Journal of
Mathematical
Inequalities
Volume 10, Number 1 (2016), 147–157 doi:10.7153/jmi-10-12

A REFINEMENT OF THE DIFFERENCE BETWEEN TWO INTEGRAL MEANS IN TERMS OF THE CUMULATIVE VARIATION AND APPLICATIONS

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(*Communicated by A. Aglić Aljinović*)

Abstract. In this paper, we establish a refinement of the difference between two integral means for functions of bounded variation in terms of the cumulative variation function. The applications for probability density functions are also given.

1. Introduction

In the past few years, many authors have considered various generalizations of some kinds of integral inequalities, which give explicit error bounds for some known and some new quadrature formulae. For example, Dragomir [7] established the following Ostrowski's inequality for mappings of bounded variation:

THEOREM 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then, for all $x\in[a,b]$, we have the following inequality*

$$\left|f(x)(b-a)-\int_a^b f(t)\mathrm{d}t\right|\leqslant\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f), \tag{1.1}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$. The constant $\frac{1}{2}$ is the best possible.

Recently, the same author [6] considered the following refinements of the inequalities (1.1) in terms of the cumulative variation function and given some applications for selfadjoint operators on complex Hilbert spaces.

Mathematics subject classification (2010): 26D15, 26A45, 60E15.
Keywords and phrases: Ostrowski's inequality, integral mean, cumulative variation, function of bounded variation, probability density function.

THEOREM 1.2. *Let* $f:[a,b]\to\mathbb{C}$ *be a function of bounded variation on* $[a,b]$. *Then*

$$\left|f(x)(b-a)-\int_a^b f(t)\mathrm{d}t\right| \leqslant \int_a^x\left(\bigvee_t^x(f)\right)\mathrm{d}t+\int_x^b\left(\bigvee_x^t(f)\right)\mathrm{d}t \tag{1.2}$$

$$\leqslant (x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)$$

$$\leqslant \begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f),\\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right](b-a),\end{cases}$$

and

$$\left|f(x)(b-a)-\int_a^b f(t)\mathrm{d}t\right| \leqslant \int_a^x\left(\bigvee_t^x(|f(x)-f|)\right)\mathrm{d}t+\int_x^b\left(\bigvee_x^t(|f(x)-f|)\right)\mathrm{d}t \tag{1.3}$$

$$\leqslant (x-a)\bigvee_a^x(|f(x)-f|)+(b-x)\bigvee_x^b(|f(x)-f|)$$

$$\leqslant (x-a)\bigvee_a^x(f)+(b-x)\bigvee_x^b(f)$$

$$\leqslant \begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f),\\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right](b-a),\end{cases}$$

for any $x\in[a,b]$.

In [2], Barnett et al. compared the difference of two integral means for absolutely continuous mapping whose first derivatives are in $L_\infty[a,b]$ and gave several applications for probability density functions, special means, Jeffrey's divergence and continuous streams. In [11], Hwang and Dragomir compared the difference of two integral means as in the following Theorem 1.3 for the functions with bounded variation. The obtained result is also a generalization of (1.1) and has been applied to probability density functions. For related results, see [1], [3]–[5], [10], [12]–[25], [26] and the references therein.

THEOREM 1.3. *Let* $f:[a,b]\to\mathbb{R}$ *be a function of bounded variation on* $[a,b]$. *Then, for all* $a\leqslant x<y\leqslant b$, *we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{1}{y-x}\int_x^y f(s)\mathrm{d}s\right| \tag{1.4}$$
$$\leqslant\frac{x-a}{b-a}\bigvee_a^x(f)+\max\left\{\frac{x-a}{b-a},\frac{b-y}{b-a}\right\}\bigvee_x^y(f)+\frac{b-y}{b-a}\bigvee_y^b(f).$$

The inequality (1.4) *is sharp.*

Motivated by the above research, the purpose of this paper is to provide a refinement of the inequality (1.4) for functions of bounded variation in terms of the cumulative variation function and give several applications for probability density functions.

2. The refinement of the difference between two integral means

As in [6, 9], for a function of bounded variation $v:[a,b]\to\mathbb{C}$, we define the Cumulative Variation Function (CVF) $V:[a,b]\to[0,\infty]$ by

$$V(t):=\bigvee_a^t(v)$$

the total variation of v on the interval $[a,t]$ with $t\in[a,b]$.

It's known that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous in a point $c\in[a,b]$ if and only if the generating function v is continuing in that point. If v is Lipschitzian with the constant $L>0$, i.e.,

$$|v(t)-v(s)|\leqslant L|t-s| \text{ for any } t,s\in[a,b]$$

then V is also Lipschitzian with the same constant.

The following lemma is of interest in itself as well, see also [8, 9].

LEMMA 2.1. *Let $f,u:[a,b]\to\mathbb{R}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)\mathrm{d}u(t)\right|\leqslant\int_a^b|f(t)|\mathrm{d}\left(\bigvee_a^t(u)\right)\leqslant\max_{t\in[a,b]}|f(t)|\bigvee_a^b(u). \tag{2.1}$$

The following result may be stated.

THEOREM 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation on $[a,b]$. Then, for all $a\leqslant x<y\leqslant b$, we have the inequalities*

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{1}{y-x}\int_x^y f(s)\mathrm{d}s\right| \tag{2.2}$$
$$\leqslant\frac{1}{b-a}\int_a^x\left(\bigvee_s^x(f)\right)\mathrm{d}s+\left(\frac{1}{y-x}-\frac{1}{b-a}\right)\left[\int_x^{s_0}\left(\bigvee_x^s(f)\right)\mathrm{d}s+\int_{s_0}^y\left(\bigvee_s^y(f)\right)\mathrm{d}s\right]$$
$$+\frac{1}{b-a}\int_y^b\left(\bigvee_y^s(f)\right)\mathrm{d}s$$

$$\leqslant \frac{x-a}{b-a}\bigvee_a^x(f)+\left(\frac{1}{y-x}-\frac{1}{b-a}\right)\left[(s_0-x)\bigvee_x^{s_0}(f)+(y-s_0)\bigvee_{s_0}^y(f)\right]+\frac{b-y}{b-a}\bigvee_y^b(f)$$
$$=\frac{x-a}{b-a}\bigvee_a^x(f)+\left[\frac{x-a}{b-a}\bigvee_x^{s_0}(f)+\frac{b-y}{b-a}\bigvee_{s_0}^y(f)\right]+\frac{b-y}{b-a}\bigvee_y^b(f)$$
$$\leqslant\begin{cases}\dfrac{x-a}{b-a}\bigvee_a^x(f)+\max\left\{\dfrac{x-a}{b-a},\dfrac{b-y}{b-a}\right\}\bigvee_x^y(f)+\dfrac{b-y}{b-a}\bigvee_y^b(f),\\ \dfrac{x-a}{b-a}\bigvee_a^x(f)+\left(1-\dfrac{y-x}{b-a}\right)\max\left\{\bigvee_x^{s_0}(f),\bigvee_{s_0}^y(f)\right\}+\dfrac{b-y}{b-a}\bigvee_y^b(f),\end{cases}$$

where $s_0=\dfrac{bx-ay}{(b-a)-(y-x)}$ *is the solution of* $\dfrac{s_0-x}{y-x}+\dfrac{a-s_0}{b-a}=0$*. The inequalities* (2.2) *are sharp.*

Proof. We start with the identity

$$\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{1}{y-x}\int_x^y f(s)\mathrm{d}s \tag{2.3}$$
$$=\int_a^x\frac{a-s}{b-a}\mathrm{d}f(s)+\int_x^y\left(\frac{s-x}{y-x}+\frac{a-s}{b-a}\right)\mathrm{d}f(s)+\int_y^b\frac{b-s}{b-a}\mathrm{d}f(s),$$

which holds for all $a\leqslant x<y\leqslant b$ and $f:[a,b]\to\mathbb{R}$ is a function of bounded variation on $[a,b]$ (see [11]). We denote

$$p(s)=\frac{s-x}{y-x}+\frac{a-s}{b-a},\quad s\in[x,y],$$

then we get that

$$p(x)=\frac{a-x}{b-a}\leqslant 0,\quad p(y)=\frac{b-y}{b-a}\geqslant 0,\quad p'(s)=\frac{1}{y-x}-\frac{1}{b-a}\geqslant 0,$$

for all $a\leqslant x<y\leqslant b$, which implies that $p(s)$ is nondecreasing on $[x,y]$.

Taking the modulus in (2.3) and using the property (2.1), we have

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{1}{y-x}\int_x^y f(s)\mathrm{d}s\right| \tag{2.4}$$
$$\leqslant\left|\int_a^x\frac{a-s}{b-a}\mathrm{d}f(s)\right|+\left|\int_x^y\left(\frac{s-x}{y-x}+\frac{a-s}{b-a}\right)\mathrm{d}f(s)\right|+\left|\int_y^b\frac{b-s}{b-a}\mathrm{d}f(s)\right|$$
$$\leqslant\int_a^x\left|\frac{a-s}{b-a}\right|\mathrm{d}\left(\bigvee_a^s(f)\right)+\int_x^y\left|\frac{s-x}{y-x}+\frac{a-s}{b-a}\right|\mathrm{d}\left(\bigvee_a^s(f)\right)+\int_y^b\left|\frac{b-s}{b-a}\right|\mathrm{d}\left(\bigvee_a^s(f)\right)$$
$$=\int_a^x\frac{s-a}{b-a}\mathrm{d}\left(\bigvee_a^s(f)\right)-\int_x^{s_0}\left(\frac{s-x}{y-x}+\frac{a-s}{b-a}\right)\mathrm{d}\left(\bigvee_a^s(f)\right)$$
$$+\int_{s_0}^y\left(\frac{s-x}{y-x}+\frac{a-s}{b-a}\right)\mathrm{d}\left(\bigvee_a^s(f)\right)+\int_y^b\frac{b-s}{b-a}\mathrm{d}\left(\bigvee_a^s(f)\right)$$

for all $a \leqslant x < y \leqslant b$.

Utilising the integration by parts for Riemann-Stieltjes integral and the fact that $\frac{s_0 - x}{y - x} + \frac{a - s_0}{b - a} = 0$, we get

$$\begin{aligned}
\int_a^x \frac{s-a}{b-a} \mathrm{d}\left(\bigvee_a^s(f)\right) &= \left(\frac{s-a}{b-a}\right) \bigvee_a^s(f)\Big|_{s=a}^x - \frac{1}{b-a}\int_a^x \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&= \frac{x-a}{b-a}\bigvee_a^x(f) - \frac{1}{b-a}\int_a^x \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&= \frac{1}{b-a}\int_a^x \left(\bigvee_a^x(f)\right) \mathrm{d}s - \frac{1}{b-a}\int_a^x \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&= \frac{1}{b-a}\int_a^x \left(\bigvee_s^x(f)\right) \mathrm{d}s,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&-\int_x^{s_0} \left(\frac{s-x}{y-x} + \frac{a-s}{b-a}\right) \mathrm{d}\left(\bigvee_a^s(f)\right) + \int_{s_0}^y \left(\frac{s-x}{y-x} + \frac{a-s}{b-a}\right) \mathrm{d}\left(\bigvee_a^s(f)\right) \qquad (2.6)\\
&= -\left(\frac{s-x}{y-x} + \frac{a-s}{b-a}\right) \bigvee_a^s(f)\Big|_{s=x}^{s_0} + \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_x^{s_0} \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&\quad + \left(\frac{s-x}{y-x} + \frac{a-s}{b-a}\right) \bigvee_a^s(f)\Big|_{s=s_0}^{y} - \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_{s_0}^{y} \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&= \frac{a-x}{b-a}\bigvee_a^x(f) + \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_x^{s_0} \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&\quad + \frac{b-y}{b-a}\bigvee_a^y(f) - \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_{s_0}^{y} \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&= \left(\frac{a-x}{b-a} - \frac{s_0-x}{y-x} - \frac{a-s_0}{b-a}\right)\bigvee_a^x(f) + \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_x^{s_0} \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&\quad + \left(\frac{b-y}{b-a} - \frac{s_0-x}{y-x} + 1 - \frac{a-s_0}{b-a} - 1\right)\bigvee_a^y(f) - \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_{s_0}^{y} \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&= -\left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_x^{s_0} \left(\bigvee_a^x(f)\right) \mathrm{d}s + \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_x^{s_0} \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&\quad + \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_{s_0}^{y} \left(\bigvee_a^y(f)\right) \mathrm{d}s - \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\int_{s_0}^{y} \left(\bigvee_a^s(f)\right) \mathrm{d}s \\
&= \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\left[\int_x^{s_0} \left(\bigvee_x^s(f)\right) \mathrm{d}s + \int_{s_0}^{y} \left(\bigvee_s^y(f)\right) \mathrm{d}s\right]
\end{aligned}$$

and

$$\int_y^b \frac{b-s}{b-a}\,\mathrm{d}\left(\bigvee_a^s(f)\right) = \frac{b-s}{b-a}\bigvee_a^s(f)\Bigg|_{s=y}^b + \frac{1}{b-a}\int_y^b\left(\bigvee_a^s(f)\right)\mathrm{d}s \tag{2.7}$$

$$= -\frac{b-y}{b-a}\bigvee_a^y(f) + \frac{1}{b-a}\int_y^b\left(\bigvee_a^s(f)\right)\mathrm{d}s$$

$$= \frac{1}{b-a}\int_y^b\left(\bigvee_a^s(f)\right)\mathrm{d}s - \frac{1}{b-a}\int_y^b\left(\bigvee_a^y(f)\right)\mathrm{d}s$$

$$= \frac{1}{b-a}\int_y^b\left(\bigvee_y^s(f)\right)\mathrm{d}s$$

for all $a \leqslant x < y \leqslant b$.

Using (2.4)-(2.7), we deduce the first inequality in (2.2).

Since

$$\bigvee_s^x(f) \leqslant \bigvee_a^x(f) \ \text{ for } s \in [a,x], \quad \bigvee_x^s(f) \leqslant \bigvee_x^{s_0}(f) \ \text{ for } s \in [x,s_0],$$
$$\bigvee_s^y(f) \leqslant \bigvee_{s_0}^y(f) \ \text{ for } s \in [s_0,y], \quad \bigvee_y^s(f) \leqslant \bigvee_y^b(f) \ \text{ for } s \in [y,b]$$

and

$$\left(\frac{1}{y-x} - \frac{1}{b-a}\right)(s_0 - x) = \frac{x-a}{b-a}, \quad \left(\frac{1}{y-x} - \frac{1}{b-a}\right)(y - s_0) = \frac{b-y}{b-a},$$

then

$$\frac{1}{b-a}\int_a^x\left(\bigvee_s^x(f)\right)\mathrm{d}s + \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\left[\int_x^{s_0}\left(\bigvee_x^s(f)\right)\mathrm{d}s + \int_{s_0}^y\left(\bigvee_s^y(f)\right)\mathrm{d}s\right]$$
$$+\frac{1}{b-a}\int_y^b\left(\bigvee_y^s(f)\right)\mathrm{d}s$$
$$\leqslant \frac{x-a}{b-a}\bigvee_a^x(f) + \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\left[(s_0 - x)\bigvee_x^{s_0}(f) + (y - s_0)\bigvee_{s_0}^y(f)\right] + \frac{b-y}{b-a}\bigvee_y^b(f)$$
$$= \frac{x-a}{b-a}\bigvee_a^x(f) + \left[\frac{x-a}{b-a}\bigvee_x^{s_0}(f) + \frac{b-y}{b-a}\bigvee_{s_0}^y(f)\right] + \frac{b-y}{b-a}\bigvee_y^b(f)$$

for all $a \leqslant x < y \leqslant b$, which proves the second inequality and the subsequence in (2.2).

The last part is obvious by the max properties and the fact that for $c,d \in \mathbb{R}$ we have

$$\max\{c,d\} = \frac{c+d+|c-d|}{2}.$$

The details are omitted.

To prove the sharpness, we consider the mapping $f:[a,b]\to\mathbb{R}$ (see [11]) given by

$$f(s)=\begin{cases}0, \text{ if } s\in[a,x),\\ 1, \text{ if } s\in[x,y],\\ 0, \text{ if } s\in(y,b].\end{cases}$$

Then f is bounded variation on $[a,b]$, and $\bigvee_a^x(f)=1$, $\bigvee_x^y(f)=0$, $\bigvee_x^{s_0}(f)=0$, $\bigvee_{s_0}^y(f)=0$, $\bigvee_y^b(f)=1$, $\int_a^b f(s)\mathrm{d}s=y-x$, $\int_x^y f(s)\mathrm{d}s=y-x$. Obviously, all the identities hold in (2.2). This completes the proofs. □

The following corollary holds.

COROLLARY 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic mapping on $[a,b]$, then we have the inequalities*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{1}{y-x}\int_x^y f(s)\mathrm{d}s\right| \qquad (2.8)\\ \leqslant&\frac{1}{b-a}\int_a^x|f(x)-f(s)|\mathrm{d}s+\left(\frac{1}{y-x}-\frac{1}{b-a}\right)\left[\int_x^{s_0}|f(s)-f(x)|\mathrm{d}s+\int_{s_0}^y|f(y)-f(s)|\mathrm{d}s\right]\\ &+\frac{1}{b-a}\int_y^b|f(s)-f(y)|\mathrm{d}s\\ \leqslant&\frac{x-a}{b-a}|f(x)-f(a)|+\left(\frac{1}{y-x}-\frac{1}{b-a}\right)[(s_0-x)|f(s_0)-f(x)|+(y-s_0)|f(y)-f(s_0)|]\\ &+\frac{b-y}{b-a}|f(b)-f(y)|\\ =&\frac{x-a}{b-a}|f(x)-f(a)|+\left[\frac{x-a}{b-a}|f(s_0)-f(x)|+\frac{b-y}{b-a}|f(y)-f(s_0)|\right]+\frac{b-y}{b-a}|f(b)-f(y)|\\ \leqslant&\begin{cases}\frac{x-a}{b-a}|f(x)-f(a)|+\max\left\{\frac{x-a}{b-a},\frac{b-y}{b-a}\right\}|f(y)-f(x)|+\frac{b-y}{b-a}|f(b)-f(y)|\\ \frac{x-a}{b-a}|f(x)-f(a)|+\max\{|f(s_0)-f(x)|,|f(y)-f(s_0)|\}+\frac{b-y}{b-a}|f(b)-f(y)|\end{cases}\end{aligned}$$

for all $a\leqslant x<y\leqslant b$.

REMARK 2.1. Corollary 2.1 gives a refinement of [11, Corollary 1].

The case of Lipschitzian mapping is embodied in the following corollary.

COROLLARY 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a L-Lipschitzian mapping on $[a,b]$, that is,*

$$|f(x)-f(y)|\leqslant L|x-y|$$

for positive number L and for all $x,y \in [a,b]$, then we have the inequalities

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s - \frac{1}{y-x}\int_x^y f(s)\mathrm{d}s\right| \qquad (2.9)\\
\leqslant&\frac{(x-a)^2L}{b-a} + \left(\frac{1}{y-x} - \frac{1}{b-a}\right)\left[(s_0-x)^2+(y-s_0)^2\right]L + \frac{(b-y)^2L}{b-a}\\
=&\frac{L}{b-a}\left[(x-a)^2+(x-a)(s_0-x)+(b-y)(y-s_0)+(b-y)^2\right]\\
\leqslant&\begin{cases} \dfrac{L}{b-a}\left[(x-a)^2+\max\{x-a,b-y\}(y-x)+(b-y)^2\right]\\ \dfrac{L}{b-a}\left[(x-a)^2+\max\{s_0-x,y-s_0\}(b-a)+(b-y)^2\right]\end{cases}
\end{aligned}$$

for all $a \leqslant x < y \leqslant b$.

REMARK 2.2. Corollary 2.2 gives a refinement of [11, Corollary 2].

Denote $\|f'\|_{1,[s,t]} = \int_s^t |f'(u)|\mathrm{d}u$, for $s,t \in [a,b]$. Using Theorem 2.1, we have the following corollary immediately.

COROLLARY 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a continuous differentiable on (a,b), and f' is integrable on (a,b), then we have the inequalities*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s - \frac{1}{y-x}\int_x^y f(s)\mathrm{d}s\right| \qquad (2.10)\\
\leqslant&\frac{1}{b-a}\int_a^x \|f'\|_{1,[s,x]}\mathrm{d}s + \left(\frac{1}{y-x}-\frac{1}{b-a}\right)\left[\int_x^{s_0}\|f'\|_{1,[x,s]}\mathrm{d}s + \int_{s_0}^y \|f'\|_{1,[s,y]}\mathrm{d}s\right]\\
&+\frac{1}{b-a}\int_y^b \|f'\|_{1,[y,s]}\mathrm{d}s\\
\leqslant&\frac{x-a}{b-a}\|f'\|_{1,[a,x]} + \left(\frac{1}{y-x}-\frac{1}{b-a}\right)\left[(s_0-x)\|f'\|_{1,[x,s_0]}+(y-s_0)\|f'\|_{1,[s_0,y]}\right]\\
&+\frac{b-y}{b-a}\|f'\|_{1,[y,b]}\\
=&\frac{x-a}{b-a}\|f'\|_{1,[a,x]} + \left[\frac{x-a}{b-a}\|f'\|_{1,[x,s_0]} + \frac{b-y}{b-a}\|f'\|_{1,[s_0,y]}\right] + \frac{b-y}{b-a}\|f'\|_{1,[y,b]}\\
\leqslant&\begin{cases}\dfrac{x-a}{b-a}\|f'\|_{1,[a,x]} + \max\left\{\dfrac{x-a}{b-a},\dfrac{b-y}{b-a}\right\}\|f'\|_{1,[x,y]} + \dfrac{b-y}{b-a}\|f'\|_{1,[y,b]}\\ \dfrac{x-a}{b-a}\|f'\|_{1,[a,x]} + \max\{\|f'\|_{1,[x,s_0]},\|f'\|_{1,[s_0,y]}\} + \dfrac{b-y}{b-a}\|f'\|_{1,[y,b]}\end{cases}
\end{aligned}$$

for all $a \leqslant x < y \leqslant b$.

REMARK 2.3. Corollary 2.3 gives a refinement of [11, Corollary 3].

3. Applications for probability density functions

In the following, assume that $f:[a,b]\to\mathbb{R}^+$ is a probability density function of a certain continuous random variable X and $F(t)=\int_a^t f(x)\mathrm{d}x$ is its cumulative distribution function. We shall give some refinements of [11, Propositions 1–4], respectively.

PROPOSITION 3.1. *Let f and F be as above, then we have*

$$\left|F(t)-\frac{t-a}{b-a}\right| \leqslant \frac{b-t}{b-a}\int_a^t\left(\bigvee_s^t(f)\right)\mathrm{d}s+\frac{t-a}{b-a}\int_t^b\left(\bigvee_t^s(f)\right)\mathrm{d}s \leqslant \frac{(b-t)(t-a)}{b-a}\bigvee_a^b(f) \tag{3.1}$$

provided that f is bounded variation on $[a,b]$.

Proof. Taking $x=a$ and $y=t$ in (2.2), we have the desired inequalities. □

PROPOSITION 3.2. *Let f and F be as above, then we have*

$$\left|F(t)-\frac{t-a}{b-a}\right| \leqslant \frac{b-t}{b-a}\int_a^t|f(t)-f(s)|\mathrm{d}s+\frac{t-a}{b-a}\int_t^b|f(s)-f(t)|\mathrm{d}s \leqslant \frac{(b-t)(t-a)}{b-a}\left(|f(t)-f(a)|+|f(b)-f(t)|\right) \tag{3.2}$$

provided that f is a monotonic mapping on $[a,b]$.

Proof. Taking $x=a$ and $y=t$ in (2.8), we have the desired inequalities. □

PROPOSITION 3.3. *Let f and F be as above, then we have*

$$\left|F(t)-\frac{t-a}{b-a}\right| \leqslant \frac{L}{2}(b-t)(t-a) \tag{3.3}$$

provided that f is a L-Lipschitzian mapping on $[a,b]$.

Proof. Taking $x=a$ and $y=t$ directly in the first inequality of (2.8) (so we have $s_0=a$), and after calculating two integrals, we get inequality (3.3). □

PROPOSITION 3.4. *Let f and F be as above, then we have*

$$\left|F(t)-\frac{t-a}{b-a}\right| \leqslant (t-a)\left[\frac{b-t}{b-a}\|f'\|_{1,[a,t]}+\frac{b-t}{b-a}\|f'\|_{1,[t,b]}\right] \leqslant \frac{(b-t)(t-a)}{b-a}\|f'\|_{1,[a,b]} \tag{3.4}$$

provided that f is a continuous differentiable on (a,b) and f' is integrable on (a,b).

Proof. Taking $x = a$ and $y = t$ in (2.10), we have the desired inequality. □

Acknowledgements. The authors wish to thank the anonymous referees and the editor for their valuable comments.

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(Received May 27, 2014)

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