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A CHARACTERIZATION OF MATRIX INEQUALITY $A \geqslant B \geqslant C$ VIA KARCHER MEAN

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Abstract. Let A,B and C be three positive definite matrices. In this paper, we show a characterization of $A \geqslant B \geqslant C$ via the Karcher mean as follows,

$$\Lambda(\omega;A^{-p_1},B^{-p_2},B^t\natural_s C^{p_3}) \leqslant C,$$
$$\Lambda(\omega;C^{-p_1},B^{-p_2},B^t\natural_s A^{p_3}) \geqslant A$$

hold for $t\in[0,1]$, $s\geqslant 1$, $p_1,p_2>0$ and $p_3>1$, where $\hat{\omega}=(\frac{1}{p_1+1},\frac{1}{p_2+1},\frac{2}{(p_3-t)s+t-1})$, $\omega=(w_1,w_2,w_3)=\frac{\hat{\omega}}{\|\hat{\omega}\|_1}$.

1. Introduction

A capital letter (such as T) stands for an $m\times m$ matrix on $\mathbb{C}$. $T>0$ and $T\geqslant 0$ mean T is a positive definite matrix and T is a positive semidefinite matrix, respectively.

For $A,B>0$, the weighted geometric mean of A and B is defined by $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ for $\alpha\in[0,1]$. In order to extend this classical definition to three or more positive definite matrices, T. Ando et al. listed ten properties (such as self-duality, arithmetic-geometric-harmonic mean inequality) which the geometric mean should satisfied. See [1] for details. Also, T. Ando et al. gave a kind of geometric mean by matrix iteration in [1]. In 2006, R. Bhatia et al. showed another kind of geometric mean by argument of minimum in matrix function as follows.

DEFINITION 1.1. ([2, 5, 9, 11]) For $A_1,A_2,\cdots,A_n>0$ and a probability vector $\omega=(w_1,w_2,\cdots,w_n)$ such that $\sum w_i=1$ and each $w_i>0$, the weighted Karcher mean of $A_1,A_2,\cdots,A_n$ is defined by

$$\Lambda(\omega;A_1,A_2,\cdots,A_n) = \underset{X>0}{arg\ min}\sum_{i=1}^{n} w_i\delta_2^2(A_i,X),$$

where $arg\ min f(X)$ means the point X_0 which attains minimum value of the function $f(X)$ and δ_2 is the Riemannian metric $\delta_2(A,B)=\|\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\|_2$.

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It was obtained in [6, 7] that the weighted Karcher mean for n positive definite matrices $A_1, A_2, \cdots, A_n$ coincides with the unique positive definite solution of the following Karcher equation:

$$\sum_{i=1}^{n} w_i \log(X^{1/2} A_i^{-1} X^{1/2}) = 0.$$

It is easy to prove that $\Lambda(1-\alpha, \alpha; A, B) = A \sharp_\alpha B$. See [3] for details.

Recently, C.-S. Lin et al. [8] and J. Yuan et al. [13] proved several characterizations of $A \geqslant B \geqslant C$ via Furuta type inequalities. As a continuation, we shall show a characterization of $A \geqslant B \geqslant C$ via the Karcher mean.

Let us recall an important theorem due to M. Ito [4] related to the main result first. For convenience, we use the notation $\natural_s$ for the binary operation

$$A \natural_s B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^s A^{\frac{1}{2}} \quad \text{for } s \notin [0,1],$$

whose formula is the same as $\sharp_s$.

THEOREM 1.1. ([4]) *Let $A_1, A_2, \cdots, A_n$ be positive definite matrices. If $A_i \geqslant A_n$ for each $i = 1, 2, \cdots, n-1$, then*

$$\Lambda(\omega; A_1^{-p_1}, A_2^{-p_2}, \cdots, A_{n-1}^{-p_{n-1}}, A_{n-1}^t \natural_s A_n^{p_n}) \leqslant A_n$$

holds for all $t \in [0,1]$, $s \geqslant 1$, $p_i \geqslant 0$ $(i = 1, 2, \cdots, n-1)$ and $p_n > 1$, where $\hat{\omega} = (\frac{1}{p_1+1}, \frac{1}{p_2+1}, \cdots, \frac{1}{p_{n-1}+1}, \frac{n-1}{(p_n-t)s+t-1})$, $\omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1}$.

2. Main result

THEOREM 2.1. *For $A, B, C > 0$. $A \geqslant B \geqslant C$ if and only if the following two Karcher mean inequalities*

$$\Lambda(\omega; A^{-p_1}, B^{-p_2}, B^t \natural_s C^{p_3}) \leqslant C \tag{2.1}$$

$$\Lambda(\omega; C^{-p_1}, B^{-p_2}, B^t \natural_s A^{p_3}) \geqslant A \tag{2.2}$$

hold for $t \in [0,1]$, $s \geqslant 1$, $p_1, p_2 > 0$ and $p_3 > 1$, where $\hat{\omega} = (\frac{1}{p_1+1}, \frac{1}{p_2+1}, \frac{2}{(p_3-t)s+t-1})$, $\omega = (w_1, w_2, w_3) = \frac{\hat{\omega}}{\|\hat{\omega}\|_1}$.

Proof. $(\Rightarrow)$ (2.1) is obvious by Theorem 1.1. Applying $C^{-1} \geqslant B^{-1} \geqslant A^{-1}$ to (2.1), we have

$$\Lambda(\omega; C^{p_1}, B^{p_2}, B^{-t} \natural_s A^{-p_3}) \leqslant A^{-1}. \tag{2.3}$$

Then (2.2) is obtained by self-duality of Karcher mean ($\Lambda(\omega; A_1^{-1}, A_2^{-1}, \cdots, A_n^{-1})^{-1} = \Lambda(\omega; A_1, A_2, \cdots, A_n)$, [11]).

$(\Leftarrow)$ According to the geometric-harmonic mean inequality of the Karcher mean ($(\sum_{i=1}^{n} w_i A_i^{-1})^{-1} \leqslant \Lambda(\omega; A_1, A_2, \cdots, A_n)$, [11]) and (2.1), we have

$$\left(w_1 A^{p_1} + w_2 B^{p_2} + w_3 (B^t \natural_s C^{p_3})^{-1}\right)^{-1} \leqslant C, \tag{2.4}$$

and hence we obtain

$$w_1A^{p_1} + w_2B^{p_2} + w_3B^{-t}\natural_s C^{-p_3} \geqslant C^{-1}. \tag{2.5}$$

Let $t = 1$ and $p_3 \to 1^+$, then for fixed p_1, p_2 and s, we have $w_1 = \frac{\frac{1}{p_1+1}}{\frac{1}{p_1+1}+\frac{1}{p_2+1}+\frac{2}{(p_3-1)s}} \to 0$, $w_2 = \frac{\frac{1}{p_2+1}}{\frac{1}{p_1+1}+\frac{1}{p_2+1}+\frac{2}{(p_3-1)s}} \to 0$, $w_3 = \frac{\frac{2}{(p_3-1)s}}{\frac{1}{p_1+1}+\frac{1}{p_2+1}+\frac{2}{(p_3-1)s}} \to 1$. Thus, $B^{-1}\natural_s C^{-1} \geqslant C^{-1}$ holds by (2.5). By the definition of $\natural_s$ and $s \geqslant 1$, we have $(B^{\frac{1}{2}}C^{-1}B^{\frac{1}{2}})^{s-1} \geqslant I$, which ensures $B \geqslant C$.

Similarly, applying the arithmetic-geometric mean inequality ($\Lambda(\omega; A_1, A_2, \cdots, A_n) \leqslant \sum_{i=1}^n w_iA_i$, [12]) to (2.2), we have

$$w_1C^{-p_1} + w_2B^{-p_2} + w_3B^t\natural_s A^{p_3} \geqslant A. \tag{2.6}$$

Let $t = 1$ and $p_3 \to 1^+$, then for fixed p_1, p_2 and s, we have $w_1 \to 0$, $w_2 \to 0$, $w_3 \to 1$. Thus, $B\natural_s A \geqslant A$ holds by (2.6). By the definition of $\natural_s$ and $s \geqslant 1$, $(B^{\frac{-1}{2}}AB^{\frac{-1}{2}})^{s-1} \geqslant I$ is obtained, which ensures $A \geqslant B$. Hence, the proof is completed. □

COROLLARY 2.1. *For $A, B > 0$. $A \geqslant B$ if and only if $\Lambda(\omega; A^{-p_1}, A^{-p_2}, A^t\natural_s B^{p_3}) \leqslant B$ hold for $t \in [0,1]$, $s \geqslant 1$, $p_1, p_2 > 0$ and $p_3 > 1$, where $\hat{\omega} = (\frac{1}{p_1+1}, \frac{1}{p_2+1}, \frac{2}{(p_3-t)s+t-1})$, $\omega = (w_1, w_2, w_3) = \frac{\hat{\omega}}{\|\hat{\omega}\|_1}$.*

Proof. Let $B = A$ and $C = B$ in (2.1) of Theorem 2.1. □

Next, we give another proof of the main result in [8] by our method above.

THEOREM 2.2. ([8]) *For $A, B, C > 0$. $A \geqslant B \geqslant C$ if and only if the following two inequalities*

$$A^{r-t} \geqslant [A^{\frac{r}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{r-t}{(p-t)s+r}}, \tag{2.7}$$

$$[C^{\frac{r}{2}}(B^{-\frac{t}{2}}A^pB^{-\frac{t}{2}})^sC^{\frac{r}{2}}]^{\frac{r-t}{(p-t)s+r}} \geqslant C^{r-t} \tag{2.8}$$

hold for all $p, s \geqslant 1$, $r \geqslant t$ and $t \in [0,1]$.

Proof. ($\Rightarrow$) (2.7) and (2.8) are obvious by extended grand Furuta inequality. See [8] and [10] for details.

($\Leftarrow$) Put $p = 1$ and $t = 1$ in (2.7), then

$$A^{r-1} \geqslant [A^{\frac{r}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^sA^{\frac{r}{2}}]^{\frac{r-1}{r}} \tag{2.9}$$

holds for all $s \geqslant 1$ and $r \geqslant 1$.

Without loss of generality, we may assume that $\|A\| \leqslant 1$ in (2.9), otherwise (2.9) is equivalent to

$$\left(\frac{A}{\|A\|}\right)^{r-1} \geqslant \left[\left(\frac{A}{\|A\|}\right)^{\frac{r}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^s\left(\frac{A}{\|A\|}\right)^{\frac{r}{2}}\right]^{\frac{r-1}{r}}.$$

According to (2.9), we have

$$A^{-1} \geqslant A^{-\frac{r}{2}}[A^{\frac{r}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^s A^{\frac{r}{2}}]^{\frac{r-1}{r}} A^{-\frac{r}{2}} = A^{-r}\sharp_{\frac{r-1}{r}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^s$$
$$= \Lambda\Big(\frac{1}{r}, \frac{r-1}{r}; A^{-r}, (B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^s\Big)$$
$$\geqslant \Big[\frac{1}{r}A^r + \frac{r-1}{r}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{-s}\Big]^{-1}.$$

The last inequality is due to the geometric-harmonic mean inequality of the Karcher mean.

Taking inverse both sides above, we have

$$A \leqslant \frac{1}{r}A^r + \frac{r-1}{r}(B^{\frac{1}{2}}C^{-1}B^{\frac{1}{2}})^s. \tag{2.10}$$

Let $r \to +\infty$, then the following inequality holds by the assumption of $\|A\| \leqslant 1$.

$$A \leqslant (B^{\frac{1}{2}}C^{-1}B^{\frac{1}{2}})^s. \tag{2.11}$$

It follows that

$$A^{\frac{1}{s}} \leqslant B^{\frac{1}{2}}C^{-1}B^{\frac{1}{2}} \tag{2.12}$$

holds for any $s \geqslant 1$ by Löwner-Heinz inequality, which yield $B \geqslant C$ if let $s \to +\infty$.

(2.8) is equivalent to

$$(C^{-1})^{r-t} \geqslant \{(C^{-1})^{\frac{r}{2}}[(B^{-1})^{-\frac{t}{2}}(A^{-1})^p(B^{-1})^{-\frac{t}{2}}]^s(C^{-1})^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}. \tag{2.13}$$

By the same method above, we can obtain $A \geqslant B$ from (2.13). □

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