Journal of
Mathematical
Inequalities
Volume 10, Number 1 (2016), 173–183
doi:10.7153/jmi-10-15

ON φ–CONVEX FUNCTIONS

M. ESHAGHI GORDJI, M. ROSTAMIAN DELAVAR AND M. DE LA SEN

(*Communicated by M. Klaričić Bakula*)

Abstract. In this paper we introduce the notion of φ-convex functions as generalization of convex functions. Some basic results under various conditions for the function φ are investigated. Moreover, we establish Jensen and Hermite-Hadamard type inequalities related to φ-convex functions. Also, the notions of φ_b-convex and φ_E-convex functions, which are generalization of φ-convex functions are introduced and some new results related to these new settings are obtained.

1. Introduction

The role of convex sets, convex functions and their generalizations are important in applied mathematics specially in nonlinear programming and optimization theory. For example in economics, convexity plays a fundamental role in equilibrium and duality theory. The convexity of sets and functions have been the object of many studies in recent years. But in many new problems encountered in applied mathematics the notion of convexity is not enough to reach favorite results and hence it is necessary to extend the notion of convexity to the new generalized notions. Recently, several extensions have been considered for the classical convex functions such that some of these new concepts are based on extension of the domain of a convex function (a convex set) to a generalized form and some of them are new definitions that there is no generalization on the domain but on the form of the definition. Some new generalized concepts in this point of view are pseudo-convex functions [6], quasi-convex functions [1], invex functions [4], preinvex functions [7], B-vex functions [5], B-preinvex functions [2] and E-convex functions [10].

In this paper, we introduce the concept of φ-convex functions as generalization of convex functions. Some basic results under various conditions for the function φ are investigated. We define φ_b-convex and φ_E-convex functions as generalized forms of φ-convex functions and prove some new results related to the new settings. Among other things, we investigate the Jensen and Hermite-Hadamard type inequalities related to φ-convex functions.

Mathematics subject classification (2010): 26A51, 26D15, 52A01.
Keywords and phrases: φ-convex function, Jensen inequality, Hermite-Hadamard inequality.

Paper JMI-10-15

2. Basic Results

Through this paper, let I be an interval in real line $\mathbb{R}$ and $\varphi : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a bifunction except for special cases.

DEFINITION 2.1. A function $f : I \to \mathbb{R}$ is called *convex with respect to* φ *(briefly* φ*-convex)*, if

$$f(\lambda x+(1-\lambda)y)\leqslant f(y)+\lambda\varphi(f(x),f(y)), \tag{2.1}$$

for all $x,y\in I$ and $\lambda\in[0,1]$. Also f is called φ*-quasiconvex*, if

$$f(\lambda x+(1-\lambda)y)\leqslant \max\{f(y),f(y)+\varphi(f(x),f(y))\}$$

for all $x,y\in I$ and $\lambda\in[0,1]$. Moreover, f is called φ-affine if

$$f(\lambda x+(1-\lambda)y)=f(y)+\lambda\varphi(f(x),f(y))$$

for all $x,y,\lambda\in\mathbb{R}$.

In the above definition if we set $\varphi(x,y)=x-y$, then we approach to the classic definition of convex, quasiconvex and affine function respectively. Note that by taking $x=y$ in (2.1) we get $\lambda\varphi(f(x),f(x))\geqslant 0$ for any $x\in I$ and $t\in[0,1]$ which implies that

$$\varphi(f(x),f(x))\geqslant 0,$$

for any $x\in I$. Also if we take $\lambda=1$ in (2.1) we get

$$f(x)-f(y)\leqslant\varphi(f(x),f(y)),$$

for any $x,y\in I$. The second condition obviously implies the first. If $f : I\to\mathbb{R}$ is a convex function and $\varphi : I\times I\to\mathbb{R}$ is an arbitrary bifunction that satisfies

$$\varphi(x,y)\geqslant x-y$$

for any $x,y\in I$, then

$$f(\lambda x+(1-\lambda)y)\leqslant f(y)+\lambda[f(x)-f(y)]\leqslant f(y)+\lambda\varphi(f(x),f(y)),$$

showing that f is φ-convex.

EXAMPLE 2.2. (1) For a convex function f, we may find another function φ other than the function $\varphi(x,y)=x-y$ such that f is φ-convex. Consider $f(x)=x^2$ and $\varphi(x,y)=2x+y$. Then we have

$$\begin{aligned}
f(\lambda x+(1-\lambda)y) &= (\lambda x+(1-\lambda)y)^2\\
&\leqslant y^2+\lambda x^2+\lambda(1-\lambda)2xy\\
&\leqslant y^2+\lambda x^2+\lambda(1-\lambda)(x^2+y^2)\\
&\leqslant y^2+\lambda(x^2+x^2+y^2)\\
&= y^2+\lambda(2x^2+y^2)\\
&= f(y)+\lambda\varphi(f(x),f(y))
\end{aligned}$$

for all $x,y \in \mathbb{R}$ and $\lambda \in (0,1)$. Also the facts $x^2 \leqslant y^2 + (2x^2 + y^2)$ and $y^2 \leqslant y^2$, for all $x,y \in \mathbb{R}$ show the correctness of inequality for $\lambda = 1$ and $\lambda = 0$ respectively which means that f is φ-convex. Note that the function $f(x) = x^2$ is φ-convex w.r.t all $\varphi(x,y) = ax + by$ with $a \geqslant 1$, $b \geqslant -1$ and $x,y \in \mathbb{R}$.

(2) There is a φ-convex function f which is not convex. Consider $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} -x, & x \geqslant 0; \\ x, & x < 0, \end{cases}$$

and $\varphi : [-\infty, 0] \times [-\infty, 0] \to \mathbb{R}$ as

$$\varphi(x,y) = \begin{cases} x, & y = 0; \\ -y, & x = 0; \\ -x-y, & x < 0, y < 0. \end{cases}$$

Then it is not hard to check that f is φ-convex. Also, it is obvious that f is not a convex function.

In the following we define various conditions for the function φ. We use these concepts frequently in our results.

DEFINITION 2.3. The function φ is said to be

(i) *nonnegatively homogeneous* if $\varphi(\gamma x, \gamma y) = \gamma\varphi(x,y)$ for all $x,y \in \mathbb{R}$ and all $\gamma \geqslant 0$.

(ii) *additive* if $\varphi(x_1,y_1) + \varphi(x_2,y_2) = \varphi(x_1 + x_2, y_1 + y_2)$ for all $x_1,x_2,y_1,y_2 \in \mathbb{R}$.

(iii) *nonnegatively linear* if satisfies conditions (i) and (ii).

(iv) *nondecreasing in first variable* if $x \leqslant y$ implies that $\varphi(x,z) \leqslant \varphi(y,z)$, for all $x,y,z \in \mathbb{R}$.

(v) *nonnegatively sublinear in first variable* if $\varphi(\gamma x + y, z) \leqslant \gamma\varphi(x,z) + \varphi(y,z)$, for all $x,y,z \in \mathbb{R}$ and $\gamma \geqslant 0$.

The proof of propositions 2.4, 2.5 and Theorem 2.6 is straightforward.

PROPOSITION 2.4. *Consider φ-convex function $f : I \to \mathbb{R}$ such that φ is nonnegatively homogeneous. Then for any $\gamma \geqslant 0$, the function $\gamma f : I \to \mathbb{R}$ is φ-convex.*

PROPOSITION 2.5. *Consider two φ-convex functions $f,g : I \to \mathbb{R}$ such that φ is additive. Then $f + g : I \to \mathbb{R}$ is φ-convex.*

THEOREM 2.6. *Consider φ-convex functions $f_i : I \to \mathbb{R}$ for $i = 1,\dots,n$, such that φ is nonnegatively linear. Then for $\gamma_i \geqslant 0$, $i = 1,\dots n$, the function $f = \sum_{i=1}^{n} \gamma_i f_i : I \to \mathbb{R}$ is φ-convex.*

The class of φ-convex and φ-quasiconvex functions with special conditions are closed under Sup operation.

THEOREM 2.7. *Suppose that* $\{f_j : I \to \mathbb{R}, j \in J\}$ *is a nonempty collection of* φ-*convex* (φ-quasiconvex) functions such that

(a) there exist $\alpha \in [0,\infty]$ and $\beta \in [-1,\infty]$ such that $\varphi(x,y) = \alpha x + \beta y$ for all $x,y \in \mathbb{R}$,

(b) for each $x \in I$, $\sup_{j\in J} f_j(x)$ exists in $\mathbb{R}$.

Then the function $f : I \to \mathbb{R}$ defined by $f(x) = \sup_{j\in J} f_j(x)$ for each $x \in I$, is φ-convex (φ-quasiconvex).

Proof. For any $x,y \in I$ and $\lambda \in [0,1]$, we can drive following relations

$$
\begin{aligned}
f(\lambda x + (1-\lambda)y) &= \sup_{j\in J} f_j(\lambda x + (1-\lambda)y) \\
&\leqslant \sup_{j\in J}\{f_j(y) + \lambda\varphi(f_j(x), f_j(y))\} \\
&= \sup_{j\in J}\{f_j(y) + \lambda(\alpha f_j(x) + \beta f_j(y))\} \\
&= \sup_{j\in J}\{(1+\beta\lambda) f_j(y) + \alpha\lambda f_j(x)\} \\
&\leqslant (1+\beta\lambda)\sup_{j\in J} f_j(y) + \alpha\lambda \sup_{j\in J} f_j(x) \\
&= (1+\beta\lambda) f(y) + \alpha\lambda f(x) \\
&= f(y) + \lambda(\alpha f(x) + \beta f(y)) \\
&= f(y) + \lambda\varphi(f(x), f(y)).
\end{aligned}
$$

In the case that f is φ-quasiconvex, the proof is similar.

Using the definition of an invex set, we introduce the definition of generalized preinvex (briefly G-preinvex) function as generalized form of a convex function. In Theorem 2.10 we prove that combination of a φ-convex function with a nondecreasing G-preinvex function w.r.t. φ and ψ, is ψ-convex.

DEFINITION 2.8. [7] We call a set $A \subseteq \mathbb{R}$ *invex with respect to* $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ if

$$x,y \in A, \lambda \in [0,1] \Rightarrow y + \lambda\eta(x,y) \in A.$$

DEFINITION 2.9. Let $A \subseteq \mathbb{R}$ be an invex set with respect to η. A function $f : A \to \mathbb{R}$ is said to be *G-preinvex w.r.t.* η *and* ψ if

$$f(y + \lambda\eta(x,y)) \leqslant f(y) + \lambda\psi(f(x), f(y)) \quad \textit{for all } x,y \in A \textit{ and } \lambda \in [0,1].$$

THEOREM 2.10. *Suppose that $U \subseteq \mathbb{R}$ is an invex set w.r.t. φ where $\varphi : U \times U \to U$ is a bifunction. Suppose that $f : I \to U$ is a φ-convex function and $g : U \to \mathbb{R}$ is a nondecreasing G-preinvex function w.r.t. φ and ψ. Then $gof : I \to \mathbb{R}$ is a ψ-convex function.*

Proof. Consider $x, y \in I$ and $\lambda \in [0,1]$. Since g is nondecreasing G-preinvex function, from inequality

$$f(\lambda x + (1-\lambda)y) \leqslant f(y) + \lambda\varphi(f(x), f(y)),$$

we have

$$\begin{aligned} g(f(\lambda x + (1-\lambda)y)) &\leqslant g(f(y) + \lambda\varphi(f(x), f(y))) \\ &\leqslant g(f(y)) + \lambda\psi(g(f(x)), g(f(y))). \end{aligned}$$

3. Main Results

We will use the following relations in the proof of Theorem 3.1 which is Jensen type inequality for φ-convex functions.
Let $f : I \to \mathbb{R}$ be a φ-convex function. For $x_1, x_2 \in I$ and $\alpha_1 + \alpha_2 = 1$ we have $f(\alpha_1 x_1 + \alpha_2 x_2) \leqslant f(x_2) + \alpha_1\varphi(f(x_1), f(x_2))$. Also when $n > 2$, for $x_1, x_2, \cdots, x_n \in I$, $\sum_{i=1}^{n} \alpha_i = 1$ and $T_i = \sum_{j=1}^{i} \alpha_j$, we have

$$\begin{aligned} f(\sum_{i=1}^{n} \alpha_i x_i) &= f((T_{n-1} \sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} x_i) + \alpha_n x_n) \\ &\leqslant f(x_n) + T_{n-1}\varphi(f(\sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} x_i), f(x_n)). \qquad (\star) \end{aligned}$$

THEOREM 3.1. *Let $f : I \to \mathbb{R}$ be a φ-convex function and φ be nondecreasing nonnegatively sublinear in first variable. If $T_i = \sum_{j=1}^{i} \alpha_j$ for $i = 1, \cdots n$ such that T_n=1, then*

$$f(\sum_{i=1}^{n} \alpha_i x_i) \leqslant f(x_n) + \sum_{i=1}^{n-1} T_i \varphi_f(x_i, x_{i+1}, \ldots, x_n),$$

where $\varphi_f(x_i, x_{i+1}, \ldots, x_n) = \varphi(\varphi_f(x_i, x_{i+1}, \ldots, x_{n-1}), f(x_n))$ and $\varphi_f(x) = f(x)$ for all $x \in I$.

Proof. Since φ is nondecreasing nonnegatively sublinear on first variable, so from $(\star)$ it follows that:

$$\begin{aligned}
f(\sum_{i=1}^{n}\alpha_i x_i) &\leqslant f(x_n)+T_{n-1}\varphi(f(\sum_{i=1}^{n-1}\frac{\alpha_i}{T_{n-1}}x_i),f(x_n))\\
&= f(x_n)+T_{n-1}\varphi\big(f(\frac{T_{n-2}}{T_{n-1}}\sum_{i=1}^{n-2}\frac{\alpha_i}{T_{n-2}}x_i+\frac{\alpha_{n-1}}{T_{n-1}}x_{n-1}),f(x_n)\big)\\
&\leqslant f(x_n)+T_{n-1}\varphi\big(f(x_{n-1})+\frac{T_{n-2}}{T_{n-1}}\varphi(f(\sum_{i=1}^{n-2}\frac{\alpha_i}{T_{n-2}}x_i),f(x_{n-1})),f(x_n)\big)\\
&\leqslant f(x_n)+T_{n-1}\varphi(f(x_{n-1}),f(x_n))+T_{n-2}\varphi\big(\varphi(f(\sum_{i=1}^{n-2}\frac{\alpha_i}{T_{n-2}}x_i),f(x_{n-1})),f(x_n)\big)\\
&\leqslant \cdots\\
&\leqslant f(x_n)+T_{n-1}\varphi(f(x_{n-1}),f(x_n))+T_{n-2}\varphi\big(\varphi(f(x_{n-2}),f(x_{n-1})),f(x_n)\big)\\
&\qquad +\cdots+T_1\varphi\big(\varphi(\cdots\varphi(\varphi(f(x_1),f(x_2)),f(x_3))\cdots)),f(x_{n-1})),f(x_n)\big)\\
&= f(x_n)+T_{n-1}\varphi_f(x_{n-1},x_n)+T_{n-2}\varphi_f(x_{n-2},x_{n-1},x_n)+\cdots\\
&\qquad +T_1\varphi_f(x_1,x_2,\cdots,x_{n-1},x_n)\\
&= f(x_n)+\sum_{i=1}^{n-1}T_i\varphi_f(x_i,x_{i+1},\cdots,x_n).
\end{aligned}$$

EXAMPLE 3.2. Consider $f(x)=x^2$ and $\varphi(x,y)=x(1+2y)$ for $x,y\in\mathbb{R}^+=[0,\infty]$. The function φ is nondecreasing nonnegatively sublinear in first variable and f is φ-convex since $(\alpha_1x_1+\alpha_2x_2)^2\leqslant x_2^2+\alpha_1x_1^2(1+2x_2^2)$, for $x_1,x_2\in\mathbb{R}^+$ and $\alpha_1,\alpha_2\geqslant 0$ with $\alpha_1+\alpha_2=1$. Now for $x_1,x_2,\cdots,x_n\in\mathbb{R}^+$ and $\alpha_1,\alpha_2,\cdots,\alpha_n$ with $\sum_{i=1}^{n}\alpha_i=1$ according to Theorem 3.1, we have

$$(\sum_{i=1}^{n}\alpha_i x_i)^2\leqslant x_n^2+\sum_{i=1}^{n-1}T_i[x_i^2(1+2x_{i+1}^2)(1+2x_{i+2}^2)...(1+2x_n^2)].$$

Some basic results are required to prove that there is Hermite-Hadamard type inequality for any φ-convex function. Under special condition for φ, any φ-convex function is continuous.

DEFINITION 3.3. [8] A function $f:[a,b]\to\mathbb{R}$ is *absolutely continuous* on $[a,b]$ if corresponding to any $\varepsilon>0$ there exists a $\delta>0$ such that for any collection $\{a_i,b_i\}_1^n$ of disjoint open intervals of $[a,b]$ with $\sum_1^n(b_i-a_i)<\delta$, $\sum_1^n\mid f(b_i)-f(a_i)\mid<\varepsilon$.

DEFINITION 3.4. [8] A function $f:[a,b]\to\mathbb{R}$ is said to satisfy the *Lipschitz condition* on $[a,b]$ if there is a constant K so that for any two points $x,y\in[a,b]$, $|f(x)-f(y)|\leqslant K|x-y|$.

THEOREM 3.5. *Suppose that $f : I \to \mathbb{R}$ is φ-convex and φ is bounded from above on $f(I) \times f(I)$ with M_φ as an upper bound. Then f satisfies the Lipschitz condition on any closed interval $[a,b]$ contained in the interior I° of I. Hence, f is absolutely continuous on $[a,b]$ and continuous on I°.*

Proof. Consider closed interval $[a,b]$ in I° and choose $\varepsilon > 0$ such that $[a-\varepsilon, b+\varepsilon]$ belongs to I. Suppose that x,y are distinct points of $[a,b]$. Set $z = y + \frac{\varepsilon}{|y-x|}(y-x)$ and $\lambda = \frac{|y-x|}{\varepsilon+|y-x|}$. So it is easy to see that $z \in [a-\varepsilon, b+\varepsilon]$ and $y = \lambda z + (1-\lambda)x$. Then $f(y) \leqslant f(x) + \lambda\varphi(f(z), f(x)) \leqslant f(x) + \lambda M_\varphi$. This implies that

$$f(y) - f(x) \leqslant \lambda M_\varphi = \frac{|y-x|}{\varepsilon+|y-x|} M_\varphi \leqslant \frac{|y-x|}{\varepsilon} M_\varphi = K\,|y-x|,$$

where $K = \frac{M_\varphi}{\varepsilon}$. Also if we change the place of x,y in above argument we have $f(x) - f(y) \leqslant K\,|y-x|$. Therefore $|f(y) - f(x)| \leqslant K\,|y-x|$. It follows that if we choose $\delta < \varepsilon/K$, then f is absolutely continuous. Finally since $[a,b]$ is arbitrary on I°, then f is continuous on I°.

THEOREM 3.6. *Suppose that $f : I \to \mathbb{R}$ is a φ-convex function such that φ is bounded from above on $f(I) \times f(I)$. Then for any $a,b \in I$ with $a < b$,*

$$2f(\frac{a+b}{2}) - M_\varphi \leqslant \frac{1}{b-a}\int_a^b f(x) \leqslant f(b) + \frac{\varphi(f(a), f(b))}{2},$$

where M_φ is an upper bound of φ on $f([a,b]) \times f([a,b])$.

Proof. Consider $a,b \in I$ with $a < b$. First we show that f has upper and lower bound on $[a,b]$. In fact

$$f(\lambda a + (1-\lambda)b) \leqslant f(b) + \lambda\varphi(f(a), f(b)) \leqslant f(b) + M_\varphi$$

shows the upper bound of f. For lower bound of f consider an arbitrary point in the form $\frac{a+b}{2} - t$ in $[a,b]$. Then

$$\begin{aligned}
f(\frac{a+b}{2}) &= f(\frac{a+b}{4} + \frac{t}{2} + \frac{a+b}{4} - \frac{t}{2}) \\
&= f\big(\frac{1}{2}(\frac{a+b}{2} + t) + \frac{1}{2}(\frac{a+b}{2} - t)\big) \\
&\leqslant f(\frac{a+b}{2} - t) + \frac{1}{2}\varphi\big(f(\frac{a+b}{2} + t), f(\frac{a+b}{2} - t)\big) \\
&\leqslant f(\frac{a+b}{2} - t) + \frac{M_\varphi}{2}.
\end{aligned}$$

Now consider $m = f(\frac{a+b}{2}) - \frac{M_\varphi}{2}$. For the right side of inequality consider arbitrary $x = \lambda a + (1-\lambda)b$, $\lambda \in [0,1]$. So $f(x) \leqslant f(b) + \lambda\varphi(f(a), f(b))$ with $\lambda = \frac{x-b}{a-b}$. It follows that

$$\frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{1}{b-a}\big(f(b)(b-a) + \frac{\varphi(f(a), f(b))}{b-a}.\frac{(b-a)^2}{2}\big) = f(b) + \frac{\varphi(f(a), f(b))}{2}.$$

Through φ-convexity of f, we have

$$
\begin{aligned}
f(\frac{a+b}{2}) &= f(\frac{a+b}{4} - \frac{t(b-a)}{4} + \frac{a+b}{4} + \frac{t(b-a)}{4}) \\
&= f\big(\frac{1}{2}(\frac{a+b-t(b-a)}{2}) + \frac{1}{2}(\frac{a+b+t(b-a)}{2}\big) \\
&\leqslant f(\frac{a+b+t(b-a)}{2}) + \frac{1}{2}\varphi\big(f(\frac{a+b-t(b-a)}{2}), f(\frac{a+b+t(b-a)}{2})\big) \\
&\leqslant f(\frac{a+b+t(b-a)}{2}) + \frac{1}{2}M_\varphi
\end{aligned}
$$

for all $t \in [0,1]$. Hence, for the left side of inequality

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)dx &= \frac{1}{b-a}\Big[\int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx\Big] \\
&= \int_0^1 \Big[f(\frac{a+b-t(b-a)}{2}) + f(\frac{a+b+t(b-a)}{2})\Big]dt \\
&\geqslant \int_0^1 \Big[f(\frac{a+b-t(b-a)}{2}) + f(\frac{a+b}{2}) - \frac{1}{2}M_\varphi\Big]dt \\
&\geqslant m + f(\frac{a+b}{2}) - \frac{1}{2}M_\varphi \\
&= 2f(\frac{a+b}{2}) - M_\varphi.
\end{aligned}
$$

The final result of this section says that the class of affine functions and φ-convex functions defined from $\mathbb{R}$ to $\mathbb{R}$ are equivalent.

THEOREM 3.7. *For a function $f:\mathbb{R}\to\mathbb{R}$, the following assertions are equivalent:*

(a) f is an affine function.
(b) f is φ-affine w.r.t some φ.

Proof. $(a)\to(b)$ is clear. For $(b)\to(a)$ consider any $t\in\mathbb{R}$. So $f(t) = f\big(t\cdot 1 + (1-t)0\big) = f(0) + t\varphi(f(1), f(0))$. Now for $x, y, \lambda \in \mathbb{R}$, we have

$$
\begin{aligned}
f(\lambda x + (1-\lambda)y) &= f(0) + (\lambda x + (1-\lambda)y)\varphi(f(1), f(0)) \\
&= \lambda f(0) + (1-\lambda)f(0) + \lambda x\varphi(f(1), f(0)) + (1-\lambda)y\varphi(f(1), f(0)) \\
&= \lambda\big(f(0) + x\varphi(f(1), f(0))\big) + (1-\lambda)\big(f(0) + y\varphi(f(1), f(0))\big) \\
&= \lambda f(x) + (1-\lambda)f(y).
\end{aligned}
$$

4. φ_b-convex and φ_E-convex functions

In this section, we define φ_b-convex and φ_E-convex functions as generalized form of φ-convex functions and give some results. Theorem 4.2 shows that the class of φ_b-convex functions and φ-quasiconvex functions are equivalent.

DEFINITION 4.1. Let $\mathbb{R}^+$ be the set of nonnegative real numbers and $b : \mathbb{R}\times\mathbb{R}\times[0,1] \to \mathbb{R}^+$ be a function with $\lambda b(x,y,\lambda) \in [0,1]$ for all $x,y \in \mathbb{R}$ and $\lambda \in [0,1]$. A function $f : I \to \mathbb{R}$ is called φ_b-convex if

$$f(\lambda x+(1-\lambda)y) \leqslant f(y)+\lambda b(x,y,\lambda)\varphi(f(x),f(y))$$

for all $x,y \in \mathbb{R}$ and $\lambda \in [0,1]$.

THEOREM 4.2. *Consider a function* $f : I \to \mathbb{R}$. *The following assertions are equivalent:*

(a) f is φ_b-convex for some function b.
(b) f is φ-quasiconvex.

Proof. (a)$\to$(b) For any $x,y \in I$ and $\lambda \in [0,1]$,

$$f(\lambda x+(1-\lambda)y) \leqslant f(y)+\lambda b(x,y,\lambda)\varphi(f(x),f(y)) \leqslant \max\{f(y), f(y)+\varphi(f(x),f(y))\}.$$

(b)$\to$(a) For $x,y \in I$ and $\lambda \in [0,1]$, define

$$b(x,y,\lambda) = \begin{cases} 1/\lambda, \text{ if } \lambda \in (0,1] \textit{ and } f(y) \leqslant f(y)+\varphi(f(x),f(y)); \\ 0, \quad\ \ \lambda = 0 \textit{ or } f(y) > f(y)+\varphi(f(x),f(y)). \end{cases}$$

Notice that $\lambda b(x,y,\lambda) \in [0,1]$. For a such function b we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\leqslant \max\{f(y), f(y)+\varphi(f(x),f(y))\} \\ &= \lambda b(x,y,\lambda)(f(y)+\varphi(f(x),f(y))+(1-\lambda b(x,y,\lambda))f(y) \\ &= f(y)+\lambda b(x,y,\lambda)\varphi(f(x),f(y)). \end{aligned}$$

DEFINITION 4.3. [9, 10] A set $A \subseteq \mathbb{R}$ is said to be E-convex iff there is a map $E : \mathbb{R} \to \mathbb{R}$ such that $\lambda E(x)+(1-\lambda)E(y) \in A$, for each $x,y \in A$ and $0 \leqslant \lambda \leqslant 1$.

LEMMA 4.4. [3] Suppose that $A \subseteq \mathbb{R}$ is E-Convex. Then $E(A) \subseteq A$.

DEFINITION 4.5. Suppose that A is an E-convex set. A function $f : A \to \mathbb{R}$ is said to be φ_E*-convex* if

$$f(\lambda E(x)+(1-\lambda)E(y)) \leqslant f(E(y))+\lambda\varphi(f(E(x)),f(E(y))).$$

Also it is called φ_E-quasiconvex if

$$f(\lambda E(x)+(1-\lambda)E(y)) \leqslant \max\{f(E(y)), f(E(y))+\varphi(f(E(x)),f(E(y)))\}.$$

The restriction of a φ_E-convex (φ_E-quasiconvex) function is a φ-convex (φ-quasiconvex) function. This fact is shown in Theorem 4.6. In Theorem 4.7 we show that if a φ_E-convex (φ_E-quasiconvex) function is restricted on a suitable domain it is equivalent to a φ-convex (φ-quasiconvex) function.

THEOREM 4.6. *Suppose $A \subseteq \mathbb{R}$ is an E-convex set and $B \subseteq E(A)$ is a nonempty convex set. If $f : A \to \mathbb{R}$ is φ_E-convex* (φ_E-quasiconvex), then it's restriction $\bar{f} : B \to \mathbb{R}$ defined as

$$\bar{f}(x') = f(x') \ \ for\ all\ x' \in B,$$

is φ-convex (φ-quasiconvex) on B.

Proof. For $x', y' \in B$ there are $x, y \in A$ such that $x' = E(x)$ and $y' = E(y)$. Since B is convex set, then for any $\lambda \in [0,1]$, we have $\lambda x' + (1-\lambda)y' \in B$. So

$$\begin{aligned}\bar{f}(\lambda x' + (1-\lambda)y') &= f(\lambda E(x) + (1-\lambda)E(y)) \\ &\leqslant f(E(y)) + \lambda\varphi(f(E(x)), f(E(y))) \\ &= \bar{f}(y') + \lambda\varphi(\bar{f}(x'), \bar{f}(y')).\end{aligned}$$

THEOREM 4.7. *Suppose $A \subseteq \mathbb{R}$ is an E-convex set and $E(A)$ is a convex set. The function $f : A \to \mathbb{R}$ is φ_E-convex* (φ_E-quasiconvex) if and only if it's restriction $\bar{f} : E(A) \to \mathbb{R}$ defined as

$$\bar{f}(x') = f(x') \ \ for\ all\ x' \in E(A),$$

is φ-convex (φ-quasiconvex) on $E(A)$.

Proof. Necessary, is the same as proof of Theorem 4.6. For sufficient, consider any $x, y \in A$. Hence $E(x), E(y) \in E(A)$. From φ-convexity of $\bar{f}$ on $E(A)$ we have

$$\bar{f}(\lambda E(x) + (1-\lambda)E(y)) \leqslant \bar{f}(E(y)) + \lambda\varphi(\bar{f}(E(x)), \bar{f}(E(y))).$$

Since $\bar{f}$ is the restriction of f to $E(A)$ then:

$$f(\lambda E(x) + (1-\lambda)E(y)) \leqslant f(E(y)) + \lambda\varphi(f(E(x)), f(E(y))).$$

Acknowledgement. The authors are very grateful to an anonymous referee for the valuable suggestions and comments.

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(Received August 20, 2014)

M. Eshaghi Gordji
Department of Mathematics
Faculty of Basic Sciences
Semnan University
P. O. Box 35195-363, Semnan, Iran
e-mail: madjid.eshaghi@gmail.com, meshaghi@semnan.ac.ir

M. Rostamian Delavar
Department of Mathematics
Bojnord University
Bojnord, Iran
e-mail: rostamian333@gmail.com

M. De La Sen
Institute of Research and Development of Processes
University of Basque Country
Campus of Leioa (Bizkaia) – Aptdo
644 – Bilbao, Bilbao, 48080, Spain
e-mail: manuel.delasen@ehu.es