IMPROVED CAUCHY–SCHWARZ NORM INEQUALITY FOR OPERATORS

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(*Communicated by M. Fujii*)

Abstract. Let A, B and X be operators on a complex seperable Hilbert space such that A and B are positive. Cauchy-Schwarz norm inequality for operators asserts that

$$\left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 \leqslant \| |AX|^r \| . \| |XB|^r \|,$$

for any real number $r > 0$ and every unitarily invariant norm $\|.\|$. In this article we derive several refinements of Cauchy-Schwarz norm inequality for operators. In particular, we show improvements for the results of Hiai and Zhan [Linear Algebra Appl. 341 (2002) 151–169]. Besides, new type inequalities close to Cauchy-Schwarz norm inequality will be introduced.

1. Introduction

Let $B(H)$ denote the space of bounded linear operators on a complex separable Hilbert space H. Let $\|.\|$ denote a unitarily invariant norm defined on a norm ideal associated with it. It has been shown by Horn and Mathias [5] that if A, $B \in B(H)$, then

$$\| |A^*B|^r \|^2 \leqslant \| (AA^*)^r \| . \| (BB^*)^r \|,$$

for every positive real number r and every unitarily invariant norm. This inequality can be considerd as an operator version of the familiar Cauchy-Schwarz inequality for real numbers. A stronger version of this inequality, which has been proved by Bhatia and Davis [1], asserts that

$$\| |A^*XB|^r \|^2 \leqslant \| |AA^*X|^r \| . \| |XBB^*|^r \|,$$

for all A, B, $X \in B(H)$, and any $r > 0$, which is equivalent to

$$\left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 \leqslant \| |AX|^r \| . \| |XB|^r \|, \tag{1}$$

for positive operators A, B and arbitrary X.

Hiai and Zhan [4] proved that if A, B, $X \in B(H)$ such that A, B are positive and $r > 0$, then the function $f(v) = \| |A^v X B^{1-v}|^r \| . \| |A^{1-v} X B^v|^r \|$ is convex on $[0,1]$,

attains its minimum at $v = \frac{1}{2}$, its maximum at $v = 0$ and $v = 1$. Moreover, $f(v) = f(1-v)$. Thus for every unitarily invariant norm, we have the following refinement of Cauchy-Schwarz norm inequality for operators

$$\left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 \leqslant \left\| \left| A^v X B^{1-v} \right|^r \right\| \cdot \left\| \left| A^{1-v} X B^v \right|^r \right\| \leqslant \left\| |AX|^r \right\| \cdot \left\| |XB|^r \right\| \qquad (2)$$

Using some basic properties of convex functions, Hu [6] obtained the following refinement of the second inequality in (2)

$$\left\| \left| A^v X B^{1-v} \right|^r \right\| \cdot \left\| \left| A^{1-v} X B^v \right|^r \right\| \leqslant 2v_0 \left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 + (1 - 2v_0) \left\| |AX|^r \right\| \cdot \left\| |XB|^r \right\|,$$

where $v_0 = \min\{v,\ 1-v\}$.

Our main task in this article is to derive other improvements of Cauchy-Schwarz norm inequality with the help of the well-known Hermite-Hadamard inequality. In addition, we establish new type inequalities close to Cauchy-Schwarz norm inequality for operators.

2. Refinements of Cauchy-Schwarz norm inequality for operators via the convexity

Hermite-Hadamard inequality, which includes a basic property of convex functions and plays a central role in our investigation, asserts that if g is a convex real valued function on the interval $[a,b]$, then

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b g(t)dt \leqslant \frac{g(a)+g(b)}{2}$$

A recent refinement of the second inequality in Hermite-Hadamard inequality, due to Feng [2], asserts that

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b g(t)dt \leqslant \frac{1}{4}\left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b)\right] \leqslant \frac{g(a)+g(b)}{2}, \quad (3)$$

where g is a convex real valued function on $[a,b]$. In the following lemma, we construct a refinement of the first inequality in the Hermite-Hadamard inequality.

LEMMA 1. *Let g be a real valued function which is convex on the interval $[a,\ b]$. Then*

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\left[g\left(\frac{3a+b}{4}\right) + g\left(\frac{a+3b}{4}\right)\right] \leqslant \frac{1}{b-a}\int_a^b g(t)dt \leqslant \frac{g(a)+g(b)}{2}$$

Proof. Using Hermite-Hadamard inequality, we have

$$g\left(\frac{a+b}{2}\right) = g\left[\frac{1}{2}\left(\frac{3a+b}{4}\right) + \frac{1}{2}\left(\frac{a+3b}{4}\right)\right] \leqslant \frac{1}{2}\left[g\left(\frac{3a+b}{4}\right) + g\left(\frac{a+3b}{4}\right)\right].$$

To prove the second inequality, note that

$$\frac{1}{2}\left[g\left(\frac{3a+b}{4}\right)+g\left(\frac{a+3b}{4}\right)\right]=\frac{1}{2}\left[g\left(\frac{\frac{a+b}{2}+a}{2}\right)+g\left(\frac{\frac{a+b}{2}+b}{2}\right)\right]$$

$$\leqslant\frac{1}{2}\left[\frac{1}{a-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^{a}g(t)dt+\frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^{b}g(t)dt\right]$$

$$=\frac{1}{2}\left[\frac{2}{b-a}\int_{a}^{\frac{a+b}{2}}g(t)dt+\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}g(t)dt\right]$$

$$=\frac{1}{b-a}\int_{a}^{b}g(t)dt.$$

This completes the proof. $\square$

In order to derive refinements of Cauchy-Schwarz norm inequality, we apply Lemma 1 and the inequality (3) to the function $f(v)=\left|\left|\left|A^{v}XB^{1-v}\right|^{r}\right|\right|.\left|\left|\left|A^{1-v}XB^{v}\right|^{r}\right|\right|$ on the interval $[\mu,1-\mu]$, when $0\leqslant\mu<\frac{1}{2}$, and on the interval $[1-\mu,\mu]$, when $\frac{1}{2}<\mu\leqslant1$.

THEOREM 2. *Let A, B, $X\in B(H)$ such that A and B are positive. Then for $0\leqslant\mu\leqslant1$, $r>0$, and for every unitarily invariant norm,*

$$\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right|\right|^{2}\leqslant\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}\right|^{r}\right|\right|.\left|\left|\left|A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|^{r}\right|\right| \tag{4}$$

$$\leqslant\frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu}\left|\left|\left|A^{v}XB^{1-v}\right|^{r}\right|\right|.\left|\left|\left|A^{1-v}XB^{v}\right|^{r}\right|\right|dv\right|$$

$$\leqslant\frac{1}{2}\left[\left|\left|\left|A^{\mu}XB^{1-\mu}\right|^{r}\right|\right|.\left|\left|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right|\right|+\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right|\right|^{2}\right]$$

$$\leqslant\left|\left|\left|A^{\mu}XB^{1-\mu}\right|^{r}\right|\right|.\left|\left|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right|\right|$$

$$\leqslant\left|\left|\left|AX\right|^{r}\right|\right|\left|\left|\left|XB\right|^{r}\right|\right|$$

Proof. First assume that $0\leqslant\mu<\frac{1}{2}$. Then it follows by applying Lemma 1 and the inequality (3) to the function $f(v)=\left|\left|\left|A^{v}XB^{1-v}\right|^{r}\right|\right|.\left|\left|\left|A^{1-v}XB^{v}\right|^{r}\right|\right|$ on the interval $[\mu,1-\mu]$ that

$$f\left(\frac{1}{2}\right)\leqslant\frac{1}{2}\left[f\left(\frac{2\mu+1}{4}\right)+f\left(\frac{3-2\mu}{4}\right)\right]\leqslant\frac{1}{1-2\mu}\int_{\mu}^{1-\mu}f(t)dt$$

$$\leqslant\frac{1}{2}\left[f(\mu)+f\left(\frac{1}{2}\right)\right]\leqslant f(\mu). \tag{5}$$

Thus,

$$\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right|\right|^{2}\leqslant\left|\left|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}\right|^{r}\right|\right|.\left|\left|\left|A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|^{r}\right|\right| \tag{6}$$

$$\leqslant\frac{1}{1-2\mu}\int_{\mu}^{1-\mu}\left|\left|\left|A^{v}XB^{1-v}\right|^{r}\right|\right|.\left|\left|\left|A^{1-v}XB^{v}\right|^{r}\right|\right|dv$$

$$\leqslant \frac{1}{2}\left[\left\| \left|A^{\mu}XB^{1-\mu}\right|^{r}\right\|\cdot\left\|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right\|+\left\|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right\|^{2}\right]$$

$$\leqslant \left\|\left|A^{\mu}XB^{1-\mu}\right|^{r}\right\|\cdot\left\|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right\|.$$

Now, assume that $\frac{1}{2}<\mu\leqslant 1$. Then by applying (6) to $1-\mu$, we get

$$\left\|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right\|^{2}\leqslant\left\|\left|A^{\frac{2\mu+1}{4}}XB^{\frac{3-2\mu}{4}}\right|^{r}\right\|\cdot\left\|\left|A^{\frac{3-2\mu}{4}}XB^{\frac{2\mu+1}{4}}\right|^{r}\right\| \tag{7}$$

$$\leqslant\frac{1}{2\mu-1}\int_{1-\mu}^{\mu}\left\|\left|A^{v}XB^{1-v}\right|^{r}\right\|\cdot\left\|\left|A^{1-v}XB^{v}\right|^{r}\right\|dv$$

$$\leqslant\frac{1}{2}\left[\left\|\left|A^{\mu}XB^{1-\mu}\right|^{r}\right\|\cdot\left\|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right\|+\left\|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right\|^{2}\right]$$

$$\leqslant\left\|\left|A^{\mu}XB^{1-\mu}\right|^{r}\right\|\cdot\left\|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right\|.$$

Since

$$\underset{\mu\longrightarrow\frac{1}{2}}{Lim}\left\|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right|^{r}\right\|^{2}=\underset{\mu\longrightarrow\frac{1}{2}}{Lim}\frac{1}{|2\mu-1|}\left|\int_{1-\mu}^{\mu}\left\|\left|A^{v}XB^{1-v}\right|^{r}\right\|\cdot\left\|\left|A^{1-v}XB^{v}\right|^{r}\right\|dv\right|,$$

the inequalities in (4) follow by combining the inequalities (6) and (7). □

Applying the inequality (3) to the function $f(v)=\left\|\left|A^{v}XB^{1-v}\right|^{r}\right\|\cdot\left\|\left|A^{1-v}XB^{v}\right|^{r}\right\|$ on the interval $[0,\mu]$, when $0<\mu\leqslant\frac{1}{2}$ and on the interval $[\mu,1]$, when $\frac{1}{2}\leqslant\mu<1$, we obtain the following theorem.

THEOREM 3. *Let $A,B,X\in B(H)$ such that A and B are positive. Then*
a) For $0\leqslant\mu\leqslant\frac{1}{2}$, $r>0$, and for every unitarily invariant norm,

$$\left\|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}\right|^{r}\right\|\cdot\left\|\left|A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|^{r}\right\| \tag{8}$$

$$\leqslant\frac{1}{\mu}\int_{0}^{\mu}\left\|\left|A^{v}XB^{1-v}\right|^{r}\right\|\cdot\left\|\left|A^{1-v}XB^{v}\right|^{r}\right\|dv$$

$$\leqslant\frac{1}{4}\left[\left\||AX|^{r}\right\|\cdot\left\||XB|^{r}\right\|+2\left\|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}\right|^{r}\right\|\cdot\left\|\left|A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|^{r}\right\|\right.$$

$$\left.+\left\|\left|A^{\mu}XB^{1-\mu}\right|^{r}\right\|\cdot\left\|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right\|\right]$$

$$\leqslant\frac{1}{2}\left[\left\||AX|^{r}\right\|\cdot\left\||XB|^{r}\right\|+\left\|\left|A^{\mu}XB^{1-\mu}\right|^{r}\right\|\cdot\left\|\left|A^{1-\mu}XB^{\mu}\right|^{r}\right\|\right]$$

b) For $\frac{1}{2}\leqslant\mu\leqslant 1$, $r>0$, and for every unitarily invariant norm,

$$\left\|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}\right|^{r}\right\|\cdot\left\|\left|A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|^{r}\right\| \tag{9}$$

$$\leqslant\frac{1}{1-\mu}\int_{\mu}^{1}\left\|\left|A^{v}XB^{1-v}\right|^{r}\right\|\cdot\left\|\left|A^{1-v}XB^{v}\right|^{r}\right\|dv$$

$$\leqslant \frac{1}{4}\left[\left\||AX|^r\right\|\cdot\left\||XB|^r\right\|+2\left\|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}\right|^r\right\|\cdot\left\|\left|A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|^r\right\|\right.$$

$$\left.+\left\|\left|A^\mu XB^{1-\mu}\right|^r\right\|\cdot\left\|\left|A^{1-\mu}XB^\mu\right|^r\right\|\right]$$

$$\leqslant \frac{1}{2}\left[\left\||AX|^r\right\|\cdot\left\||XB|^r\right\|+\left\|\left|A^\mu XB^{1-\mu}\right|^r\right\|\cdot\left\|\left|A^{1-\mu}XB^\mu\right|^r\right\|\right]$$

In view of the fact that the function $f(v)=\left\|\left|A^v XB^{1-v}\right|^r\right\|\cdot\left\|\left|A^{1-v}XB^v\right|^r\right\|$ is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$, and based on the inequalities (8) and (9), we obtain the following series of refinements of Cauchy-Schwarz norm inequality.

THEOREM 4. *Let A, B, $X\in B(H)$ such that A and B are positive. Then*
a) For $0\leqslant\mu\leqslant\frac{1}{2}$, $r>0$, and for every unitarily invariant norm,

$$\left\|\left|A^\mu XB^{1-\mu}\right|^r\right\|\cdot\left\|\left|A^{1-\mu}XB^\mu\right|^r\right\|$$

$$\leqslant\left\|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}\right|^r\right\|\cdot\left\|\left|A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|^r\right\|$$

$$\leqslant\frac{1}{\mu}\int_0^\mu\left\|\left|A^v XB^{1-v}\right|^r\right\|\cdot\left\|\left|A^{1-v}XB^v\right|^r\right\|dv$$

$$\leqslant\frac{1}{4}\left[\left\||AX|^r\right\|\cdot\left\||XB|^r\right\|+2\left\|\left|A^{\frac{\mu}{2}}XB^{1-\frac{\mu}{2}}\right|^r\right\|\cdot\left\|\left|A^{1-\frac{\mu}{2}}XB^{\frac{\mu}{2}}\right|^r\right\|\right.$$

$$\left.+\left\|\left|A^\mu XB^{1-\mu}\right|^r\right\|\cdot\left\|\left|A^{1-\mu}XB^\mu\right|^r\right\|\right]$$

$$\leqslant\frac{1}{2}\left[\left\||AX|^r\right\|\cdot\left\||XB|^r\right\|+\left\|\left|A^\mu XB^{1-\mu}\right|^r\right\|\cdot\left\|\left|A^{1-\mu}XB^\mu\right|^r\right\|\right].$$

b) For $\frac{1}{2}\leqslant\mu\leqslant 1$, $r>0$, and for every unitarily invariant norm,

$$\left\|\left|A^\mu XB^{1-\mu}\right|^r\right\|\cdot\left\|\left|A^{1-\mu}XB^\mu\right|^r\right\|$$

$$\leqslant\left\|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}\right|^r\right\|\cdot\left\|\left|A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|^r\right\|$$

$$\leqslant\frac{1}{1-\mu}\int_\mu^1\left\|\left|A^v XB^{1-v}\right|^r\right\|\cdot\left\|\left|A^{1-v}XB^v\right|^r\right\|dv$$

$$\leqslant\frac{1}{4}\left[\left\||AX|^r\right\|\cdot\left\||XB|^r\right\|+2\left\|\left|A^{\frac{1+\mu}{2}}XB^{\frac{1-\mu}{2}}\right|^r\right\|\cdot\left\|\left|A^{\frac{1-\mu}{2}}XB^{\frac{1+\mu}{2}}\right|^r\right\|\right.$$

$$\left.+\left\|\left|A^\mu XB^{1-\mu}\right|^r\right\|\cdot\left\|\left|A^{1-\mu}XB^\mu\right|^r\right\|\right]$$

$$\leqslant\frac{1}{2}\left[\left\||AX|^r\right\|\cdot\left\||XB|^r\right\|+\left\|\left|A^\mu XB^{1-\mu}\right|^r\right\|\cdot\left\|\left|A^{1-\mu}XB^\mu\right|^r\right\|\right].$$

3. Inequalities close to Cauchy-Schwarz norm inequality

In this section, we prove that the function $f(v)=\left\|\left|A^v XB^v\right|^r\right\|\cdot\left\|\left|A^{1-v}XB^{1-v}\right|^r\right\|$ is convex on the interval $[0,1]$ and use this convexity to obtain some inequalities that are related to Cauchy-Schwarz norm inequality.

THEOREM 5. *Let A, B, $X \in B(H)$ such that A and B are positive. Then for $r > 0$, and for every unitarily invariant norm, the function*

$$f(v) = \||A^v X B^v|^r\| \cdot \||A^{1-v} X B^{1-v}|^r\|$$

is convex on $[0,1]$ and attains its minimum at $v = \frac{1}{2}$. consequently, it is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$.

Proof. Without loss of generality, we may assume that $A > 0$ and $B > 0$. Since $f(v)$ is continuous and symmetric with respect to $v = \frac{1}{2}$, all the conclusions will follow after we show that

$$f(v) \leqslant \frac{1}{2}[f(v-s) + f(v+s)] \tag{10}$$

for $v \pm s \in [0,1]$. By the inequality (2), we have

$$\||A^v X B^v|^r\| = \||A^s(A^{v-s} X B^{v+s})B^{-s}|^r\|$$
$$\leqslant \left[\||A^{2s}(A^{v-s} X B^{v+s})|^r\| \cdot \||(A^{v-s} X B^{v+s})B^{-2s}|^r\|\right]^{\frac{1}{2}}$$
$$= \left[\||A^{v+s} X B^{v+s}|^r\| \cdot \||A^{v-s} X B^{v-s}|^r\|\right]^{\frac{1}{2}}$$

and

$$\left\|\left|A^{1-v} X B^{1-v}\right|^r\right\| = \left\|\left|A^s(A^{1-v-s} X B^{1-v+s})B^{-s}\right|^r\right\|$$
$$\leqslant \left[\left\|\left|A^{2s}(A^{1-v-s} X B^{1-v+s})\right|^r\right\| \cdot \left\|\left|(A^{1-v-s} X B^{1-v+s})B^{-2s}\right|^r\right\|\right]^{\frac{1}{2}}$$
$$= \left[\left\|\left|A^{1-(v-s)} X B^{1-(v-s)}\right|^r\right\| \cdot \left\|\left|A^{1-(v+s)} X B^{1-(v+s)}\right|^r\right\|\right]^{\frac{1}{2}}$$

Upon multiplication of the above two inequalities we obtain

$$\||A^v X B^v|^r\| \cdot \left\|\left|A^{1-v} X B^{1-v}\right|^r\right\| \leqslant [f(v-s) \cdot f(v+s)]^{\frac{1}{2}}. \tag{11}$$

Applying the arithmetic-geometric mean inequality to the right hand side of (11) yields (10). This completes the proof. □

Since the function $f(v) = \||A^v X B^v|^r\| \||A^{1-v} X B^{1-v}|^r\|$ is convex on $[0,1]$, attains its minimum at $v = \frac{1}{2}$, and its maximum at $v = 0$ and $v = 1$, and $f(v) = f(1-v)$ for $0 \leqslant v \leqslant 1$, we have the following results.

THEOREM 6. *Let A, B, $X \in B(H)$ such that A and B are positive. Then for $0 \leqslant v \leqslant 1$, $r > 0$, and for every unitarily invariant norm,*

$$\left\|\left|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right|^r\right\|^2 \leqslant \||A^v X B^v|^r\| \cdot \left\|\left|A^{1-v} X B^{1-v}\right|^r\right\| \leqslant \||X|^r\| \cdot \||AXB|^r\| \tag{12}$$

A special case of the inequality (12) can be obtained as follows.

THEOREM 7. *Let $A,\ B \in B(H)$ such that A and B are positive. Then for $0 \leqslant v \leqslant 1,\ r > 0$, and for the operator norm,*

$$\left\| \left| A^{\frac{1}{2}} B^{\frac{1}{2}} \right|^r \right\|^2 \leqslant \left\| \left| A^v B^v \right|^r \right\| \cdot \left\| \left| A^{1-v} B^{1-v} \right|^r \right\| \leqslant \left\| |AB|^r \right\| \tag{13}$$

In particular,

$$\left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\|^2 \leqslant \left\| A^v B^v \right\| \cdot \left\| A^{1-v} B^{1-v} \right\| \leqslant \|AB\|$$

REMARK 8. The operator norm inequality $\left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\| \leqslant \|AB\|^{\frac{1}{2}}$ for $A,\ B \geqslant 0$ mentioned above is equivalent to the Löwner-Heinz inequality, Heinz-Kato inequality and moreover,

$$\|A^r B^r\| \leqslant \|AB\|^r \text{ for } r \in [0,1],$$

see [3].

An equivalent formulation of the inequality (13) is obtained in the following theorem.

THEOREM 9. *Let $A,\ B \in B(H)$ such that A and B are positive. Then for $t \geqslant 1,\ r > 0$, and for the operator norm,*

$$\left\| \left| A^{\frac{t}{2}} B^{\frac{t}{2}} \right|^r \right\|^2 \leqslant \left\| |AB|^r \right\| \cdot \left\| \left| A^{t-1} B^{t-1} \right|^r \right\| \leqslant \left\| \left| A^t B^t \right|^r \right\|$$

Proof. From Theorem 7, we have

$$\left\| \left| A^{\frac{1}{2}} B^{\frac{1}{2}} \right|^r \right\|^2 \leqslant \left\| \left| A^{\frac{1}{t}} B^{\frac{1}{t}} \right|^r \right\| \cdot \left\| \left| A^{1-\frac{1}{t}} B^{1-\frac{1}{t}} \right|^r \right\| \leqslant \left\| |AB|^r \right\|, \text{ for } t \geqslant 1.$$

Replace $A,\ B$ by $A^t,\ B^t$, respectively. This completes the proof. $\square$

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