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SIMULTANEOUS APPROXIMATION WITH THE RAFU METHOD

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Abstract. Let f be a function k times continuously differentiable in $[a,b]$, then we will prove that the RAFU method provides a sequence $(H_n)_n$ defined in $[a,b]$ such that for each $j=0,\ldots,k$

$$\left\|f^{(j)}-H_n^{(j)}\right\| \leqslant \left[\frac{M-m}{\sqrt{n}}+\omega\left(f^{(k)},\frac{b-a}{n}\right)\right](b-a)^{k-j}$$

being $n \geqslant 2$, $\|.\|$ the uniform norm, M and m the maximum and the minimum of $f^{(k)}$ in $[a,b]$ respectively and $\omega\left(f^{(k)},\frac{b-a}{n}\right)$ its modulus of continuity. The called RAFU remainder in Taylor's formula will be presented. The simultaneous approximation problem will be solved from average samples, from linear combinations, from local average samples given by convolution, from approximate values and in the case of non-uniformly spaced data of $f^{(k)}$. We will also study the numerical differentiation case. Our approach is easily realizable in computations. Some examples will be given.

1. Introduction

Let f be an arbitrary function defined in $[a,b]$ and let $a=x_0<x_1<\ldots<x_n=b$ be a partition of $[a,b]$ for each natural n. The RAFU (radical functions) method on approximation is an approximation procedure to the function f by a sequence $(C_n)_n$ of radical functions defined by the formula

$$C_n(x)=f(x_1)+\sum_{p=2}^{n}[f(x_p)-f(x_{p-1})]\cdot F_n(x_{p-1},x) \tag{1}$$

being $F_n(x_p,x)=\frac{\sqrt[2n+1]{x_p-a}+\sqrt[2n+1]{x-x_p}}{\sqrt[2n+1]{b-x_p}+\sqrt[2n+1]{x_p-a}}$, $p=1,\ldots,n-1$. For details about RAFU approximation, we refer to reader to [1, 2, 3].

Most of existed studies on the simultaneous approximation problem were only concerned with the density, so theoretical results are very difficult to apply in practice. For example, from the result due to K. Kopotun [6], one observes that it is not easy to find the algebraic polynomials (more precisely the linear operators) which approximate a function and its derivatives. In [4] the hypothesis of the procedure of Hermite interpolation given by B. Della Vecchia, G. Mastroianni and P. Vértesi are difficult to use. The problem is solved using feed-forward neural networks by N. Hahm and B. I. Hong

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in [5] and T.F. Xie and F.L. Cao in [7]. Both approximation methods are constructed from the initial function f.

Our proposal is different. Our approach is original and unknown. This paper deals with a constructive approximation procedure in which the terms of $(H_n)_n$ will be defined from $f^{(j)}(a)$, $j=0, 1, ..., k$ and the values of $f^{(k)}(x)$ at some points of $[a,b]$. So, this way to solve the simultaneous approximation problem can be useful to solve, for instance, differential equations. Moreover, we will also see we can obtain the above main result by considering average samples, linear combinations, local average samples given by convolution or approximate values of the previous data. The mentioned theorem holds using numerical approximations of the derivatives if necessary. We will also give an error formula in the case of non-uniformly spaced data.

A trivial corollary will show that in simultaneous approximation Taylor's polynomial and RAFU approximation are connected and this is another special contribution of this work.

The paper does not impress with the difficulties it overcomes. It does not contain complicated calculations or reasonings, but we think that the importance of this technique to solve these problems will balance the deficiency of difficulties. The paper is organized as follows. In Section 2 we give the main results. In Section 3 we will show that concise algorithms can be easily implemented to solve the simultaneous approximation problem and some examples will be shown. All proofs are in Section 4.

2. Main results

An important question that remained to be answered in RAFU approximation dealt with differentiability, namely each radical function $F_n(x_p,x)$ is not differentiable at $x=x_p$. In the works published about the RAFU method until now [1, 2, 3], all function f was approximated by using a sequence $(C_n)_n$, defined as (1), but from now on, if f is a smooth function, this approximation procedure provides a sequence of functions $(H_n)_n$ with the same smoothness that f and which converges uniformly to it.

THEOREM 1. *Let f be a function k times continuously differentiable in $[a,b]$, then there exists a sequence $(H_n)_n$ defined in $[a,b]$ such that for each $j=0,...,k$*

$$\left\|f^{(j)}-H_n^{(j)}\right\| \leqslant \left[\frac{M-m}{\sqrt{n}}+\omega\left(f^{(k)},\frac{b-a}{n}\right)\right](b-a)^{k-j}$$

being $n \geqslant 2$, $\|.\|$ the uniform norm, M and m the maximum and the minimum of $f^{(k)}$ in $[a,b]$, $\omega\left(f^{(k)},\frac{b-a}{n}\right)$ its modulus of continuity and $H_n(x)=\sum_{i=0}^{k-1} f^{(i)}(a)\frac{(x-a)^i}{i!}+G_n(x)$, where $G_n(x)=\int_a^x G_n'(t)dt$, $G_n'(x)=\int_a^x G_n''(t)dt$,..., $G_n^{(k-1)}(x)=\int_a^x C_n(t)dt$ and

$$C_n(x)=f^{(k)}(x_1)+\sum_{p=2}^{n}[f^{(k)}(x_p)-f^{(k)}(x_{p-1})]\cdot F_n(x_{p-1},x) \tag{2}$$

being $F_n(x_p,x)=\frac{\sqrt[2n+1]{x_p-a}+\sqrt[2n+1]{x-x_p}}{\sqrt[2n+1]{b-x_p}+\sqrt[2n+1]{x_p-a}}$, $p=1, ..., n-1$, $x_i=a+ih$, $i=0, ...,n$ and $h=\frac{b-a}{n}$.

If f is a function k times differentiable in $[a,b]$, it is well-known that by the Taylor's formula, $f(x) = T(f;a,x) + R(f;a,x)$, where $T(f;a,x) = \sum_{i=0}^{k-1} \frac{f^{(i)}(a)(x-a)^i}{i!}$ is Taylor's polynomial of degree $k-1$ of f at $x=a$ and $R(f;a,x) = \frac{f^{(k)}(\alpha)(x-a)^k}{k!}$, for some α between a and x, is Taylor's remainder. In simultaneous approximation, Taylor's polynomial and RAFU approximation are connected. More exactly,

COROLLARY 1. *With the hypothesis of Theorem 1, we have*

$$\|R(f;a,x) - G_n(x)\| \leqslant \left[\frac{M-m}{\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)\right](b-a)^k$$

where $R(f;a,x)$ is Taylor's remainder of f.

DEFINITION 1. Let f be a function k times continuously differentiable in $[a,b]$ and let $(G_n)_n$ be the sequence uniformly convergent to $R(f;a,x) = \frac{f^{(k)}(\alpha)(x-a)^k}{k!}$ on $[a,b]$. We will say that the function G_n is the RAFU remainder of degree n of f.

The function f and its derivatives can be approximated from average samples of the data $f^{(k)}(x_p)$, $p=1, ..., n$ used in Theorem 1. In fact,

COROLLARY 2. *If the data $f^{(k)}(x_p)$, $p=1, ..., n$ in (2) are substituted by $k_p = \frac{f^{(k)}(x_{p1})n_1+...+f^{(k)}(x_{ps})n_s}{n_1+...+n_s}$, $x_{1q} \in [a,x_1]$ or $x_{pq} \in (x_{p-1},x_p]$, $p=2,...,n$, $q=1,...,s$, $n_1+...+n_s \neq 0$, then Theorem 1 holds.*

In many applications it is more realistic to assume that the available samples are local average samples near a certain x. We consider the special case in which we know data of the type $\left(\chi_{[-h,h]} \star f^{(k)}\right)(x) = \int_{-\infty}^{+\infty} \chi_{[-h,h]}(y) f^{(k)}(x-y)dy = \int_{x-h}^{x+h} f^{(k)}(z)dz$ where $\star$ denotes the convolution of the functions $\chi_{[-h,h]}$ and $f^{(k)}$. From these data, an analogous assertion to Theorem 1 can be established.

COROLLARY 3. *If the data $f^{(k)}(x_p)$, $p=1, ..., n$ in (2) are defined by $k_p = \frac{\int_{\tilde{x}_p-h}^{\tilde{x}_p+h} f^{(k)}(z)dz}{2h}$, with $[\tilde{x}_1-h, \tilde{x}_1+h] \subseteq [a,x_1]$ or $[\tilde{x}_p-h, \tilde{x}_p+h] \subseteq (x_{p-1},x_p]$, $p=2,...,n$, then Theorem 1 follows.*

The simultaneous approximation problem can also be solved from linear combinations of the values $f^{(k)}(x_p)$, $p=1, ..., n$ given in Theorem 1.

COROLLARY 4. *If the values $f^{(k)}(x_p)$, $p=1, ..., n$ in (2) are changed by $k_p = \frac{f^{(k)}(\tilde{x}_p)-f^{(k)}(\tilde{x}_{p-1})}{\tilde{x}_p-\tilde{x}_{p-1}} \cdot \left(x'_p - \tilde{x}_{p-1}\right) + f^{(k)}(\tilde{x}_{p-1})$ with $x'_1 \in [\tilde{x}_0, \tilde{x}_1] \subseteq [a,x_1]$ or $x'_p \in [\tilde{x}_{p-1}, \tilde{x}_p] \subseteq (x_{p-1},x_p]$, $p=2,..., n$, then Theorem 1 holds.*

When we want to approach f and its derivatives from approximate values of $f^{(k)}(x_p)$, $p=1, ..., n$, the following result can be useful.

COROLLARY 5. *With the hypothesis of Theorem* 1*, if the values* $f^{(k)}(x_p)$, $p=1$, ..., n *in (2) are unknown but we know* $f^{(k)}(x_p)+\eta_p$*, with* $\left|\eta_p\right|<\eta$, $p=1,...,n$ *then*

$$\left\|f^{(j)}-H_n^{(j)}\right\| \leqslant \left[\frac{M-m+2\eta}{\sqrt{n}}+\omega\left(f^{(k)},\frac{b-a}{n}\right)+\eta\right](b-a)^{k-j}$$

To constructing the functions H_n we need to know $f^{(j)}(a)$, $j=0,\ 1,\ ...,\ k-1$ and $f^{(k)}(x_p)$, $p=1,\ ...,\ n$. This can be a problem in practice. Next, we give a solution to this drawback using numerical approximations of the first and second derivatives.

THEOREM 2. *If a function* f *has four continuous derivatives in* $[a,b]$*, then there exists a sequence* $(H_n)_n$ *defined in* $[a,b]$ *such that* $\left\|f''-H_n''\right\|\leqslant K_n$, $\left\|f'-H_n'\right\|\leqslant K_n(b-a)+\frac{h}{2}M_2$ *and* $\|f-H_n\|\leqslant K_n(b-a)^2+\frac{h}{2}M_2(b-a)$ *where*

$$K_n=\left[\frac{M_2-m_2+\frac{(M_1-m_1)(b-a)^2}{48n^2}}{\sqrt{n}}+\omega\left(f'',\frac{b-a}{2n}\right)+\frac{(b-a)^2M_1}{48n^2}\right]$$

$n\geqslant 2$, M_1, m_1 *and* M_2, m_2 *are the maximum and the minimum of* $f^{(4)}$ *and* f'' *in* $[a,b]$ *respectively,* $\omega\left(f'',\frac{b-a}{2n}\right)$ *the modulus of continuity of* f'', $H_n(x)=\frac{f(a+h)-f(a)}{h}(x-a)+f(a)+G_n(x)$*, being* $G_n(x)=\int_a^x G_n'(t)dt$, $G_n'(x)=\int_a^x C_n(t)dt$ *and*

$$C_n(x)=k_1+\sum_{p=2}^{n}\left[k_p-k_{p-1}\right]\cdot F_n(x_{p-1},x)$$

where $F_n(x_p,x)=\frac{\sqrt[2n+1]{x_p-a}+\sqrt[2n+1]{x-x_p}}{\sqrt[2n+1]{b-x_p}+\sqrt[2n+1]{x_p-a}}$, $p=1,\ ...\ ,\ n-1$, $x_i=a+ih$, $i=0,\ ...\ ,n$, $h=\frac{b-a}{n}$ *and the* k_j *are given by the formulas* $k_p=\frac{f(x_p)-2f\left(\frac{x_p+x_{p-1}}{2}\right)+f(x_{p-1})}{\left(\frac{h}{2}\right)^2}$, $p=1$, ..., n.

COROLLARY 6. *If we define in Theorem* 2 *the data* k_p *by the formula* $k_p=\frac{f(x_{p+1})-2f(x_p)+f(x_{p-1})}{h^2}$, $p=1,\ ...,\ n-1$ *and* $k_n=k_{n-1}$*, then we only need to know the values* $f(x_p)$ $p=0,\ ...,\ n$ *and the mentioned theorem is also valid. In this case,*

$$K_n=\left[\frac{M_2-m_2+\frac{(M_1-m_1)(b-a)^2}{12n^2}}{\sqrt{n}}+\omega\left(f'',\frac{b-a}{n}\right)+\frac{(b-a)^2M_1}{12n^2}\right]$$

Finally, the RAFU method can be used to solve the simultaneous approximation problem in the case of non-uniformly spaced data.

THEOREM 3. *Let* $P_n=\{a=x_0,x_1,...,x_{s_n}=b\}$ *be a partition of* $[a,b]$ *with* $\delta(s_n)=\min_{1\leqslant j\leqslant s_n}\left|x_j-x_{j-1}\right|$ *and* $\Delta(s_n)=\max_{1\leqslant j\leqslant s_n}\left|x_j-x_{j-1}\right|$ *such that* $\frac{3(b-a)}{n^K}\leqslant\delta(s_n)\leqslant\Delta(s_n)\leqslant h$

being $h = \frac{b-a}{n}$ *and* $K \geqslant 2$ *a positive integer. Let* f *be a function* k *times continuously differentiable in* $[a,b]$, *then there exists a sequence* $(H_n)_n$ *defined in* $[a,b]$ *such that for each* $j = 0, \dots, k$

$$\left\| f^{(j)} - H_n^{(j)} \right\| \leqslant \left[\frac{6K}{5} \frac{M-m}{\sqrt{n}} + \omega\left(f^{(k)}, \Delta(s_n) \right) \right] (b-a)^{k-j}$$

being $n \geqslant 2$, M *and* m *the maximum and the minimum of* $f^{(k)}$ *in* $[a,b]$ *respectively and* $\omega\left(f^{(k)}, \Delta(s_n)\right)$ *its modulus of continuity,* H_n *as usual and*

$$C_n(x) = f^{(k)}(x_1) + \sum_{p=2}^{s_n} [f^{(k)}(x_p) - f^{(k)}(x_{p-1})] \cdot F_n(x_{p-1}, x)$$

with $F_n(x_p, x) = \frac{\sqrt[2n+1]{x_p - a} + \sqrt[2n+1]{x - x_p}}{\sqrt[2n+1]{b - x_p} + \sqrt[2n+1]{x_p - a}}$, $p = 1, \dots, s_n - 1$.

3. Algorithms and examples

We want to know if the approximation procedure is easy to implement. This will be true if the expressions of the functions $G_n^{(j)}(x)$ can be obtained easily. The next result shows that this it is possible.

LEMMA 1. *If we define* $G_n(x) = \int_a^x G_n'(t)dt$, $G_n'(x) = \int_a^x G_n''(t)dt, \dots, G_n^{(k-1)}(x) = \int_a^x C_n(t)dt$ *where*

$$C_n(x) = M + N \sqrt[2n+1]{x}$$

being M *and* N *real numbers, then for all* $j = 0, \dots, k-1$ *it verifies that*

$$G_n^{(j)}(x) = M \frac{x^{k-j}}{(k-j)!} + N \frac{(2n+1)^{k-j} x^{k-j} \sqrt[2n+1]{x}}{\prod_{i=1}^{k-j} (2ni + i + 1)}$$

$$- \left[K_{1,a} \frac{x^{k-j-1}}{(k-j-1)} + \dots + K_{k-j-1,a} x + K_{k-j,a} \right]$$

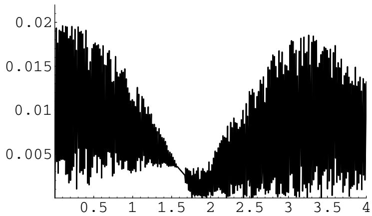
where $K_{1,a}, \dots, K_{k-j,a}$ *are real numbers.*

Proof. It is trivial. □

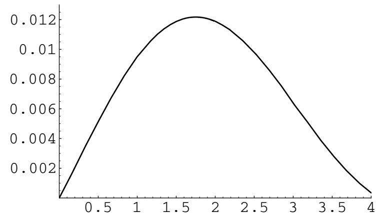
From this result, the formulas of the functions $G_n^{(j)}$ in the case in which $x \in [x_{p-1}, x_p]$ and the function C_n is defined as (2) are given by

$$G_n^{(j)}(x) = \left[k_1 + \sum_{i=2}^{p} \frac{(k_i - k_{i-1}) \sqrt[2n+1]{x_{i-1} - a}}{\sqrt[2n+1]{b - x_{i-1}} + \sqrt[2n+1]{x_{i-1} - a}} \right] \frac{x^{k-j}}{(k-j)!} +$$

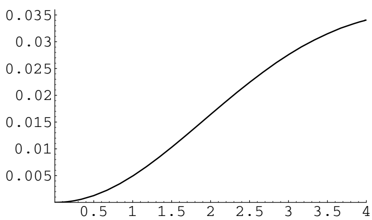
$$\frac{(2n+1)^{k-j}}{\prod_{i=1}^{k-j} (2ni + i + 1)} \left[\sum_{i=2}^{p} \frac{(k_i - k_{i-1}) (x - x_{i-1})^{k-j} \sqrt[2n+1]{x - x_{i-1}}}{\sqrt[2n+1]{b - x_{i-1}} + \sqrt[2n+1]{x_{i-1} - a}} \right]$$



(a) Error approximation to f”



(b) Error approximation to f’



(c) Error approximation to f

Figure 1: Simultaneous approximation with Theorem 1

$$-\left[K_{1,a}\frac{x^{k-j-1}}{(k-j-1)}+...+K_{k-j-1,a}x+K_{k-j,a}\right]$$

and this provides that easy algorithms can be implemented.

Consider the function $f(x)=sin(x)$ and its first and second derivatives defined in $[0,4]$.

In Figure 1 we show the approximation errors by using Theorem 1 and $n=200$.

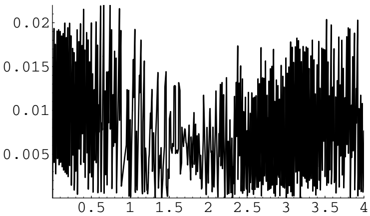
The simultaneous approximation errors in case of Corollary 5 and $n=200$ are in Figure 2. Here $f^{(k)}(x_p)+\eta_p=f^{(k)}(x_p)\left(1+0.01Random[Real,\{-1,1\}]\right)$, $p=1,...,n$ where Random denotes a random number with uniform distribution on $[-1,1]$ and 0.01 is the considered relative error level of the data.

In Figure 3 we give the approximation errors by using the numerical differentiation formulas mentioned in Corollary 6 for $n=200$.

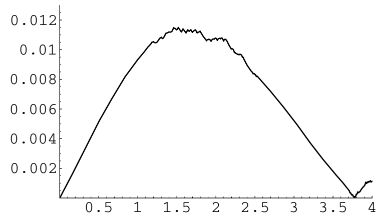
Theorem 3 is used to approach the $f(x)=sin(x)$ and its first and second derivatives in $[0,0.5]$ by considering $n=10$, $K=2$ and $s_n=15$. In this example we have used the values of f'' at the points of the partition

$$P_n=\{0,0.02,0.05,0.09,0.14,0.17,0.21,0.26,0.31,0.34,0.39,0.42,0.44,0.48,0.5\}$$

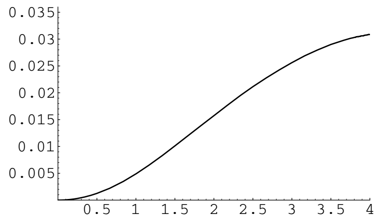
which verify that $0.015\leqslant\delta(s_n)\leqslant\Delta(s_n)\leqslant0.05$. The results are in Figure 4.



(a) Error approximation to f”

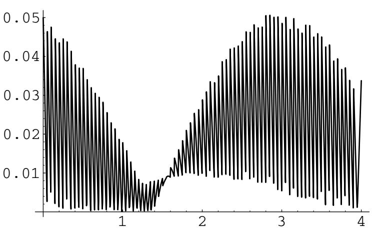


(b) Error approximation to f’

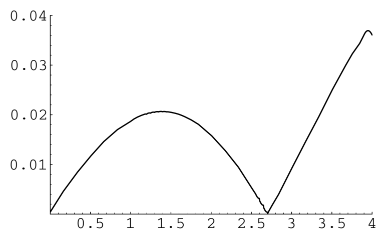


(c) Error approximation to f

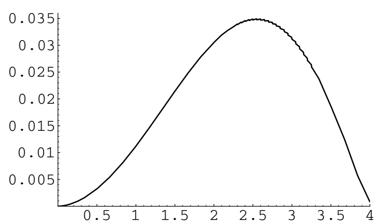
Figure 2: Simultaneous approximation with approximate data



(a) Error approximation to f”

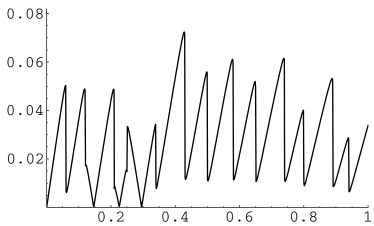


(b) Error approximation to f’

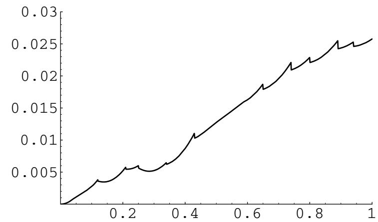


(c) Error approximation to f

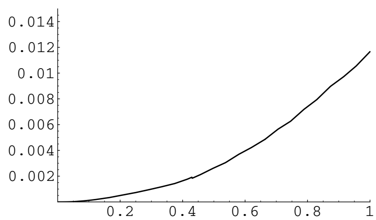
Figure 3: Simultaneous approximation with numerical differentiation formulas



(a) Error approximation to f”



(b) Error approximation to f’



(c) Error approximation to f

Figure 4: Simultaneous approximation in the case of non-uniformly spaced data

4. Proofs

Until the proof of Corollary 6 we will consider partitions P_n of $[a,b]$ with $x_j = a + j\cdot\frac{b-a}{n}$, $j = 0,1,...,n$. Moreover, each interval $[x_{k-1},x_k]$ of length $\frac{b-a}{n}$ will be divided into three equal parts: $\left[x_{k-1},x_{k-1}+\frac{b-a}{3n}\right]$, $\left[x_{k-1}+\frac{b-a}{3n},x_k-\frac{b-a}{3n}\right]$, $\left[x_k-\frac{b-a}{3n},x_k\right]$

PROPOSITION 1. *Let P_n be a partition of $[a,b]$ and let E_n be the step function defined by* $E_n(x) = \begin{cases} k_1 & x\in[a,x_1] \\ k_2 & x\in(x_1,x_2] \\ \dots & \\ k_n & x\in(x_{n-1},b] \end{cases}$ *being k_j real numbers. Let C_n be the radical function associated to E_n, $C_n(x) = k_1 + \sum_{j=2}^{n}[k_j - k_{j-1}]\cdot F_n\left(x_{j-1},x\right)$. Then, for all $n \geqslant 2$ it follows that*

$|C_n(x) - E_n(x)| \leqslant \frac{M_n - m_n}{\sqrt{n}}$ *if* $x \in [a,b]\setminus \cup_{k=1}^{n-1}\left(x_k - \frac{b-a}{3n}, x_k + \frac{b-a}{3n}\right)$

$\left|C_n(x) - \left[k_j(1-\alpha_x) + k_{j+1}\alpha_x\right]\right| \leqslant \frac{M_n - m_n}{\sqrt{n}}$ *if* $x \in \left(x_j - \frac{b-a}{3n}, x_j + \frac{b-a}{3n}\right)$ *and* $j = 1,..., n-1$

being M_n and m_n the maximum and the minimum of the k_j respectively and $\alpha_x \in (0,1)$ a number that depends upon x.

For a proof, the reader can see [2].

PROPOSITION 2. *Let f be a continuous function defined in $[a,b]$. Then there* *exists a sequence of radical functions* $(C_n)_n$ *defined in* $[a,b]$ *as (1), such that*

$$\|C_n - f\| \leqslant \frac{M-m}{\sqrt{n}} + \omega\left(f, \frac{b-a}{n}\right)$$

for all $n \geqslant 2$*, where* $\|.\|$ *denotes the uniform norm,* M *and* m *are the maximum and the minimum of* f *in* $[a,b]$ *respectively and* $\omega\left(f, \frac{b-a}{n}\right)$ *its modulus of continuity.*

For details, the reader can see [2].

Proof. Theorem 1.

It is well-known that if $g \in C[a,b]$, then $G(x) = \int_a^x g(t)dt$ is well-defined, $G'(x) = g(x)$ on $[a,b]$ and $\int_a^b g(t)dt = G(b) - G(a)$. Moreover, it is also known that if $(g_n)_n$ is a sequence of continuous functions defined on $[a,b]$ which converges uniformly to the function g on $[a,b]$ and we define $G_n(x) = \int_a^x g_n(t)dt$, then the sequence $(G_n)_n$ converges uniformly to the function G on $[a,b]$.

Define the continuous functions C_n in $[a,b]$ from the continuous function $f^{(k)}$ as (1). Moreover, we consider the following well-defined functions in $[a,b]$ as $G_n^{(j)}(x) = \int_a^x G_n^{(j+1)}(t)dt$ for $j = 0, ..., k-2$ and $G_n^{(k-1)}(x) = \int_a^x C_n(t)dt$. Then, by Proposition 2 and the above properties, the following limits are uniform on $[a,b]$

$$\lim_{n\to\infty} C_n(x) \to f^{(k)}(x)$$

and for all $j = 0, ..., k-1$

$$\lim_{n\to\infty} G_n^{(j)}(x) = f^{(j)}(x) - f^{(k-1)}(a)\frac{(x-a)^{k-1-j}}{(k-1-j)!} - ... - f^{(j+1)}(a)(x-a) - f^{(j)}(a)$$

Now, we define $H_n(x) = \sum_{i=0}^{k-1} f^{(i)}(a)\frac{(x-a)^i}{i!} + G_n(x)$ and we obtain for the k index

$$\left\| f^{(k)}(x) - \left(\sum_{i=0}^{k-1} f^{(i)}(a)\frac{(x-a)^i}{i!} + G_n(x)\right)^{(k)} \right\|$$

$$= \left\| f^{(k)}(x) - C_n(x) \right\| \leqslant \frac{M-m}{\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)$$

and for the $k-1$ index

$$\left\| f^{(k-1)}(x) - \left(\sum_{i=0}^{k-1} f^{(i)}(a)\frac{(x-a)^i}{i!} + G_n(x)\right)^{(k-1)} \right\|$$

$$= \left\| f^{(k-1)}(x) - f^{(k-1)}(a) - G_n^{(k-1)}(x) \right\|$$

$$= \left\| \int_a^x f^{(k)}(t)dt - \int_a^x C_n(t)dt \right\| \leqslant \left[\frac{M-m}{\sqrt{n}} + \omega\left(f^{(k)}, \frac{b-a}{n}\right)\right](b-a)$$

Proceeding in this way, we finish obtaining that

$$\left\| f(x) - \left(\sum_{i=0}^{k-1} f^{(i)}(a) \frac{(x-a)^i}{i!} + G_n(x) \right) \right\|$$

$$= \left\| f(x) - f^{(k-1)}(a) \frac{(x-a)^{k-1}}{(k-1)!} - \ldots - f^{'}(a)(x-a) - f(a) - G_n(x) \right\|$$

$$\leqslant \left[\frac{M-m}{\sqrt{n}} + \omega \left(f^{(k)}, \frac{b-a}{n} \right) \right] (b-a)^k$$

This completes the proof. □

Proof of Corollary 1 is trivial by using Taylor's formula and Theorem 1.

Proof. Corollaries 2, 3, 4 and 5.

Results of Corollaries 2, 3 and 4 are obtained taking into account that in all these cases the sequence $(C_n)_n$ verifies that $\left| C_n(x) - f^{(k)}(x) \right| \leqslant \frac{M-m}{\sqrt{n}} + \omega \left(f^{(k)}, \frac{b-a}{n} \right)$ and this can be easily proved from Propositions 1 and 2. Corollary 5 holds because the sequence $(C_n)_n$ obtained from approximate data verifies that $\left| C_n(x) - f^{(k)}(x) \right| \leqslant \frac{M-m+2\eta}{\sqrt{n}} + \omega \left(f^{(k)}, \frac{b-a}{n} \right) + \eta$ and this can be easily proved from Propositions 1 and 2 too. See [2] for details. □

Next, we give

Proof. Theorem 2.

Define

$$E_n(x) = \begin{cases} k_1 & x \in [a, x_1] \\ k_2 & x \in (x_1, x_2] \\ \ldots & \\ k_n & x \in (x_{n-1}, b] \end{cases}$$

where $k_p = \frac{f(x_p) - 2f\left(\frac{x_p + x_{p-1}}{2}\right) + f(x_{p-1})}{\left(\frac{h}{2}\right)^2}$, $p = 1, \ldots, n$. Taking into account the second-order approximation of the second derivative,

$$\left| E_n(x) - f^{''}(x) \right| = \left| \frac{f(x_{p+1}) - 2f\left(\frac{x_p + x_{p+1}}{2}\right) + f(x_p)}{\left(\frac{h}{2}\right)^2} - f''(x) \right| =$$

$$\left| f^{''}\left(\frac{x_p + x_{p+1}}{2}\right) + \frac{\left(\frac{h}{2}\right)^2}{12} f^{(4)}(\eta_p) - f^{''}(x) \right| \leqslant \omega \left(f^{''}, \frac{b-a}{2n} \right) + \frac{(b-a)^2 M_1}{48n^2}$$

where η_p is a real number belongs to (x_p, x_{p+1}) and $x \in (x_p, x_{p+1}]$ for some p.

On the other hand, in proofs of Propositions 1 and 2, for all $p = 1, ..., n$ we can take $k_p = \frac{f(x_p)-2f\left(\frac{x_p+x_{p-1}}{2}\right)+f(x_{p-1})}{\left(\frac{h}{2}\right)^2}$ and obtain for $x \in [a,b]$ that

$$|C_n(x) - E_n(x)| \leqslant \frac{1}{\sqrt{n}}\left[M_2 - m_2 + \frac{(M_1 - m_1)(b-a)^2}{48n^2}\right]$$

Thus, $\left\|C_n - f''\right\| \leqslant K_n$ where

$$K_n = \frac{M_2 - m_2 + \frac{(M_1-m_1)(b-a)^2}{48n^2}}{\sqrt{n}} + \omega\left(f'', \frac{b-a}{2n}\right) + \frac{(b-a)^2 M_1}{48n^2}$$

Define $G_n(x) = \int_a^x G_n'(t)dt$, $G_n'(x) = \int_a^x C_n(t)dt$ and

$$H_n(x) = \frac{f(a+h) - f(a)}{h}(x-a) + f(a) + G_n(x)$$

In this way,

$$\left\|f''(x) - H_n''(x)\right\| = \left\|f''(x) - C_n(x)\right\| \leqslant K_n$$

Moreover

$$\begin{aligned}
&\left\|f'(x) - H_n'(x)\right\| \\
&= \left\|f'(x) - G_n'(x) - \frac{f(a+h)-f(a)}{h}\right\| \\
&= \left\|\int_a^x f''(t)dt - \int_a^x C_n(t)dt - \frac{f(a+h)-f(a)}{h} + f'(a)\right\| \\
&\leqslant K_n(b-a) + \frac{h}{2}M_2
\end{aligned}$$

and finally

$$\begin{aligned}
&\|f(x) - H_n(x)\| \\
&= \left\|f(x) - G_n(x) - \frac{f(a+h)-f(a)}{h}(x-a) - f(a)\right\| \\
&= \left\|f(x) - G_n(x) - \left(f'(a) + \frac{h}{2}f''(\eta)\right)(x-a) - f(a)\right\| \\
&\leqslant K_n(b-a)^2 + \frac{h}{2}M_2(b-a)
\end{aligned}$$

and this completes the proof. □

Proof of Corollary 6 is trivial.

From now on, we will consider partitions $P_n = \{x_0 = a, x_1, ..., x_n = b\}$ of $[a,b]$ with non-uniformly spaced data. Before proving Theorem 3 we have to prove some new results.

LEMMA 2. *Let k be a positive integer. Then, for $n \geqslant 2$ it verifies that* $\left| \sqrt[2n+1]{n^k} - 1 \right| \leqslant \frac{(2k-1)\sqrt[7]{3}}{2\sqrt{n}}$ *and* $\left| \sqrt[2n+1]{\frac{1}{n^k}} - 1 \right| \leqslant \frac{k}{3\sqrt{n}}$

Proof. By induction on k. Cases $k = 1$ are in [3]. The proof finishes taking into account that

$$\left| \sqrt[2n+1]{n^{\pm k}} - 1 \right| = \left| \sqrt[2n+1]{n^{\pm k}} - \sqrt[2n+1]{n^{\pm 1}} + \sqrt[2n+1]{n^{\pm 1}} - 1 \right|. \quad \square$$

LEMMA 3. *Let $P_n = \{a = x_0, x_1, \ldots, x_s = b\}$ be a partition of $[a,b]$ with $\delta(s) = \min_{1 \leqslant j \leqslant s} |x_j - x_{j-1}|$. Then, for any $k = 1, \ldots, s-1$ and $x \in [a,b] \setminus \left(x_k - \frac{\delta(s)}{3}, x_k + \frac{\delta(s)}{3} \right)$ it follows that:*

1. $\sqrt[2n+1]{\frac{\delta(s)}{b-a}} \frac{1 + \sqrt[2n+1]{\frac{1}{3}}}{2} \leqslant F_n(x_k, x) \leqslant 1$ *if* $x - x_k > 0$

2. $0 \leqslant F_n(x_k, x) \leqslant \frac{\sqrt[2n+1]{\frac{b-a}{\delta(s)}} - \sqrt[2n+1]{\frac{1}{3}}}{2}$ *if* $x - x_k < 0$

The proof can be obtained by elementary estimates.

LEMMA 4. *Let $K \geqslant 2$ be a positive integer such that $\frac{3(b-a)}{n^K} \leqslant \delta(s)$. Then, for all $n \geqslant 2$, it verifies that* $\left| 1 - \sqrt[2n+1]{\frac{\delta(s)}{b-a}} \frac{1 + \sqrt[2n+1]{\frac{1}{3}}}{2} \right| \leqslant \frac{K}{3\sqrt{n}}$ *and* $\left| \frac{\sqrt[2n+1]{\frac{b-a}{\delta(s)}} - \sqrt[2n+1]{\frac{1}{3}}}{2} - 0 \right| \leqslant \frac{(6K-3)\sqrt[7]{3}+2}{12\sqrt{n}}$. *Moreover,* $\max\left\{ \frac{K}{3}, \frac{(6K-3)\sqrt[7]{3}+2}{12} \right\} \leqslant \frac{3K}{5}$

The proof can be obtained easily from Lemma 2.

PROPOSITION 3. *Let $P_s = \{a = x_0, x_1, \ldots, x_s = b\}$ be a partition of $[a,b]$ and let E_s be a step function defined in $[a,b]$ by*

$$E_s(x) = k_1 \cdot \chi_{[x_0, x_1]} + \sum_{i=2}^{s} k_i \cdot \chi_{(x_{i-1}, x_i]} \quad k_i \in \mathbb{R}$$

If $\frac{3(b-a)}{n^K} \leqslant \delta(s)$, being $\delta(s) = \min_{1 \leqslant j \leqslant s} |x_j - x_{j-1}|$ and $K \geqslant 2$ a positive integer, then for all $n \geqslant 2$ it follows that:

1. $|C_n(x) - E_s(x)| \leqslant \frac{6K}{5} \frac{M_s - m_s}{\sqrt{n}}$ *if* $x \in [a,b] \setminus \cup_{j=1}^{s-1} \left(x_j - \frac{\delta(s)}{3}, x_j + \frac{\delta(s)}{3} \right)$

2. $\left| C_n(x) - [k_j(1 - \alpha_x) + k_{j+1}\alpha_x] \right| \leqslant \frac{6K}{5} \frac{M_s - m_s}{\sqrt{n}}$ *if* $x \in \left(x_j - \frac{\delta(s)}{3}, x_j + \frac{\delta(s)}{3} \right)$, $j = 1, \ldots, s-1$.

where M_s and m_s are the maximum and the minimum of the k_j, $\alpha_x \in (0,1)$ is a number which depends only on x and $(C_n)_n$ is the sequence of radical functions associated to E_s.

Proof. It is analogous to the proof given in [3] (p. 115-117) but now we use Lemmas 2, 3 and 4. □

Finally, we prove Theorem 3.

Proof. In [3] (p. 117-118) there is a similar proof for the inequality

$$\left\| f^{(k)} - H_n^{(k)} \right\| \leqslant \frac{6K}{5} \frac{M-m}{\sqrt{n}} + \omega\left(f^{(k)}, \Delta(s_n)\right)$$

being $n \geqslant 2$, M and m the maximum and the minimum of $f^{(k)}$ in $[a,b]$ respectively, $\omega\left(f^{(k)}, \Delta(s_n)\right)$ its modulus of continuity and

$$H_n^{(k)}(x) = C_n(x) = f^{(k)}(x_1) + \sum_{p=2}^{s_n} [f^{(k)}(x_p) - f^{(k)}(x_{p-1})] \cdot F_n(x_{p-1}, x)$$

In this case, this proof would take into account Lemmas 2, 3 and 4 and Proposition 3. The remainder of the proof is the same as Theorem 1. □

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