\sigma_0(1) \neq \sigma_0(0) = 0$ and the map d is continuous, endomorphism $f\colon \mathbb{C} \to \mathbb{C}$ is also continuous. Recall that there are only two nonzero continuous endomorphisms of field $\mathbb{C}$: the identity map and the complex conjugation. Thus $f(\lambda) = \lambda$ for all $\lambda \in \mathbb{C}$ or $f(\lambda) = \overline{\lambda}$ for all $\lambda \in \mathbb{C}$.

Denote $T_2(A) = (T_1(A))^f$. Here for $X = (x_{ij})$ we denote $X^f = (f(x_{ij}))$, i.e. the matrix for which f is applied elementwise. So, the map $T_2\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is injective and continuous and

$$T_2(A) = \sum_{\lambda \in \mathbb{C}} \sigma(\lambda) S_A^2(\lambda) \text{ for all } A \in \mathscr{D}_n(\mathbb{C}),$$

where $\sigma(\lambda) := f(\sigma_0(\lambda))$.

3. Let us prove that the function $\sigma\colon \mathbb{C} \to \mathbb{C}$ is linear. First we note that the function $\sigma\colon \mathbb{C} \to \mathbb{C}$ is continuous and injective. Indeed,

$$T_2(\mu E_{11}) = \sum_{\lambda \in \mathbb{C}} \sigma(\lambda) S_{\mu E_{11}}^2(\lambda) = \sigma(0) S_{\mu E_{11}}^2(0) + \sigma(\mu) S_{\mu E_{11}}^2(\mu) = \sigma(\mu) E_{11},$$

and the map T_2 is injective and continuous.

Let $\mu,\nu \in \mathbb{C}\setminus\{0\}$, $\mu \neq \nu$. Set

$$A_{\mu,\nu} = \mu E_{11} + E_{12} + \nu E_{22},$$

$$X_{\mu,\nu} = \mu E_{11} - \frac{\mu}{\nu-\mu} E_{12},$$

$$Y_{\mu,\nu} = \nu E_{22} + \frac{\nu}{\nu-\mu} E_{12}.$$

Then $A_{\mu,\nu} = X_{\mu,\nu} + Y_{\mu,\nu}$, $X_{\mu,\nu}^2 = \mu X_{\mu,\nu}$, $Y_{\mu,\nu}^2 = \nu Y_{\mu,\nu}$, $X_{\mu,\nu} \perp Y_{\mu,\nu}$.
By inequality $\mu \neq \nu$ we have

$$S^1_{A_{\mu,\nu}}(\mu) = X_{\mu,\nu}, \quad S^1_{A_{\mu,\nu}}(\nu) = Y_{\mu,\nu}.$$

Compute $T_2(A_{\mu,\nu})$:

$$\begin{aligned} T_2(A_{\mu,\nu}) &= \sum_{\lambda\in\mathbb{C}} \sigma(\lambda) S^2_{A_{\mu,\nu}}(\lambda) = \sigma(\mu) S^2_{A_{\mu,\nu}}(\mu) + \sigma(\nu) S^2_{A_{\mu,\nu}}(\nu) \\ &= \frac{\sigma(\mu)}{\mu} S^1_{A_{\mu,\nu}}(\mu) + \frac{\sigma(\nu)}{\nu} S^1_{A_{\mu,\nu}}(\nu) = \frac{\sigma(\mu)}{\mu} X_{\mu,\nu} + \frac{\sigma(\nu)}{\nu} Y_{\mu,\nu} \\ &= \sigma(\mu) E_{11} + \sigma(\nu) E_{22} + \frac{\sigma(\nu)-\sigma(\mu)}{\nu-\mu} E_{12}. \end{aligned}$$

Now we are going to apply the methods from the complex analysis.

All the definitions and results which will be used below can be found in [22], see also [3]. By continuity of T_2 for all $\mu \in \mathbb{C}\setminus\{0\}$ there exists $\lim\limits_{\nu\to\mu} \frac{\sigma(\nu)-\sigma(\mu)}{\nu-\mu}$. By [22, p. 34] the function σ is analytic on the domain $\mathbb{C}\setminus\{0\}$. Moreover, by continuity of $\sigma\colon \mathbb{C}\to\mathbb{C}$ there exists $\lim\limits_{\mu\to 0} \sigma(\mu)$ and a single point of removable singularity $0 \in \mathbb{C}$. Therefore σ is an entire function and hence it is analytic on the whole complex plane.

By [22, p. 187] any analytic function σ satisfy the following property (open mapping theorem): if σ is not a constant, then for each domain $\Omega \subseteq C$ the set $\sigma(\Omega)$ is also a domain.

Since σ is injective it is not a constant. Denote $\Omega_0 = \{z \in \mathbb{C} \mid |z| < 1\}$ then $\Omega_1 = \sigma(\Omega_0)$ is a domain and $\sigma(0) = 0 \in \Omega_1$. Since the set Ω_1 is open then there exists $\varepsilon > 0$ such that $\Omega^{(\varepsilon)} = \left\{z \in \mathbb{C} \middle| |z| < \varepsilon\right\} \subseteq \Omega_1$.

Thus all values from the domain $\Omega^{(\varepsilon)}$ are attained by the function σ on the domain Ω_0. However the function σ is injective and $\sigma(z) \notin \Omega^{(\varepsilon)}$ for all

$$z \in \mathbb{C}\setminus\Omega_0 = \left\{z \in \mathbb{C} \middle| |z| \geqslant 1\right\}.$$

The point ∞ is a unique singular point of the analytic function σ. Assume, this point is a removable singularity. Then σ is a constant which is not true. Let us show that ∞ is not an essential singularity of the function σ.

Assume the opposite. Thus by Casorati–Weierstrass (Sokhotski's) Theorem (see [22], p. 123) there exists a sequence of points z_m, $m = 1,2,\ldots$ such that $z_m \to \infty$ as $m \to \infty$ and $\lim\limits_{m\to\infty} \sigma(z_m) = 0$. Therefore there exists N_1 such that for $m > N_1$ we have $\sigma(z_m) \in \Omega^{(\varepsilon)}$. On the opposite side since $z_m \to \infty$ as $m \to \infty$ there exists N_2 such that $|z_m| > 1$ for all $m > N_2$. Set $N = \max\{N_1, N_2\} + 1$ and obtain $z_N \notin \Omega_0$, $\sigma(z_N) \in \Omega^{(\varepsilon)}$ which is impossible. The obtained contradiction shows that ∞ is not essential singularity point of the function σ.

Thus the only existing possibility is that point ∞ is a pole. In this case the function σ is a polynomial.

Set $\sigma(z) = \alpha \prod\limits_{s=1}^{k} (z - z_s)$ where $\alpha \in \mathbb{C} \setminus \{0\}$, k is a degree of this polynomial, $z_s \in \mathbb{C}$, $s = 1,\ldots,k$ are the roots. Since $\sigma(z_s) = 0 = \sigma(0)$ if $s = 1,\ldots,k$ then $z_1 = z_2 = \cdots = z_k = 0$ by the injectivity of σ. Hence $\sigma(z) = \alpha z^k$. In this case $\sigma(1) = \sigma(e^{\frac{2\pi i}{k}})$. Since σ is injective it follows that $k = 1$. Hence $\sigma(z) = \alpha z$, this shows the linearity of σ.

4. Since $\sigma(z) = \alpha z$ for all $z \in \mathbb{C}$ we have

$$T_2(A) = \sum_{\lambda\in\mathbb{C}} \sigma(\lambda) S_A^2(\lambda) = \sum_{\lambda\in\mathbb{C}} (\alpha\lambda) S_A^2(\lambda) = \alpha \sum_{\lambda\in\mathbb{C}} \lambda S_A^2(\lambda) = \alpha A$$

for all $A \in \mathscr{D}_n(\mathbb{C})$.

Let $X \in M_n(\mathbb{C})$ be an arbitrary matrix. Then there exists a family $\{A_q\} \subseteq \mathscr{D}_n(\mathbb{C})$ of diagonalizable matrices such that $\lim\limits_{q\to\infty} A_q = X$. By the continuity assumption on T_2 we have

$$T_2(X) = \lim_{q\to\infty} T(A_q) = \lim_{q\to\infty} \alpha A_q = \alpha X,$$

i.e. $T_2(X) = \alpha X$ for all matrices $X \in M_n(\mathbb{C})$.

Therefore by the definitions of T_2 and T_1 the map T has the required form.

Let us assume now that the condition b) is satisfied. Let $A \in \mathscr{D}_n(\mathbb{C})$ be an arbitrary diagonalizable matrix, $\mathrm{rk} A = r$. It can easily be checked that there exist diagonalizable matrices $A_1,\ldots,A_n$, such that $A_r = A$ and

$$0 \overset{\mathrm{cn}}{<} A_1 \overset{\mathrm{cn}}{<} \cdots \overset{\mathrm{cn}}{<} A_n.$$

Thus

$$0 \overset{\mathrm{cn}}{<} T(A_1) \overset{\mathrm{cn}}{<} \cdots \overset{\mathrm{cn}}{<} T(A_n).$$

Therefore by [2, Lemma 3.4] matrices $T(A_1),\ldots,T(A_n)$ are also diagonalizable. In this case $T(A) = T(A_r)$ is diagonalizable and $T(\mathscr{D}_n(\mathbb{C})) \subseteq \mathscr{D}_n(\mathbb{C})$. Since $\overset{\sharp}{\leqslant}$- and $\overset{\mathrm{cn}}{\leqslant}$-orders coincide on the set of diagonalizable matrices $\mathscr{D}_n(\mathbb{C})$ the rest of the proof coincides with the proof in the case a).

c) We see that $T(0) = T(0) + T(0)$, whence $T(0) = 0$. Since the map T is injective we have $T(X) \neq 0$ for $X \neq 0$.

Let $A \in \mathscr{D}_n(\mathbb{C})$, $k = \mathrm{rk} A$. Then there exist matrices $A_1,\ldots,A_n \in \mathscr{D}_n(\mathbb{C})$ such that $A = A_1 + \ldots + A_k$, $\mathrm{rk} A_i = 1$ for all $i = 1,\ldots,n$ and $A_i \perp A_j$ for all $i \neq j$. Indeed, since the matrix A is diagonalizable then there exists $P_A \in GL_n(\mathbb{C})$ such that matrix $\widetilde{A} = P_A^{-1}AP_A$ is diagonal and has the form $\widetilde{A} = \operatorname{diag}(\lambda_1^{(A)},\dots,\lambda_k^{(A)},0,\dots,0)$. Set $\lambda_s^{(A)} = 1$ for $k < s \leqslant n$ and $\widetilde{A}_s = \operatorname{diag}(0,\dots,0,\lambda_s^{(A)},0,\dots,0)$ where $\lambda_s^{(A)}$ is on the position s. For $1 \leqslant s \leqslant n$ we have $\widetilde{A}_i \perp \widetilde{A}_j$ if $i \neq j$. Moreover if $A_s = P_A\widetilde{A}_sP_A^{-1}$ for $1 \leqslant s \leqslant n$, then $A_i \perp A_j$ for $i \neq j$ and $A = A_1 + \ldots + A_k$.

By 0-additivity of T we have $T(A_i) \perp T(A_j)$ for $i \neq j$ and $T(A) = T(A_1) + \ldots + T(A_k)$. In addition $T(A_i) \neq 0$ if $i = 1,\dots,n$. Therefore $\operatorname{rk} T(A_i) = 1$ and $T(A_i) \in \mathscr{D}_n(\mathbb{C})$ for all i. In this case $T(A) \in \mathscr{D}_n(\mathbb{C})$ thus $T(\mathscr{D}_n(\mathbb{C})) \subseteq \mathscr{D}_n(\mathbb{C})$.

By [9, Lemma 4.1] the map $T|_{\mathscr{D}_n(\mathbb{C})}$ is monotone with respect to $\overset{\sharp}{\leqslant}$-order. Thus the statement of the Theorem follows from item a). □

4. Corollaries and examples

The results below follow directly from Theorem 3.

COROLLARY 1. *The map T is automatically surjective and $\mathbb{R}$-linear in the conditions of Theorem 3.*

COROLLARY 2. *Let the conditions of Theorem 3 are satisfied. Then the assumptions (a) and (b) are equivalent.*

The following example shows that the assumption of injectivity is indispensable in Theorem 3. Indeed, the following maps are non-standard and non-injective:

EXAMPLE 1. Let $\|\cdot\|$ be a norm in the space $M_n(\mathbb{C})$ and $\varepsilon > 0$ is such that ε-neighborhood of the matrix I in norm $\|\cdot\|$ does not contain singular matrices. We define T_ε as follows

$$T_\varepsilon(X) = \max\{1 - \varepsilon^{-1}\|X - I\|, 0\}I.$$

Then $T_\varepsilon\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is non-injective continuous map which is monotone with respect to $\overset{\sharp}{\leqslant}$-order and is not 0-additive. In particular, it is not $\mathbb{R}$-linear and does not have the form as in the statement of the theorem.

Proof. Let $X, Y \in M_n(\mathbb{C})$, $\operatorname{Ind} X = 1$ and $X \overset{\sharp}{\leqslant} Y$. Let us show that $T_\varepsilon(X) \overset{\sharp}{\leqslant} T_\varepsilon(Y)$. If X does not belong to the ε-neighborhood of the matrix I then

$$T_\varepsilon(X) = 0 \overset{\sharp}{\leqslant} T_\varepsilon(Y).$$

In the opposite case $\operatorname{rk} X = n$. Hence $X = Y$ and $T_\varepsilon(X) = T_\varepsilon(Y)$.

Moreover, T_ε is not 0-additive. Indeed,

$$T_\varepsilon(E_{11}) + T_\varepsilon(I - E_{11}) = 0 \neq I = T_\varepsilon(I). \quad \square$$

The following example shows us that the assumption of continuity is indispensable in Theorem 3.

EXAMPLE 2. [10, Example 4.2] Let $T\colon M_n(\overline{\mathbb{F}}) \to M_n(\mathbb{F})$ be defined as

$$T(A) = \sum_{\lambda \in \mathbb{F}} (\lambda S_A^2(\lambda) - S_A^3(\lambda)).$$

Then

(1) T is bijective,

(2) T is strongly monotone with respect to the $\overset{\sharp}{\leqslant}$-order,

(3) T is not continuous,

(4) on the whole $M_n(\mathbb{F})$ the map T is not additive, hence T is not of the form described in Theorem 3.

The following example shows that without the continuity assumption the conditions (a) and (b) of Theorem 3 may not be equivalent even for bijective maps.

EXAMPLE 3. Let $n \geqslant 3$, $T\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be defined as

$$T(X) = \begin{cases} E_{12} + E_{23}, & \text{if } X = E_{12}; \\ E_{12}, & \text{if } X = E_{12} + E_{23}; \\ X, & \text{otherwise.} \end{cases}$$

Then T is bijective and strongly monotone with respect to $\overset{\sharp}{<}$-order, but T is not monotone with respect to $\overset{\text{cn}}{\leqslant}$-order.

Proof. Since $E_{12} \overset{\text{cn}}{<} E_{12} + E_{23}$, and $T(E_{12}) = E_{12} + E_{23} \overset{\text{cn}}{\not<} E_{12} = T(E_{12} + E_{23})$, we can conclude that T is not monotone with respect to $\overset{\text{cn}}{\leqslant}$-order.

Let $X \overset{\sharp}{<} Y$. If $T(X) = X$ and $T(Y) = Y$ then there is nothing to prove. On the other hand, if $T(X) \neq X$, then X is nilpotent, $X \neq 0$, which contradict to $X \overset{\sharp}{<} Y$. So $T(Y) \neq Y$, whence $X = 0$ and $T(X) \overset{\sharp}{<} T(Y)$. $\square$

Below we demonstrate that there are bijective strong preservers of $\overset{\text{cn}}{\leqslant}$-order, which are not continuous and are not of the form described in Theorem 3.

EXAMPLE 4. Let $\mathbb{F} = \overline{\mathbb{F}}$ be an algebraically closed field, $n \geqslant 1$. By M we denote $M = \{A \in M_n(\mathbb{F}) \mid k_A(\lambda, n) = 1 \text{ for some } \lambda \in \mathbb{F} \setminus \{0\}\}$. Suppose a map $S\colon M \to M$ is bijective. We define the map T by the following rule:

$$T(X) = \begin{cases} S(X), & \text{if } X \in M; \\ X, & \text{otherwise.} \end{cases}$$

Then T is bijective and strongly monotone with respect to $\overset{\sharp}{<}$- and $\overset{\text{cn}}{<}$-orders.

Proof. Obviously, T is bijective. It is easy to see that $\operatorname{rk} A = n$ for all matrices $A \in M$. Let $A \overset{\text{cn}}{<} B$. Since $\operatorname{rk} A < n$, $A \notin M$, $T(A) = A$. Moreover, if $B \in M$ then $A = 0$ (in another case, the matrix B can be represented as a sum of two non-zero orthogonal matrices, which contradicts to condition $k_B(\lambda, n) = 1$ for some $\lambda \in \mathbb{F} \setminus \{0\}$). Therefore, $T(A) = 0 \overset{\text{cn}}{<} T(B)$, and T is monotone with respect to $\overset{\text{cn}}{<}$-order. Similarly, T is monotone with respect to $\overset{\sharp}{<}$-order. Since T is invertible and an inverse of T has the same form, T is strongly monotone with respect to $\overset{\sharp}{<}$- and $\overset{\text{cn}}{<}$-orders. □

Now we are ready to provide an example of a continuous transformation which preserve the $\overset{\sharp}{<}$-order. This transformation is not injective and is not of the form described in Theorem 3, although it is not "wild".

EXAMPLE 5. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, $f(\lambda) = \lambda e^{\lambda} : \mathbb{F} \to \mathbb{F}$. Then $T(A) = f(A)$ is monotone with respect to $\overset{\sharp}{<}$-order, continuous but not injective, where $f(A)$ is defined as a formal power series in A with the usual convention $A^0 = I$.

Proof. It is routine to check that T is continuous.

Since $f(0) = 0$, then it is straightforward to see that $T(A)$ is a 0-additive map. Therefore, see [9, Lemma 4.1], T is monotone with respect to $\overset{\sharp}{\leqslant}$-order.

We are going to show now that T is also monotone with respect to $\overset{\sharp}{<}$-order, not only $\overset{\sharp}{\leqslant}$-order. So, different comparable matrices can not be mapped to the same matrix.

In order to do this we firstly have to check that the kernel of T is zero. Indeed, it is straightforward that $f(\lambda) = 0$ if and only if $\lambda = 0$. Note that $f'(\lambda) \neq 0$ in the point $\lambda = 0$, here f' is a derivative of f. For any $A \neq 0$ we have that either A has a non-zero eigenvalue or A is nilpotent. If A has a non-zero eigenvalue λ, then consequently the consideration of the Jourdan normal form of A implies that $T(A)$ has a non-zero eigenvalue $f(\lambda)$. If A is nilpotent, then $T(A)$ is similar to the matrix with $f'(0) \neq 0$ in the first superdiagonal (the diagonal above the main diagonal). Therefore, $T(A) = 0$ if and only if $A = 0$.

Now, suppose $A \overset{\sharp}{<} B$. Then $A \perp (B - A)$ and $(B - A) \neq 0$ by [9, Lemma 2.5.1]. Since T is 0-additive map, we have $T(B) = T(A) + T(B - A) \neq T(A)$, and thus $T(A) \overset{\sharp}{<} T(B)$.

T is not injective since f is not injective, indeed,

$$f(-\ln 2) = -\ln 2/2 = -\ln 4/4 = f(-\ln 4). \quad \square$$

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REFERENCES

[1] A. BEN-ISRAEL, T. GREVILLE, *Generalized Inverses: Theory and Applications*, John Wiley and Sons, New York (1974).

[2] I. I. BOGDANOV, A. E. GUTERMAN, *Monotone matrix maps defined by the group inverse and simultaneous diagonalizability*, Matematicheskii Sbornik [in Russian] **198**, 1 (2007), 3–20. Translated by Sb. Math. **198**, 1–2 (2007) 1–16.
[3] J. B. CONWAY, *Functions of One Complex Variable I*, second ed., Springer Verlag, New York (1991).
[4] G. DOLINAR, J. MAROVT, *Star partial order on $B(H)$*, Linear Algebra Appl. **434**, (2011), 319–326.
[5] G. DOLINAR, A. GUTERMAN, J. MAROVT, *Automorphisms of $K(H)$ with respect to the star partial order*, Operators and Matrices **7**, 1 (2013), 225–239.
[6] G. DOLINAR, A. GUTERMAN, J. MAROVT, *Automorphisms of $B(H)$ with respect to the left-star and the right-star partial order*, Mathematical Inequalities and Applications **17**, 2 (2014), 573–589.
[7] M. A. EFIMOV, *Linear matrix transformations that are monotone with respect to the $\overset{\sharp}{\leqslant}$- or $\overset{cn}{\leqslant}$-order*, Fundamentalnaya I Prikladnaya Matematika [in Russian] **13**, 4 (2007), 53–66. Translated by Journal of Mathematical Sciences (New-York) **155**, 6 (2008), 830–838.
[8] M. A. EFIMOV, *Additive matrix maps that are monotone with respect to the orders induced by group inverse*, Fundamentalnaya i Prikladnaya Matematika [in Russian] **17**, 6 (2011/2012), 23–40. Translated by Journal of Mathematical Sciences (New-York) **193**, 5 (2013), 659–670.
[9] M. A. EFIMOV, A. E. GUTERMAN, *Monotone maps on index one matrices*, Zapiski POMI [in Russian] **405**, (2012), 67–96. Translated by Journal of Mathematical Sciences (New-York) **191**, 1 (2013), 36–51
[10] M. A. EFIMOV, A. E. GUTERMAN, *Monotone maps on diagonalizable matrices*, Mathematical Inequalities and Applications **17**, 4 (2014), 1441–1452.
[11] R. E. HARTWIG, *How to partially order regular elements*, Math. Japonica **25**, 1 (1980), 1–13.
[12] R. E. HARTWIG, S. K. MITRA, *Partial orders based on outer inverses*, Linear Algebra Appl. **176**, (1982), 3–20.
[13] P. LEGIŠA, *Automorphisms of M_n, partially ordered by rank subtractivity ordering*, Linear Algebra Appl. **389**, (2004), 147–158.
[14] P. LEGIŠA, *Automorphisms of M_n, partially ordered by the star order*, Linear and Mult. Algebra. **54**, 3 (2006), 157–188.
[15] S. K. MITRA, *On group inverses and the sharp order*, Linear Algebra Appl. **92**, (1987), 17–37.
[16] K. S. S. NAMBOORIPAD, *The natural partial order on a regular semigroup*, Proceedings of the Edinburgh Math. Soc. **23**, (1980), 249–260.
[17] P. G. OVCHINNIKOV, *Automorphisms of the poset of skew projections*, J. of Functional Analysis **115**, (1993), 184–189.
[18] C. R. RAO, S. K. MITRA, *Generalized Inverse of Matrices and its Applications*, Wiley, New York, (1971).
[19] P. ŠEMRL, *Order-preserving maps on the poset of idempotent matrices*, Acta Sci. Math. (Szeged) **69**, (2003), 481–490.
[20] P. ŠEMRL, *Non-linear commutativity preserving maps*, Acta Sci. Math. (Szeged) **71**, (2005), 781–819.
[21] P. ŠEMRL, *Automorphisms of $B(H)$ with respect to minus partial order*, J. Math. Anal. Appl. **369**, (2010), 205–213.
[22] B. V. SHABAT, *Introduction to complex analysis*, Nauka, Moscow, (1969), [in Russian]. Translation is available at `http://en.books4study.biz/f3067.html`, `http://math.stanford.edu/~ryzhik/shabat-all.pdf`.

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