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COEFFICIENT ESTIMATES OF NEW CLASSES OF q–STARLIKE AND q–CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract. We introduce new classes of q-starlike and q-convex functions of complex order involving the q-derivative operator defined in the open unit disc. Furthermore, we find estimates on the coefficients for second and third coefficients of these classes.

1. Introduction

Simply, quantum calculus is ordinary classical calculus without the notion of limits. It defines q-calculus and h-calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Recently, the area of q-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [15, 14]. He was the first to develop q-integral and q-derivative in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. Aral and Gupta [8, 9, 10] defined and studied the q-analogue of Baskakov Durrmeyer operator which is based on q-analogue of beta function. Another important q-generalization of complex operators is q-Picard and q-Gauss-Weierstrass singular integral operators discussed in [4, 5, 7]. Mohammed and Darus [18] studied approximation and geometric properties of these q-operators in some subclasses of analytic functions in compact disk. These q-operators are defined by using convolution of normalized analytic functions and q-hypergeometric functions, where several interesting results are obtained (see also [3, 2]). A comprehensive study on applications of q-calculus in operator theory may be found in [11].

Let $\mathscr{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written as $f \prec g$ in $\mathbb{U}$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$,

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if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence holds (see [17] and [12]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For function $f \in \mathscr{A}$ given by (1.1) and $0 < q < 1$, the q-derivative of a function f is defined by (see [15, 14])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0), \tag{1.2}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.2), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \tag{1.3}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{1.4}$$

As $q \to 1^-$, $[k]_q \to k$. For a function $h(z) = z^k$, we observe that

$$D_q(h(z)) = D_q\left(z^k\right) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q \, z^{k-1},$$

$$\lim_{q \to 1} (D_q h(z)) = \lim_{q \to 1} \left([k]_q \, z^{k-1}\right) = k z^{k-1} = h'(z),$$

where h' is the ordinary derivative.

As a right inverse, Jackson [14] introduced the q-integral

$$\int_0^z f(t) \, d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f\left(zq^k\right),$$

provided that the series converges. For a function $h(z) = z^k$, we observe that

$$\int_0^z h(t) \, d_q t = \int_0^z t^k d_q t = \frac{z^{k+1}}{[k+1]_q} (k \neq -1),$$

$$\lim_{q \to 1^-} \int_0^z h(t) \, d_q t = \lim_{q \to 1^-} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z h(t) \, dt,$$

where $\int_0^z h(t) \, dt$ is the ordinary integral.

Making use of the the q-derivative $D_q f(z)$, we introduce the subclasses $\mathscr{S}_q(\alpha)$ and $\mathscr{C}_q(\alpha)$ of the class $\mathscr{A}$ for $0 \leqslant \alpha < 1$ which are defined by

$$\mathscr{S}_q^*(\alpha) = \left\{ f \in \mathscr{A} : \operatorname{Re} \frac{z D_q f(z)}{f(z)} > \alpha, \, z \in \mathbb{U} \right\}, \tag{1.5}$$

$$\mathscr{C}_q(\alpha)=\left\{f\in\mathscr{A}:\mathrm{Re}\,\frac{D_q(zD_qf(z))}{D_qf(z)}>\alpha, z\in\mathbb{U}\right\}. \tag{1.6}$$

We note that

$$f\in\mathscr{C}_q(\alpha)\Leftrightarrow zD_qf\in\mathscr{S}_q^*(\alpha), \tag{1.7}$$

and

$$\lim_{q\to1^-}\mathscr{S}_q^*(\alpha)=\left\{f\in\mathscr{A}:\lim_{q\to1^-}\mathrm{Re}\,\frac{zD_qf(z)}{f(z)}>\alpha, z\in\mathbb{U}\right\}=\mathscr{S}^*(\alpha),$$
$$\lim_{q\to1^-}\mathscr{C}_q(\alpha)=\left\{f\in\mathscr{A}:\lim_{q\to1^-}\mathrm{Re}\,\frac{D_q(zD_qf(z))}{D_qf(z)}>\alpha, z\in\mathbb{U}\right\}=\mathscr{C}(\alpha),$$

where $\mathscr{S}(\alpha)$ and $\mathscr{C}(\alpha)$ are, respectively, the classes of starlike of order α and convex of order α in $\mathbb{U}$ (see Robertson [22]).

By making use of the q-derivative of a function $f\in\mathscr{A}$ and the principle of subordination, we now introduce the following classes of q-starlike and q-convex analytic functions of complex order.

DEFINITION 1. Let $\mathscr{P}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0)=1$ and $\mathrm{Re}\,\phi(z)>0$ for $z\in\mathbb{U}$. A function $f\in\mathscr{A}$ is said to be in the class $\mathscr{S}_{q,b}(\phi)$ if it satisfies the following subordination condition:

$$1+\frac{1}{b}\left[\frac{zD_qf(z)}{f(z)}-1\right]\prec\phi(z)\ \ (b\in\mathbb{C}^*;\phi\in\mathscr{P}). \tag{1.8}$$

DEFINITION 2. A function $f\in\mathscr{A}$ is said to be in the class $\mathscr{C}_{q,b}(\phi)$ if it satisfies the following subordination condition:

$$1+\frac{1}{b}\left[\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right]\prec\phi(z)\,(b\in\mathbb{C}^*;\phi\in\mathscr{P}). \tag{1.9}$$

We note that:

(i) $\lim_{q\to1^-}\mathscr{S}_{q,b}(\phi)=\mathscr{S}_b(\phi)$ and $\lim_{q\to1^-}\mathscr{C}_{q,b}(\phi)=\mathscr{C}_b(\phi)\,(b\in\mathbb{C}^*)$ (Ravichandran et al. [21]);

(ii) $\lim_{q\to1^-}\mathscr{S}_{q,1}(\phi)=\mathscr{S}^*(\phi)$ and $\lim_{q\to1^-}\mathscr{C}_{q,1}(\phi)=\mathscr{C}(\phi)$ (Ma and Minda [16]);

(iii) $\lim_{q\to1^-}\mathscr{S}_{q,b}\left(\frac{1+(1-2\alpha)z}{1-z}\right)=\mathscr{S}_\alpha^*(b)$ and $\lim_{q\to1^-}\mathscr{C}_{q,b}\left(\frac{1+(1-2\alpha)z}{1-z}\right)$ $=\mathscr{C}_\alpha(b)\ \ (b\in\mathbb{C}^*,\ 0\leqslant\alpha<1)$ (Frasin [13]);

(iv) $\lim_{q\to1^-}\mathscr{S}_{q,b}\left(\frac{1+z}{1-z}\right)=\mathscr{S}^*(b)\,(b\in\mathbb{C}^*)$ (Nasr and Aouf [20] and Wiatrowski [23]);

(v) $\lim_{q\to1^-}\mathscr{C}_{q,b}\left(\frac{1+z}{1-z}\right)=\mathscr{C}(b)\,(b\in\mathbb{C}^*)$ (Nasr and Aouf [19] and Wiatrowski [23]);

(vi) $\lim_{q\to 1^-} \mathscr{S}_{q,1-\alpha}\left(\frac{1+z}{1-z}\right) = \mathscr{S}^*(\alpha)$ and $\lim_{q\to 1^-} \mathscr{C}_{q,1-\alpha}\left(\frac{1+z}{1-z}\right) = \mathscr{C}(\alpha)$ $(0 \leqslant \alpha < 1)$ (Robertson [22]);

(vii) $\lim_{q\to 1^-} \mathscr{S}_{q,be^{-i\theta}\cos\theta}\left(\frac{1+z}{1-z}\right) = \mathscr{S}^\theta(b)$ and $\lim_{q\to 1^-} \mathscr{C}_{q,be^{-i\theta}\cos\theta}\left(\frac{1+z}{1-z}\right) = \mathscr{C}^\theta(b)$ $(|\theta| < \frac{\pi}{2},\ b \in \mathbb{C}^*)$ (Al-Oboudi and Haidan [1] and Aouf et al. [6]).

In order to establish our main results, we need the following lemma.

LEMMA 1. [16] *If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $\mathbb{U}$ and μ is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \leqslant 2\max\{1; |2\mu - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z}.$$

LEMMA 2. [16] *If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with a positive real part in $\mathbb{U}$, then*

$$\left|c_2 - \nu c_1^2\right| \leqslant \begin{cases} -4\nu + 2 & \text{if} \quad \nu \leqslant 0, \\ 2 & \text{if} \quad 0 \leqslant \nu \leqslant 1, \\ 4\nu - 2 & \text{if} \quad \nu \geqslant 1, \end{cases}$$

when $\upsilon < 0$ or $\nu > 1$, the equality holds if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z} \qquad (0 \leqslant \lambda \leqslant 1)$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 \leqslant 2 \qquad \left(0 \leqslant \nu \leqslant \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \leqslant 2 \qquad \left(\frac{1}{2} \leqslant \nu \leqslant 1\right).$$

In the present paper, we obtain the Fekete-Szegö inequalities for the classes $\mathscr{S}_{q,b}(\phi)$ and $\mathscr{C}_{q,b}(\phi)$. The motivation of this paper is to generalize previously results.

2. Main results

Unless otherwise mentioned, we assume throughout this paper that the function $0 < q < 1$, $b \in \mathbb{C}^*$, $\phi \in \mathscr{P}$, $[k]_q$ is given by (1.4) and $z \in \mathbb{U}$.

THEOREM 1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ *with* $B_1 \neq 0$. *If* f *given by (1.1) belongs to the class* $\mathscr{S}_{q,b}(\phi)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{|B_1 b|}{[3]_q - 1} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1 b}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)\right|\right\}. \tag{2.1}$$

The result is sharp.

Proof. If $f \in \mathscr{S}_{q,b}(\phi)$, then there is a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$\frac{zD_q f(z)}{f(z)} = \phi(\omega(z)). \tag{2.2}$$

Define the function $p(z)$ by

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + \ldots. \tag{2.3}$$

Since $\omega(z)$ is a Schwarz function, we see that $\operatorname{Re} p(z) > 0$ and $p(0) = 1$. Therefore,

$$\begin{aligned} \phi(\omega(z)) &= \phi\left(\frac{p(z) - 1}{p(z) + 1}\right) \\ &= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \ldots\right]\right) \\ &= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots. \end{aligned} \tag{2.4}$$

Now by substituting (2.4) in (2.2), we have

$$1 + \frac{1}{b}\left[\frac{zD_q f(z)}{f(z)} - 1\right] = 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots.$$

From this equation, we obtain

$$\frac{[2]_q - 1}{b} a_2 = \frac{B_1 c_1}{2},$$

$$\frac{[3]_q - 1}{b} a_3 - \frac{[2]_q - 1}{b} a_2^2 = \frac{B_1 c_2}{2} - \frac{B_1 c_1^2}{4} + \frac{B_2 c_1^2}{4},$$

or, equivalently,

$$a_2 = \frac{B_1 c_1 b}{2\left([2]_q - 1\right)},$$

$$a_3 = \frac{B_1 b}{2\left([3]_q - 1\right)} \left\{ c_2 - \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_q - 1}\right] c_1^2 \right\}.$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{B_1 b}{2\left([3]_q - 1\right)} \left\{ c_2 - \nu c_1^2 \right\}, \tag{2.5}$$

where

$$\nu = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)\right]. \tag{2.6}$$

Our result now follows by an application of Lemma 1. The result is sharp for the functions

$$\frac{zD_q f(z)}{f(z)} = \phi\left(z^2\right) \quad \text{and} \quad \frac{zD_q f(z)}{f(z)} = \phi(z).$$

This completes the proof of Theorem 1. □

Similarly, we can prove the following theorem for the class $\mathscr{C}_{q,b}(\phi)$.

THEOREM 2. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathscr{C}_{q,b}(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{|B_1 b|}{[3]_q\left([3]_q - 1\right)} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1 b}{[2]_q - 1}\left(1 - \frac{[3]_q\left([3]_q - 1\right)}{\left([2]_q\right)^2\left([2]_q - 1\right)}\mu\right)\right|\right\}. \tag{2.7}$$

The result is sharp.

Taking $q \to 1^-$ in Theorem 1, we obtain the following result for the functions belonging to the class $\mathscr{S}_b(\phi)$ which improves the result of Ravichandran et al. [21, Theorem 4.1].

COROLLARY 1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathscr{S}_b(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{|B_1||b|}{2} \max\left\{1; \left|\frac{B_2}{B_1} + (1 - 2\mu) B_1 b\right|\right\}.$$

The result is sharp.

Taking $q \to 1^-$ in Theorem 2, we obtain the following result for the functions belonging to the class $\mathscr{C}_b(\phi)$.

COROLLARY 2. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ *with* $B_1 \neq 0$. *If* f *given by* (1.1) *belongs to the class* $\mathscr{C}_b(\phi)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{|B_1|\,|b|}{6} \max\left\{1; \left|\frac{B_2}{B_1} + \left(1 - \frac{3}{2}\mu\right) B_1 b\right|\right\}.$$

The result is sharp.

THEOREM 3. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ *with* $B_1 > 0$ *and* $B_2 \geqslant 0$. *Let*

$$\sigma_1 = \frac{\left([2]_q - 1\right) b B_1^2 + \left([2]_q - 1\right)^2 (B_2 - B_1)}{\left([3]_q - 1\right) b B_1^2}, \tag{2.8}$$

$$\sigma_2 = \frac{\left([2]_q - 1\right) b B_1^2 + \left([2]_q - 1\right)^2 (B_2 + B_1)}{\left([3]_q - 1\right) b B_1^2}, \tag{2.9}$$

$$\sigma_3 = \frac{\left([2]_q - 1\right) b B_1^2 + \left([2]_q - 1\right)^2 B_2}{\left([3]_q - 1\right) b B_1^2}. \tag{2.10}$$

If f *given by* (1.1) *belongs to the class* $\mathscr{S}_{q,b}(\phi)$ *with* $b > 0$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} \dfrac{B_2 b}{[3]_q - 1} + \dfrac{B_1^2 b^2}{[2]_q - 1}\left(\dfrac{1}{[3]_q - 1} - \dfrac{\mu}{[2]_q - 1}\right) & \text{if } \mu \leqslant \sigma_1, \\ \dfrac{B_1 b}{[3]_q - 1} & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ -\dfrac{B_2 b}{[3]_q - 1} - \dfrac{B_1^2 b^2}{[2]_q - 1}\left(\dfrac{1}{[3]_q - 1} - \dfrac{\mu}{[2]_q - 1}\right) & \text{if } \mu \geqslant \sigma_2, \end{cases} \tag{2.11}$$

Further, if $\sigma_1 \leqslant \mu \leqslant \sigma_3$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{\left([2]_q - 1\right)^2}{\left([3]_q - 1\right) B_1^2 b}\left[B_1 - B_2 - \frac{B_1^2 b}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)\right] |a_2|^2 \leqslant \frac{B_1 b}{[3]_q - 1}. \tag{2.12}$$

If $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{\left([2]_q - 1\right)^2}{\left([3]_q - 1\right) B_1^2 b}\left[B_1 + B_2 + \frac{B_1^2 b}{[2]_q - 1}\left(1 - \frac{[3]_q - 1}{[2]_q - 1}\mu\right)\right] |a_2|^2 \leqslant \frac{B_1 b}{[3]_q - 1}. \tag{2.13}$$

The result is sharp.

Proof. Applying Lemma 2 to (2.5) and (2.6), we can obtain our results. To show that the bounds are sharp, we define the functions $\mathscr{K}_{\phi n}\,(n=2,3,4,\ldots)$ by

$$1+\frac{1}{b}\left[\frac{zD_q\mathscr{K}_{\phi n}(z)}{\mathscr{K}_{\phi n}(z)}-1\right]=\phi\left(z^{n-1}\right),\quad \mathscr{K}_{\phi n}(0)=0=\mathscr{K}_{\phi n}'(0)-1$$

and the functions $\mathscr{F}_\lambda$ and $\mathscr{G}_\lambda$ $(0\leqslant\lambda\leqslant 1)$ by

$$1+\frac{1}{b}\left[\frac{zD_q\mathscr{F}_\lambda(z)}{\mathscr{F}_\lambda(z)}-1\right]=\phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),\quad \mathscr{F}_\lambda(0)=0=\mathscr{F}_\lambda'(0)-1$$

and

$$1+\frac{1}{b}\left[\frac{zD_q\mathscr{G}_\lambda(z)}{\mathscr{G}_\lambda(z)}-1\right]=\phi\left(-\frac{1+\lambda z}{z(z+\lambda)}\right),\quad \mathscr{G}_\lambda(0)=0=\mathscr{G}_\lambda'(0)-1.$$

Clearly, the functions $\mathscr{K}_{\phi n},\mathscr{F}_\lambda$ and $\mathscr{G}_\lambda\in\mathscr{S}_{q,b}(\phi)$. If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality holds if and only if f is $\mathscr{K}_{\phi 2}$, or one of its rotations.When $\sigma_1<\mu<\sigma_2$, the equality holds if and only if f is $\mathscr{K}_{\phi 3}$, or one of its rotations. If $\mu=\sigma_1$, then the equality holds if and only if f is $\mathscr{F}_\lambda$, or one of its rotations. If $\mu=\sigma_2$, then the equality holds if and only if f is $\mathscr{G}_\lambda$, or one of its rotations. □

Similarly, we can obtain the following theorem

THEOREM 4. *Let* $\phi(z)=1+B_1z+B_2z^2+\ldots$ *with* $B_1>0$ *and* $B_2\geqslant 0$. *Let*

$$\chi_1=\frac{\left([2]_q\right)^2\left([2]_q-1\right)\left[bB_1^2+\left([2]_q-1\right)(B_2-B_1)\right]}{\left([3]_q-1\right)bB_1^2},$$

$$\chi_2=\frac{\left([2]_q\right)^2\left([2]_q-1\right)\left[bB_1^2+\left([2]_q-1\right)(B_2+B_1)\right]}{\left([3]_q-1\right)bB_1^2},$$

$$\chi_3=\frac{\left([2]_q\right)^2\left([2]_q-1\right)\left[bB_1^2+\left([2]_q-1\right)B_2\right]}{\left([3]_q-1\right)bB_1^2}.$$

If f *given by* (1.1) *belongs to* $\mathscr{C}_{q,b}(\phi)$ *with* $b>0$, *then*

$$\left|a_3-\mu a_2^2\right|\leqslant\begin{cases}\frac{B_2b}{[3]_q\left([3]_q-1\right)}+\frac{B_1^2b^2}{[3]_q\left([3]_q-1\right)\left([2]_q-1\right)}\left(1-\frac{[3]_q\left([3]_q-1\right)}{\left([2]_q\right)^2\left([2]_q-1\right)}\mu\right) & \text{if } \mu\leqslant\chi_1,\\ \frac{B_1b}{[3]_q\left([3]_q-1\right)} & \text{if } \chi_1\leqslant\mu\leqslant\chi_2,\\ -\frac{B_2b}{[3]_q\left([3]_q-1\right)}-\frac{B_1^2b^2}{[3]_q\left([3]_q-1\right)\left([2]_q-1\right)}\left(1-\frac{[3]_q\left([3]_q-1\right)}{\left([2]_q\right)^2\left([2]_q-1\right)}\mu\right) & \text{if } \mu\geqslant\chi_2,\end{cases}$$

Further, if $\chi_1 \leqslant \mu \leqslant \chi_3$, then

$$\left|a_3-\mu a_2^2\right|+\frac{\left([2]_q\right)^2\left([2]_q-1\right)^2}{[3]_q\left([3]_q-1\right)B_1^2 b}\left[B_1-B_2-\frac{B_1^2 b}{[2]_q-1}\left(1-\frac{[3]_q\left([3]_q-1\right)}{\left([2]_q\right)^2\left([2]_q-1\right)}\mu\right)\right]|a_2|^2 \leqslant \frac{B_1 b}{[3]_q\left([3]_q-1\right)}.$$

If $\chi_3 \leqslant \mu \leqslant \chi_2$, then

$$\left|a_3-\mu a_2^2\right|+\frac{\left([2]_q\right)^2\left([2]_q-1\right)^2}{[3]_q\left([3]_q-1\right)B_1^2 b}\left[B_1+B_2+\frac{B_1^2 b}{[2]_q-1}\left(1-\frac{[3]_q\left([3]_q-1\right)}{\left([2]_q\right)^2\left([2]_q-1\right)}\mu\right)\right]|a_2|^2 \leqslant \frac{B_1 b}{[3]_q\left([3]_q-1\right)}.$$

The result is sharp.

Taking $q \to 1^-$ in Theorem 3, we obtain the following result for the functions belonging to the class $\mathscr{S}_b(\phi)$.

COROLLARY 3. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 > 0$ and $B_2 \geqslant 0$. Let*

$$\sigma_4 = \frac{bB_1^2 + (B_2 - B_1)}{2bB_1^2}, \quad \sigma_5 = \frac{bB_1^2 + (B_2 + B_1)}{2bB_1^2}, \quad \sigma_6 = \frac{bB_1^2 + B_2}{2bB_1^2}.$$

If f given by (1.1) belongs to the class $\mathscr{S}_b(\phi)$ with $b > 0$, then

$$\left|a_3-\mu a_2^2\right| \leqslant \begin{cases} \frac{B_2 b}{2} + \frac{B_1^2 b^2}{2}(1-2\mu) & \text{if } \mu \leqslant \sigma_4, \\ \frac{B_1 b}{2} & \text{if } \sigma_4 \leqslant \mu \leqslant \sigma_5, \\ -\frac{B_2 b}{2} - \frac{B_1^2 b^2}{2}(1-2\mu) & \text{if } \mu \geqslant \sigma_5, \end{cases}$$

Further, if $\sigma_4 \leqslant \mu \leqslant \sigma_6$, then

$$\left|a_3-\mu a_2^2\right| + \frac{1}{2B_1^2 b}\left[B_1 - B_2 - B_1^2 b(1-2\mu)\right]|a_2|^2 \leqslant \frac{B_1 b}{2}.$$

If $\sigma_6 \leqslant \mu \leqslant \sigma_5$, then

$$\left|a_3-\mu a_2^2\right| + \frac{1}{2B_1^2 b}\left[B_1 + B_2 + B_1^2 b(1-2\mu)\right]|a_2|^2 \leqslant \frac{B_1 b}{2}.$$

The result is sharp.

Taking $q \to 1^-$ in Theorem 4, we obtain the following result for the functions belonging to the class $\mathscr{C}_b(\phi)$.

COROLLARY 4. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ *with* $B_1 > 0$ *and* $B_2 \geqslant 0$. *Let*

$$\chi_4 = \frac{2\left[bB_1^2 + B_2 - B_1\right]}{bB_1^2}, \quad \chi_5 = \frac{2\left[bB_1^2 + B_2 + B_1\right]}{bB_1^2}, \quad \chi_6 = \frac{2\left[bB_1^2 + B_2\right]}{bB_1^2}.$$

If f *given by* (1.1) *belongs to the class* $\mathscr{C}_b(\phi)$ *with* $b > 0$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \begin{cases} \frac{B_2 b}{6} + \frac{B_1^2 b^2}{6}\left(1 - \frac{3}{2}\mu\right) & \text{if } \mu \leqslant \chi_4, \\ \frac{B_1 b}{6} & \text{if } \chi_4 \leqslant \mu \leqslant \chi_5, \\ -\frac{B_2 b}{6} - \frac{B_1^2 b^2}{6}\left(1 - \frac{3}{2}\mu\right) & \text{if } \mu \geqslant \chi_5, \end{cases}$$

Further, if $\chi_4 \leqslant \mu \leqslant \chi_6$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2}{3B_1^2 b}\left[B_1 - B_2 - B_1^2 b\left(1 - \frac{3}{2}\mu\right)\right]|a_2|^2 \leqslant \frac{B_1 b}{6}.$$

If $\chi_3 \leqslant \mu \leqslant \chi_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2}{3B_1^2 b}\left[B_1 + B_2 + B_1^2 b\left(1 - \frac{3}{2}\mu\right)\right]|a_2|^2 \leqslant \frac{B_1 b}{6}.$$

The result is sharp.

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