A CHARACTERIZATION OF MATRIX INEQUALITY $A \geq B \geq C$ VIA KARCHER MEAN

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Abstract. Let $A, B$ and $C$ be three positive definite matrices. In this paper, we show a characterization of $A \geq B \geq C$ via the Karcher mean as follows,

$$
\Lambda(\omega; A^{-p_1}, B^{-p_2}, B'^tC^{p_3}) \leq C,
$$

$$
\Lambda(\omega; C^{-p_1}, B^{-p_2}, B'^tA^{p_3}) \geq A
$$

hold for $t \in [0,1]$, $s \geq 1$, $p_1, p_2 > 0$ and $p_3 > 1$, where $\hat{\omega} = \left( \frac{1}{p_1 + 1}, \frac{1}{p_2 + 1}, \frac{2}{(p_3 - r)s + r - 1} \right)$, $\omega = (w_1, w_2, w_3) = \frac{\hat{\omega}}{||\hat{\omega}||_1}$.

1. Introduction

A capital letter (such as $T$) stands for an $m \times m$ matrix on $\mathbb{C}$. $T > 0$ and $T \geq 0$ mean $T$ is a positive definite matrix and $T$ is a positive semidefinite matrix, respectively.

For $A, B > 0$, the weighted geometric mean of $A$ and $B$ is defined by $A^\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ for $\alpha \in [0,1]$. In order to extend this classical definition to three or more positive definite matrices, T. Ando et al. listed ten properties (such as self-duality, arithmetic-geometric-harmonic mean inequality) which the geometric mean should satisfied. See [1] for details. Also, T. Ando et al. gave a kind of geometric mean by matrix iteration in [1]. In 2006, R. Bhatia et al. showed another kind of geometric mean by argument of minimum in matrix function as follows.

DEFINITION 1.1. ([2, 5, 9, 11]) For $A_1, A_2, \cdots, A_n > 0$ and a probability vector $\omega = (w_1, w_2, \cdots, w_n)$ such that $\sum w_i = 1$ and each $w_i > 0$, the weighted Karcher mean of $A_1, A_2, \cdots, A_n$ is defined by

$$
\Lambda(\omega; A_1, A_2, \cdots, A_n) = \arg \min_{X > 0} \sum_{i=1}^n w_i \delta_2^2 (A_i, X),
$$

where $\arg \min f(X)$ means the point $X_0$ which attains minimum value of the function $f(X)$ and $\delta_2$ is the Riemannian metric $\delta_2 (A, B) = \| \log A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \|_2$.

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It was obtained in [6, 7] that the weighted Karcher mean for \( n \) positive definite matrices \( A_1, A_2, \cdots, A_n \) coincides with the unique positive definite solution of the following Karcher equation:

\[
\sum_{i=1}^{n} w_i \log(X^{1/2} A_i^{-1} X^{1/2}) = 0.
\]

It is easy to prove that \( \Lambda(1 - \alpha, \alpha; A, B) = A^{\# \alpha} B \). See [3] for details.

Recently, C.-S. Lin et al. [8] and J. Yuan et al. [13] proved several characterizations of \( A \geq B \geq C \) via Furuta type inequalities. As a continuation, we shall show a characterization of \( A \geq B \geq C \) via the Karcher mean.

Let us recall an important theorem due to M. Ito [4] related to the main result

\[
\Lambda(\alpha_1, \alpha_2, \cdots, \alpha_n) = A^{\# \alpha_1} B^{\# \alpha_2} \cdots C^{\# \alpha_n} \Rightarrow (2.1)
\]

is obvious by Theorem 1.1. Applying \( C^{-1} \geq B^{-1} \geq A^{-1} \) to (2.1), we have

\[
\Lambda(\alpha; A^{-p_1}, B^{-p_2}, C^{p_3}) \leq C
\]

for each \( i = 1, 2, \cdots, n \), where \( \hat{\alpha} = (1/p_1 + 1/p_2 + 1/n - 1/p_n - 1/n - 1) \).

Theorem 1.1. \((\alpha)\) Let \( A_1, A_2, \cdots, A_n \) be positive definite matrices. If \( A_i \geq A_n \) for each \( i = 1, 2, \cdots, n - 1 \), then

\[
\Lambda(\alpha; A_1^{-p_1}, A_2^{-p_2}, \cdots, A_n^{-p_n}, A_n^{p_n}) \leq A_n
\]

holds for all \( t \in [0, 1], s \geq 1, p_i \geq 0 \) \((i = 1, 2, \cdots, n - 1)\) and \( p_n > 1 \), where \( \hat{\alpha} = (1/p_1 + 1/p_2 + 1/n - 1/p_n - 1/n - 1) \), \( \omega = \hat{\alpha} / \| \hat{\omega} \|_1 \).

2. Main result

Theorem 2.1. For \( A, B, C > 0 \). \( A \geq B \geq C \) if and only if the following two Karcher mean inequalities

\[
\Lambda(\omega; A^{-p_1}, B^{-p_2}, C^{p_3}) \leq C
\]

(2.1)

\[
\Lambda(\omega; C^{-p_1}, B^{-p_2}, C^{p_3}) \geq A
\]

(2.2)

hold for \( t \in [0, 1], s \geq 1, p_1, p_2 > 0 \) and \( p_3 > 1 \), where \( \hat{\alpha} = (1/p_1 + 1/p_2 + 1/n - 1/p_n - 1/n - 1) \), \( \omega = (w_1, w_2, w_3) = \hat{\omega} / \| \hat{\omega} \|_1 \).

Proof. \((\Rightarrow)\) is obvious by Theorem 1.1. Applying \( C^{-1} \geq B^{-1} \geq A^{-1} \) to (2.1), we have

\[
\Lambda(\omega; C^{p_1}, B^{p_2}, A^{p_3}) \leq A^{-1}
\]

(2.3)

Then (2.2) is obtained by self-duality of Karcher mean \( (\Lambda(\omega; A_1^{-1}, A_2^{-1}, \cdots, A_n^{-1})^{-1} = \Lambda(\omega; A_1, A_2, \cdots, A_n), [11]) \).

\((\Leftarrow)\) According to the geometric-harmonic mean inequality of the Karcher mean \( (\sum_{i=1}^{n} w_i A_i^{-1})^{-1} \leq \Lambda(\omega; A_1, A_2, \cdots, A_n), [11]) \) and (2.1), we have

\[
(w_1 A^{p_1} + w_2 B^{p_2} + w_3 (B^{p_3} C^{p_3})^{-1})^{-1} \leq C,
\]

(2.4)
and hence we obtain
\[ w_1A^{p_1} + w_2B^{p_2} + w_3B^{-\frac{1}{t}}A\Lambda \geq C^{-1}. \] (2.5)

Let \( t = 1 \) and \( p_3 \to 1^+ \), then for fixed \( p_1, p_2 \) and \( s \), we have \( w_1 = \frac{1}{p_1 + 1 + (p_3 - 1)s} \to 0 \), \( w_2 = \frac{1}{p_1 + 1 + (p_3 - 1)s} \to 0 \), \( w_3 = \frac{(p_3 - 1)s}{p_1 + 1 + (p_3 - 1)s} \to 1 \). Thus, \( B^{-1}B_\alpha C^{-1} \geq C^{-1} \) holds by (2.5). By the definition of \( \alpha \) and \( s \geq 1 \), we have \( (B^{-1}C^{-1}B^{1})^{s-1} \geq 1 \), which ensures \( B \geq C \).

Similarly, applying the arithmetic-geometric mean inequality \((\Lambda(\omega;A_1,A_2,\cdots,A_n) \leq \sum_{i=1}^{n} w_iA_i, [12])\) to (2.2), we have
\[ w_1C^{-p_1} + w_2B^{-p_2} + w_3B_\alpha A^{p_3} \geq A. \] (2.6)

Let \( t = 1 \) and \( p_3 \to 1^+ \), then for fixed \( p_1, p_2 \) and \( s \), we have \( w_1 \to 0 \), \( w_2 \to 0 \), \( w_3 \to 1 \). Thus, \( B_\alpha A \geq A \) holds by (2.6). By the definition of \( \alpha \) and \( s \geq 1 \), \( (B^{-1}A B^{1})^{s-1} \geq 1 \) is obtained, which ensures \( A \geq B \). Hence, the proof is completed. \( \square \)

**Corollary 2.1.** For \( A, B > 0 \). \( A \geq B \) if and only if \( \Lambda(\omega;A^{-p_1},A^{-p_2},A_\alpha B^{p_3}) \leq B \) hold for \( t \in [0,1] \), \( s \geq 1 \), \( p_1, p_2 > 0 \) and \( p_3 > 1 \), where \( \omega = (\frac{1}{p_1 + 1}, \frac{1}{p_2 + 1}, \frac{1}{(p_3 - t)s + t - 1}) \), \( \omega = (w_1,w_2,w_3) = \frac{\omega}{\|\omega\|_1} \).

**Proof.** Let \( B = A \) and \( C = B \) in (2.1) of Theorem 2.1. \( \square \)

Next, we give another proof of the main result in [8] by our method above.

**Theorem 2.2.** ([8]) For \( A, B, C > 0 \). \( A \geq B \geq C \) if and only if the following two inequalities
\[ A^{r-t} \geq \left[ A^{\frac{s}{r}} (B^{-\frac{1}{2}}C^{p}B^{-\frac{1}{2}})^{s} A^{\frac{s}{r}} \right]^{\frac{r-t}{s+r}}, \] (2.7)
\[ [C^{\frac{s}{r}} (B^{-\frac{1}{2}}C^{p}B^{-\frac{1}{2}})^{s} C^{\frac{s}{r}}]^{\frac{r-t}{(p-r)+t}} \geq C^{r-t} \] (2.8)
hold for all \( p, s \geq 1 \), \( r \geq t \) and \( t \in [0,1] \).

**Proof.** \((\Rightarrow) (2.7) \) and (2.8) are obvious by extended grand Furuta inequality. See [8] and [10] for details.

\((\Leftarrow) \) Put \( p = 1 \) and \( t = 1 \) in (2.7), then
\[ A^{r-1} \geq \left[ A^{\frac{s}{r}} (B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{s} A^{\frac{s}{r}} \right]^{\frac{r-1}{r}} \] (2.9)
holds for all \( s \geq 1 \) and \( r \geq 1 \).

Without loss of generality, we may assume that \( \|A\| \leq 1 \) in (2.9), otherwise (2.9) is equivalent to
\[ \left( \frac{A}{\|A\|} \right)^{r-1} \geq \left[ \left( \frac{A}{\|A\|} \right)^{\frac{s}{r}} (B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{s} \left( \frac{A}{\|A\|} \right)^{\frac{s}{r}} \right]^{\frac{r-1}{r}}. \]
According to (2.9), we have

\[
A^{-\frac{r}{2}}\left[A^{\frac{r}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{s}A^{\frac{r}{2}}\right]^{-\frac{1}{r}}A^{-\frac{r}{2}} = A^{-\frac{r}{2}}\left(B^{-\frac{1}{2}}CB^{-\frac{1}{2}}\right)^{s}
\]

\[
= \Lambda\left(\frac{1}{r}, \frac{r-1}{r}; A^{-r}, (B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{s}\right)
\]

\[
\geq \left[\frac{1}{r}A^{r} + \frac{r-1}{r}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{-s}\right]^{-1}.
\]

The last inequality is due to the geometric-harmonic mean inequality of the Karcher mean.

Taking inverse both sides above, we have

\[
A \leq \frac{1}{r}A^{r} + \frac{r-1}{r}(B^{\frac{1}{2}}C^{-1}B^{\frac{1}{2}})^{s}.
\]  

(2.10)

Let \( r \to +\infty \), then the following inequality holds by the assumption of \( \|A\| \leq 1 \).

\[
A \leq (B^{\frac{1}{2}}C^{-1}B^{\frac{1}{2}})^{s}.
\]  

(2.11)

It follows that

\[
A^{\frac{1}{s}} \leq B^{\frac{1}{s}}C^{-1}B^{\frac{1}{s}}
\]  

(2.12)

holds for any \( s \geq 1 \) by Löwner-Heinz inequality, which yield \( B \geq C \) if let \( s \to +\infty \).

(2.8) is equivalent to

\[
(C^{-1})^{r-t} \geq \{(C^{-1})^{\frac{r}{2}}[(B^{-1})^{-\frac{1}{2}}(A^{-1})^{p}(B^{-1})^{-\frac{1}{2}}]^{s}(C^{-1})^{\frac{r}{2}}\}^{\frac{r}{(p-1)(r-t)}}.
\]  

(2.13)

By the same method above, we can obtain \( A \geq B \) from (2.13).  

\[\square\]

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References


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