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AN ALMOST SURE CENTRAL LIMIT THEOREM FOR SELF–NORMALIZED WEIGHTED SUMS OF THE ϕ MIXING RANDOM VARIABLES

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Abstract. In this paper, an almost sure central limit theorem is obtained for self-normalized weighted sums of the ϕ mixing random variables. Our results extend and give substantial improvement for the result obtained by Zhang [12] and our results also extend the earlier work on almost sure central limit theorem such as Wu [13].

1. Introduction and main results

Let $\{X, X_n; n \geqslant 1\}$ be a sequence of random variables and define $S_n = \sum_{i=1}^{n} X_i$, $V_n^2 = \sum_{i=1}^{n} X_i^2$. In recent years, many results about the limit theory for self-normalized sums S_n/V_n were obtained. We can refer to Griffin and Kuelbs [1] for laws of iterated logarithm, Bentkus and Götze [2] for Berry-Esseen bound, Lin [3] for Chung-type laws of iterated logarithm, Gine and Götze [4] for the the asymptotic normality, Shao [5] for large deviations, Hu et al. [6] for the Cramer type moderate deviations, Csörgó et al. [7], [8] for Darling-Erdös theorem and Donsker's theorem, Liu and Lin [9] and Pang et al. [10] for asymptotics for self-normalized random products of sums for i.i.d. and mixing sequences. We can also refer to Liu et al. [11] for central limit theorem (CLT) for self-normalized weighted sums of mixing sequences, Zhang [12] for almost sure central limit theorem (ASCLT) for self-normalized sums of mixing random variables and Wu [13] for ASCLT about self-normalized partial sums for i.i.d. random variables. However, there are few results about ASCLT for self-normalized weighted sums of mixing random variables. Therefore, in the paper we shall investigate ASCLT for self-normalized weighted sums of mixing random variables.

The almost sure central limit theorem (ASCLT) was first introduced independently by Brosamler [14] and Schatte [15]. Since then many results about ASCLT have been discovered. The classical ASCLT (Lacey and Philipp [16]) for $\{X_n; n \geqslant 1\}$, a sequence of i.i.d. random variables with zero means, states that when $\mathrm{Var}(X_n) = \sigma^2$,

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$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{S_k}{\sqrt{k}\sigma}\leqslant x\right\}=\Phi(x)\ \text{ a.s. for any } x\in\mathbb{R}.$$

Here and in the following, $I\{.\}$ denotes the indicator function, and $\Phi(x)$ is the distribution function of the standard normal random variable. We refer the readers to Gonchigdanzan and Rempala [17], Li and Wang [18], Huang and Pang [19], Peng et al. [20] for some ASCLT results with logarithmic averages. The purpose of this article is to study and establish the ASCLT for self-normalized weighted sums of ϕ mixing random variables. We will prove that the ASCLT still holds with other sequence $\{D_n;n\geqslant 1\}$ tending faster to infinity than the logarithmic averages.

In the following, C denotes a positive constant which may differ from one place to another. The notation $a_n\sim b_n$ means that $\lim_{n\to\infty}a_n/b_n=1$. We first introduce the notation of ϕ mixing random variables sequence.

DEFINITION 1. A sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables is said to be ϕ mixing if $\lim_{n\to\infty}\phi(n)=0$, where

$$\phi(n)=\sup_{k\geqslant 1}\phi(\mathscr{F}_1^k,\mathscr{F}_{k+n}^{\infty})$$

with

$$\phi(\mathscr{F}_1^k,\mathscr{F}_{k+n}^{\infty})=\sup_{\substack{A\in\mathscr{F}_1^k,\ B\in\mathscr{F}_{k+n}^{\infty},\\ P(A)>0}}\{|P(B|A)-P(B)|\},$$

and $\mathscr{F}_a^b$ denotes the $\sigma-$ field generated by $X_a,X_{a+1},\ldots,X_b$.

In this paper, we consider a strictly stationary sequence $\{X,X_n\}_{n\in\mathbb{N}}$ of ϕ mixing random variables that satisfies

$$\sum_{n=1}^{\infty}\phi^{\frac{1}{2}}(2^n)<\infty, \tag{1.1}$$

Let $\{a_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ be an array of real numbers that satisfies

$$\sum_{i=1}^{n}|a_{ni}|=O(n), \tag{1.2}$$

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}a_{ni}^2}{n}=A>0, \tag{1.3}$$

$$n|a_{n,i+j}-a_{ni}|\leqslant Cj,\quad 1\leqslant i\leqslant n. \tag{1.4}$$

And let

$$l(x)=\mathbb{E}X^2I\{|X|\leqslant x\},\ \ A_n^2=\operatorname{Var}\left(\sum_{j=1}^{n}X_jI\{|X_j|\leqslant\eta_n\}\right),\ \ B_n^2=n\mathbb{E}X_1^2I\{|X_1|\leqslant\eta_n\},$$

where

$$\eta_j=\inf\left\{s:s\geqslant b+1,\frac{l(s)}{s^2}\leqslant\frac{1}{j}\right\},\quad b=\inf\{x\geqslant 1:l(x)>0\},\quad j=1,2,3\cdot$$

We can easily get that $B_n^2 = nl(\eta_n) \sim \eta_n^2$ as $n \to \infty$. And we assume that

$$A_n^2 \sim \beta B_n^2 \text{ as } n \to \infty \text{ and } l(x) = \mathbb{E}X^2 I\{|X| \leqslant x\} \text{ is slowly varying at} \infty. \tag{1.5}$$

for some $0 < \beta < \infty$.

Our main result is the following theorem.

THEOREM 1.1. *Let $\{X, X_n\}_{n\in\mathbb{N}}$ be a strictly stationary sequence of ϕ mixing random variables with zero means. Assume that (1.1) and (1.5) are satisfied, and $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of real numbers that satisfies (1.2)–(1.4). Let $0 \leqslant \alpha < 1/2$ and denote $d_k = \frac{1}{k}\exp((\log k)^\alpha)$, $D_n = \sum_{k=1}^n d_k$, let*

$$T_n = \sum_{i=1}^n a_{ni} X_i, \quad V_n^2 = \sum_{i=1}^n X_i^2.$$

Then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\left\{ \frac{T_k}{\sqrt{A\beta} V_k} \leqslant x \right\} = \Phi(x) \ \text{a.s. for any } x \in \mathbb{R}. \tag{1.6}$$

In Theorem 1.1, letting $a_{ni} = 1, (1 \leqslant i \leqslant n)$, we can obtain the following result:

COROLLARY 1.2. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a strictly stationary sequence of ϕ mixing random variables with zero means. Assume that (1.1) and (1.5) are satisfied. Let*

$$S_n = \sum_{i=1}^n X_i, \quad V_n^2 = \sum_{i=1}^n X_i^2,$$

then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\left\{ \frac{S_k}{\sqrt{\beta} V_k} \leqslant x \right\} = \Phi(x) \ \text{a.s. for any } x \in \mathbb{R}. \tag{1.7}$$

In Theorem 1.1, letting $a_{ni} = (n+1-i)/n$, $(1 \leqslant i \leqslant n)$, we have

COROLLARY 1.3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a strictly stationary sequence of ϕ mixing random variables with zero means. Assume that (1.1) and (1.5) are satisfied. Let*

$$T_n = \sum_{i=1}^n \frac{n+1-i}{n} X_i, \quad V_n^2 = \sum_{i=1}^n X_i^2,$$

then

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^n d_k I\left\{ \frac{\sqrt{3} T_k}{\sqrt{\beta} V_k} \leqslant x \right\} = \Phi(x) \ \text{a.s. for any } x \in \mathbb{R}. \tag{1.8}$$

In Theorem 1.1, letting $a_{ni} = i/n$, $(1 \leqslant i \leqslant n)$, we can get

COROLLARY 1.4. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a strictly stationary sequence of ϕ mixing random variables with zero means. Assume that (1.1) and (1.5) are satisfied. Let*

$$T_n = \sum_{i=1}^n \frac{i}{n} X_i, \quad V_n^2 = \sum_{i=1}^n X_i^2,$$

then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\frac{\sqrt{3}T_k}{\sqrt{\beta}V_k}\leqslant x\right\}=\Phi(x)\ \text{ a.s. } \textit{for any } x\in\mathbb{R}. \tag{1.9}$$

REMARKS 1. Our results give substantial improvement for the result obtained in Theorem 1 of Zhang [12] and extend the work done by him. In fact our Corollary 1.2 is Theorem 1 of Zhang [12].

REMARKS 2. Theorem 1.1 and Corollaries remain valid if we replace the weight sequence d_k by d_k^* with $0<d_k^*\leqslant d_k$ and $\sum_{k=1}^n d_k^*\to\infty$, $n\to\infty$.

2. Preliminaries

In order to prove the Theorem 1.1, we need the following lemmas.

Lemma 2.1 is due to Csörgó et al. [8].

LEMMA 2.1. *Let X be a random variable with $\mathbb{E}X=0$, then the following statements are equivalent:*

$$(a) l(x)=\mathbb{E}X^2I\{|X|\leqslant x\}\quad \textit{is a slowly varying function at } \infty;$$
$$(b) x^2\mathbb{P}(|X|>x)=o(l(x));$$
$$(c) x\mathbb{E}|X|I\{|X|>x\}=o(l(x));$$
$$(d)\mathbb{E}|X|^\alpha I\{|X|\leqslant x\}=o(x^{\alpha-2}l(x))\ \textit{ for } \alpha>2.$$

Lemma 2.2 is due to Ibragimov [21].

LEMMA 2.2. *Let $\{X_n\}_{n\in\mathbb{Z}}$ be a sequence of ϕ mixing random variables, $X\in L_p(\mathscr{F}_{-\infty}^k)$, $Y\in L_q(\mathscr{F}_{k+n}^\infty)$, and $p,q>1$ with $1/p+1/q=1$. Then*

$$|\mathbb{E}XY-\mathbb{E}X\mathbb{E}Y|\leqslant 2\phi^{1/p}(n)\|X\|_p\|Y\|_q.$$

Lemma 2.3 is due to Zhang [12].

LEMMA 2.3. *Let $\{\xi_n,n\geqslant 1\}$ be a sequence of uniformly bounded random variables and $\{d_k\}$ and $\{D_n\}$ be defined as in Theorem 1.1. If there exist constants $C>0$ and $\delta>0$ and a sequence of positive numbers $\{a(k)\}$ such that $\sum_{n=1}^\infty a(2^n)<\infty$ and*

$$\mathbb{E}|\xi_k\xi_j|\leqslant C\left(\left(\frac{k}{j}\right)^\delta+a(k)\right)\qquad \textit{for}\quad \frac{j}{k}>(\log D_n)^{2/\delta},$$

then

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^n d_k\xi_k=0\ \text{ a.s.}$$

We will use the following notations. For every $1 \leqslant i \leqslant k$, let

$$\overline{X}_{ki} = a_{ki}X_iI\{|X_i| \leqslant \eta_k\}, \qquad \widetilde{X}_{ki} = a_{ki}X_iI\{|X_i| > \eta_k\}, \qquad X_{ki}^* = \overline{X}_{ki} - \mathbb{E}\overline{X}_{ki},$$

$$\widetilde{X}_{ki}^* = \widetilde{X}_{ki} - \mathbb{E}\widetilde{X}_{ki}, \qquad T_k^* = \sum_{i=1}^{k} X_{ki}^*, \qquad \widetilde{T}_k^* = \sum_{i=1}^{k} \widetilde{X}_{ki}^*, \qquad \overline{V}_k^2 = \sum_{i=1}^{k} X_i^2 I\{|X_i| \leqslant \eta_k\}.$$

LEMMA 2.4. *Let f be a non-negative, bounded Lipschitz function and suppose that the assumptions of Theorem 1.1 hold, then we have*

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left[f\left(\frac{\widetilde{T}_k^*}{\sqrt{A\beta k l(\eta_k)}} \right) - \mathbb{E} f\left(\frac{\widetilde{T}_k^*}{\sqrt{A\beta k l(\eta_k)}} \right) \right] = 0 \text{ a.s.} \tag{2.1}$$

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left[f\left(\frac{\overline{V}_k^2}{k l(\eta_k)} \right) - \mathbb{E} f\left(\frac{\overline{V}_k^2}{k l(\eta_k)} \right) \right] = 0 \text{ a.s.} \tag{2.2}$$

$$\lim_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left[I\left\{ \bigcup_{l=1}^{k} (|X_l| > \eta_k) \right\} - \mathbb{E} I\left\{ \bigcup_{l=1}^{k} (|X_l| > \eta_k) \right\} \right] = 0 \text{ a.s.} \tag{2.3}$$

Proof. Let

$$\xi_k = f\left(\frac{\widetilde{T}_k^*}{\sqrt{A\beta k l(\eta_k)}} \right) - \mathbb{E} f\left(\frac{\widetilde{T}_k^*}{\sqrt{A\beta k l(\eta_k)}} \right),$$

for any $j/k > (\log D_n)^{2/\delta} > 2$. Noting the fact that f be a non-negative, bounded Lipschitz function and $kl(\eta_k) \sim \eta_k^2$, then by Lemma 2.1 , Lemma 2.2 and (1.2) we have

$$\begin{aligned}
|\mathbb{E}\xi_k\xi_j| &= \left| \operatorname{cov}\left(f\left(\frac{\widetilde{T}_k^*}{\sqrt{A\beta k l(\eta_k)}} \right), f\left(\frac{\widetilde{T}_j^*}{\sqrt{A\beta j l(\eta_j)}} \right) \right) \right| \\
&\leqslant \left| \operatorname{cov}\left(f\left(\frac{\widetilde{T}_k^*}{\sqrt{A\beta k l(\eta_k)}} \right), f\left(\frac{\widetilde{T}_j^*}{\sqrt{A\beta j l(\eta_j)}} \right) - f\left(\frac{\sum_{i=2k+1}^{j} \widetilde{X}_{ji}^*}{\sqrt{A\beta j l(\eta_j)}} \right) \right) \right| \\
&\quad + \left| \operatorname{cov}\left(f\left(\frac{\widetilde{T}_k^*}{\sqrt{A\beta k l(\eta_k)}} \right), f\left(\frac{\sum_{i=2k+1}^{j} \widetilde{X}_{ji}^*}{\sqrt{A\beta j l(\eta_j)}} \right) \right) \right| \\
&\leqslant C\mathbb{E}\left| f\left(\frac{\widetilde{T}_j^*}{\sqrt{A\beta j l(\eta_j)}} \right) - f\left(\frac{\sum_{i=2k+1}^{j} \widetilde{X}_{ji}^*}{\sqrt{A\beta j l(\eta_j)}} \right) \right| + C\phi^{1/2}(k) \\
&\leqslant C \frac{\mathbb{E}\left| \sum_{i=1}^{2k} \widetilde{X}_{ji}^* \right|}{\sqrt{j l(\eta_j)}} + C\phi^{1/2}(k) \\
&= C \frac{\mathbb{E}\left| \sum_{i=1}^{2k} a_{ji} \left[X_i I\{|X_i| > \eta_j\} - \mathbb{E} X_i I\{|X_i| > \eta_j\} \right] \right|}{\sqrt{j l(\eta_j)}} + C\phi^{1/2}(k)
\end{aligned}$$

$$\leqslant C\frac{\sum_{i=1}^{2k}|a_{ji}|\mathbb{E}|X|I\{|X|>\eta_j\}}{\sqrt{jl(\eta_j)}}+C\phi^{1/2}(k)\leqslant C\frac{2k\mathbb{E}XI\{|X|>\eta_j\}}{\sqrt{jl(\eta_j)}}+C\phi^{1/2}(k)$$
$$\leqslant C\frac{k}{\sqrt{jl(\eta_j)}}\frac{o(l(\eta_j))}{\eta_j}+C\phi^{1/2}(k)\leqslant C\left[\frac{k}{j}+\phi^{1/2}(k)\right].$$

Then by (1.1) and Lemma 2.3 with $\delta=1$ we obtain (2.1).

To prove (2.2), let

$$Y_k=f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right)-\mathbb{E}f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right),$$

Note that $j/k>(\log D_n)^{2/\delta}>2$ and the fact that f be a non-negative, bounded Lipschitz function. Then by Lemma 2.2 and (1.2) we have

$$\begin{aligned}
\left|\mathbb{E}Y_kY_j\right| &= \left|\mathrm{cov}\left(f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right),f\left(\frac{\overline{V}_j^2}{jl(\eta_j)}\right)\right)\right|\\
&\leqslant \left|\mathrm{cov}\left(f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right),f\left(\frac{\overline{V}_j^2}{jl(\eta_j)}\right)-f\left(\frac{\sum_{i=2k+1}^{j}X_i^2I\{|X_i|\leqslant\eta_j\}}{jl(\eta_j)}\right)\right)\right|\\
&\quad+\left|\mathrm{cov}\left(f\left(\frac{\overline{V}_k^2}{kl(\eta_k)}\right),f\left(\frac{\sum_{i=2k+1}^{j}X_i^2I\{|X_i|\leqslant\eta_j\}}{jl(\eta_j)}\right)\right)\right|\\
&\leqslant C\frac{\sum_{i=1}^{2k}\mathbb{E}X^2I\{|X|\leqslant\eta_j\}}{jl(\eta_j)}+C\phi^{1/2}(k)\\
&\leqslant C\frac{2k\mathbb{E}X^2I\{|X|\leqslant\eta_j\}}{jl(\eta_j)}+C\phi^{1/2}(k)\\
&\leqslant C\left[\frac{k}{j}+\phi^{1/2}(k)\right].
\end{aligned}$$

Hence by (1.1) and Lemma 2.3 with $\delta=1$ we obtain (2.2).

Now we prove (2.3). Let

$$Z_k=I\left\{\bigcup_{l=1}^{k}(|X_l|>\eta_k)\right\}-\mathbb{E}I\left\{\bigcup_{l=1}^{k}(|X_l|>\eta_k)\right\},$$

Note that $j/k>(\log D_n)^{2/\delta}>2$ and $I(A\cup B)-I(B)\leqslant I(A)$ for any sets A and B. Then by Lemmas 2.1 and 2.2 we get

$$\begin{aligned}
\left|\mathbb{E}Z_kZ_j\right| &= \left|\mathrm{cov}\left(I\left\{\bigcup_{l=1}^{k}(|X_l|>\eta_k)\right\},I\left\{\bigcup_{m=1}^{j}(|X_m|>\eta_j)\right\}\right)\right|\\
&\leqslant \left|\mathrm{cov}\left(I\left\{\bigcup_{l=1}^{k}(|X_l|>\eta_k)\right\},I\left\{\bigcup_{m=1}^{j}(|X_m|>\eta_j)\right\}-I\left\{\bigcup_{m=2k+1}^{j}(|X_m|>\eta_j)\right\}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\operatorname{cov}\left(I\left\{\bigcup_{l=1}^{k}(|X_l|>\eta_k)\right\},I\left\{\bigcup_{m=2k+1}^{j}(|X_m|>\eta_j)\right\}\right)\right|\\
&\leqslant \mathbb{E}\left|I\left\{\bigcup_{m=1}^{j}(|X_m|>\eta_j)\right\}-I\left\{\bigcup_{m=2k+1}^{j}(|X_m|>\eta_j)\right\}\right|+C\phi^{1/2}(k)\\
&\leqslant \mathbb{E}\left|I\left\{\bigcup_{m=1}^{2k}(|X_m|>\eta_j)\right\}\right|+C\phi^{1/2}(k)\leqslant 2kP(|X_l|>\eta_j)+C\phi^{1/2}(k)\\
&\leqslant Ck\frac{1}{\eta_j^2}o(l(\eta_j))+C\phi^{1/2}(k)\leqslant C\left[\frac{k}{j}+\phi^{1/2}(k)\right].
\end{aligned}
$$

Hence by (1.1) and Lemma 2.3 with $\delta = 1$ we obtain (2.3). □

LEMMA 2.5. *Suppose that the assumptions of Theorem 1.1 hold. Then we have*

$$
\frac{T_k}{\sqrt{A\beta kl(\eta_k)}} \xrightarrow{d} \mathscr{N}. \tag{2.4}
$$

Here and in the sequel, $\mathscr{N}$ is a standard normal random variable, and $\xrightarrow{d}$ denotes the convergence in distribution.

Proof. Suppose that the assumptions of Theorem 1.1 hold. Liu et al. [11] obtained

$$
\frac{T_k}{\sqrt{A\beta}V_k} \xrightarrow{d} \mathscr{N}.
$$

On the other hand, suppose that the assumptions of Theorem 1.1 hold. Liu and Lin [9] obtained

$$
\frac{V_k^2}{B_k^2} \xrightarrow{P} 1.
$$

Here $B_k^2 = kl(\eta_k)$. By Slutsky theorem, hence we can obtain (2.4). □

LEMMA 2.6. *Suppose that the assumptions of Theorem 1.1 hold. Then we have*

$$
\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\frac{T_k^*}{\sqrt{A\beta kl(\eta_k)}}\leqslant x\right\} = \Phi(x)\ \text{ a.s. } \textit{for any } x\in\mathbb{R}. \tag{2.5}
$$

Proof. For arbitrary $\varepsilon > 0$, by Markov's inequality and lemma 2.1, there exists sufficient large N such that when $k > N$ we have

$$
\begin{aligned}
\mathbb{P}\left(\frac{|T_k - T_k^*|}{\sqrt{A\beta kl(\eta_k)}}\geqslant \varepsilon\right) &\leqslant C\frac{\sum_{i=1}^{k}|a_{ki}|\mathbb{E}|X|I\{|X|>\eta_k\}}{\sqrt{A\beta kl(\eta_k)}}\\
&\leqslant C\frac{ko(l(\eta_k))}{\sqrt{A\beta kl(\eta_k)}\eta_k}\leqslant C\frac{ko(l(\eta_k))}{\sqrt{A\beta}kl(\eta_k)} = o(1).
\end{aligned} \tag{2.6}
$$

By lemma 2.5, (2.6) and Slutsky theorem we have

$$\frac{T_k^*}{\sqrt{A\beta kl(\eta_k)}} \xrightarrow{d} \mathscr{N}. \tag{2.7}$$

Let f be a bounded Lipschitz function having a derivative h bounded by C. From (2.7) we have

$$\mathbb{E}f\left(\frac{T_k^*}{\sqrt{A\beta kl(\eta_k)}}\right) \to \mathbb{E}f(\mathscr{N}). \tag{2.8}$$

On the other hand, noting that (2.5) is equivalent to

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k f\left(\frac{T_k^*}{\sqrt{A\beta kl(\eta_k)}}\right) = \mathbb{E}f(\mathscr{N}) \quad \text{a.s.} \tag{2.9}$$

Hence, to prove (2.5), it suffices to show that

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k \left[f\left(\frac{T_k^*}{\sqrt{A\beta kl(\eta_k)}}\right) - \mathbb{E}f\left(\frac{T_k^*}{\sqrt{A\beta kl(\eta_k)}}\right)\right] = 0 \quad \text{a.s.} \tag{2.10}$$

The proof of (2.10) is similar to that of (2.1). So we omit it here. □

3. Proof of Theorem 1.1

In order to prove (1.6), it suffices to prove the following two inequalities:

$$\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\frac{T_k}{\sqrt{A\beta}V_k} \leqslant x\right\} \leqslant \Phi(x) \quad \text{a.s. for any } x\in\mathbb{R}. \tag{3.1}$$

$$\liminf_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\frac{T_k}{\sqrt{A\beta}V_k} \leqslant x\right\} \geqslant \Phi(x) \quad \text{a.s. for any } x\in\mathbb{R}. \tag{3.2}$$

To prove (3.1), noting that for $x \geqslant 0$ and $0<\delta<1/2$ we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\frac{T_k}{\sqrt{A\beta}V_k} \leqslant x\right\}\\
\leqslant &\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\frac{T_k}{\sqrt{A\beta}} \leqslant x\sqrt{(1+\delta)kl(\eta_k)}\right\}\\
&+\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{V_k^2 > (1+\delta)kl(\eta_k)\right\}\\
\leqslant &\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\frac{T_k^*}{\sqrt{A\beta}} \leqslant \left(x+\frac{\delta}{\sqrt{1+\delta}}\right)\sqrt{(1+\delta)kl(\eta_k)}\right\}\\
&+\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n} d_k I\left\{\left|\frac{T_k-T_k^*}{\sqrt{A\beta}}\right| > \delta\sqrt{kl(\eta_k)}\right\}
\end{aligned}$$

$$+\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\overline{V}_k^2>(1+\delta)kl(\eta_k)\right\}$$
$$+\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\bigcup_{l=1}^{k}(|X_l|>\eta_k)\right\}$$
$$=\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k(M_1+M_2+M_3+M_4). \tag{3.3}$$

For $x<0$, similar to (3.3),we have

$$\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\frac{T_k}{\sqrt{A\beta}V_k}\leqslant x\right\}$$
$$\leqslant\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\frac{T_k}{\sqrt{A\beta}}\leqslant x\sqrt{(1-\delta)kl(\eta_k)}\right\}$$
$$+\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{V_k^2<(1-\delta)kl(\eta_k)\right\}$$
$$\leqslant\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\frac{T_k^*}{\sqrt{A\beta}}\leqslant\left(x+\frac{\delta}{\sqrt{1-\delta}}\right)\sqrt{(1-\delta)kl(\eta_k)}\right\}$$
$$+\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{|\frac{T_k-T_k^*}{\sqrt{A\beta}}|>\delta\sqrt{kl(\eta_k)}\right\}$$
$$+\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\overline{V}_k^2<(1-\delta)kl(\eta_k)\right\}$$
$$+\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\bigcup_{l=1}^{k}(|X_l|>\eta_k)\right\}$$
$$=\limsup_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_k(K_1+K_2+K_3+K_4). \tag{3.4}$$

We only need to prove that (3.1) holds when $x\geqslant 0$ since for $x<0$ we have the same conclusion. By Lemma 2.6 we can easily get

$$\lim_{n\to\infty}\frac{1}{D_n}\sum_{k=1}^{n}d_kI\left\{\frac{T_k^*}{\sqrt{A\beta}}\leqslant\left(x+\frac{\delta}{\sqrt{1+\delta}}\right)\sqrt{(1+\delta)kl(\eta_k)}\right\}=\Phi(\sqrt{1+\delta}x+\delta)\ \text{a.s.} \tag{3.5}$$

Let f_1 be a real-valued functions such that

$$I\{|x|\geqslant\delta\}\leqslant f_1(x)\leqslant I\{|x|\geqslant\delta/2\}\ \text{and}\quad \sup_x|f'(x)|<\infty.$$

By Lemma 2.1 and $kl(\eta_k) \sim \eta_k^2$, for arbitrary $\varepsilon_1 > 0$, there exists K_1 such that when $k > K_1$

$$\mathbb{E}|X|I\{|X| > \eta_k\} \leqslant \frac{\varepsilon_1 l(\eta_k))}{\eta_k}, \qquad kl(\eta_k) \leqslant 2\eta_k^2, \tag{3.6}$$

for every $k > K_1$. Then by Markov's inequality, Lemma 2.4 and (3.6) we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k M_2 &= \limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k I\left\{\left|\frac{T_k - T_k^*}{\sqrt{A\beta}}\right| > \delta\sqrt{kl(\eta_k)}\right\} \\
&\leqslant \limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k f_1\left(\frac{\tilde{T}_k^*}{\sqrt{A\beta kl(\eta_k)}}\right) \\
&\leqslant \limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k \left[f_1\left(\frac{\tilde{T}_k^*}{\sqrt{A\beta kl(\eta_k)}}\right) - \mathbb{E}f_1\left(\frac{\tilde{T}_k^*}{\sqrt{A\beta kl(\eta_k)}}\right)\right] \\
&\quad + \limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k \mathbb{E}f_1\left(\frac{\tilde{T}_k^*}{\sqrt{A\beta kl(\eta_k)}}\right) \\
&= \limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k \mathbb{E}f_1\left(\frac{\tilde{T}_k^*}{\sqrt{A\beta kl(\eta_k)}}\right) \\
&\leqslant \limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k \mathbb{E}I\left\{\frac{|\tilde{T}_k^*|}{\sqrt{A\beta}} > \frac{\delta\sqrt{kl(\eta_k)}}{2}\right\} \\
&\leqslant \limsup_{n\to\infty} \frac{C}{D_n}\sum_{k=1}^n d_k \frac{\mathbb{E}|\tilde{T}_k^*|}{\sqrt{kl(\eta_k)}} \\
&\leqslant \limsup_{n\to\infty} \frac{C}{D_n}\sum_{k=1}^n d_k \frac{\sum_{i=1}^k |a_{ki}|\mathbb{E}|X|I\{|X| > \eta_k\}}{\sqrt{kl(\eta_k)}} \\
&\leqslant \limsup_{n\to\infty} \frac{C}{D_n}\sum_{k=1}^n d_k \frac{k\mathbb{E}|X|I\{|X| > \eta_k\}}{\sqrt{kl(\eta_k)}} \\
&\leqslant \limsup_{n\to\infty} \frac{C}{D_n}\sum_{k=1}^{K_1} \frac{kd_k}{\sqrt{kl(\eta_k)}} + \limsup_{n\to\infty} \frac{C}{D_n}\sum_{k=K_1+1}^n \frac{kd_k}{\sqrt{kl(\eta_k)}}\frac{\varepsilon_1 l(\eta_k)}{\eta_k} \\
&\leqslant 0 + \limsup_{n\to\infty} \frac{C}{D_n}\sum_{k=K_1+1}^n \varepsilon_1 d_k = \varepsilon_1 \ \text{ a.s.}
\end{aligned}$$

Now, letting $\varepsilon_1 \to 0$, we obtain

$$\limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k M_2 = 0 \ \text{ a.s.} \tag{3.7}$$

By the same arguments as above we can get from Lemmas 2.1 and 2.4 that

$$\limsup_{n\to\infty} \frac{1}{D_n}\sum_{k=1}^n d_k M_3 = 0 \ \text{ a.s.} \tag{3.8}$$

$$\limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k M_4 = 0 \ \text{ a.s.} \tag{3.9}$$

By (3.3)–(3.9), letting $\delta \to 0$ we can get that (3.1) holds.

Now to prove (3.2). Similarly to (3.3) and (3.4), for $x \geqslant 0$, we can obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \frac{T_k}{\sqrt{A\beta} V_k} \leqslant x \right\} \\
\geqslant &\liminf_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \frac{T_k}{\sqrt{A\beta}} \leqslant x\sqrt{(1-\delta)kl(\eta_k)} \right\} \\
&- \limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ V_k^2 < (1-\delta)kl(\eta_k) \right\} \\
\geqslant &\liminf_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \frac{T_k^*}{\sqrt{A\beta}} \leqslant \left(x - \frac{\delta}{\sqrt{1-\delta}} \right) \sqrt{(1-\delta)kl(\eta_k)} \right\} \\
&- \limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \left| \frac{T_k - T_k^*}{\sqrt{A\beta}} \right| > \delta \sqrt{kl(\eta_k)} \right\} \\
&- \limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \overline{V}_k^2 < (1-\delta)kl(\eta_k) \right\} \\
&- \limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \bigcup_{l=1}^{k} (|X_l| > \eta_k) \right\}.
\end{aligned} \tag{3.10}$$

If $x < 0$, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \frac{T_k}{\sqrt{A\beta} V_k} \leqslant x \right\} \\
\geqslant &\liminf_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \frac{T_k}{\sqrt{A\beta}} \leqslant x\sqrt{(1+\delta)kl(\eta_k)} \right\} \\
&- \limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ V_k^2 > (1+\delta)kl(\eta_k) \right\} \\
\geqslant &\liminf_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \frac{T_k^*}{\sqrt{A\beta}} \leqslant \left(x - \frac{\delta}{\sqrt{1+\delta}} \right) \sqrt{(1-\delta)kl(\eta_k)} \right\} \\
&- \limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \left| \frac{T_k - T_k^*}{\sqrt{A\beta}} \right| > \delta \sqrt{kl(\eta_k)} \right\} \\
&- \limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \overline{V}_k^2 > (1+\delta)kl(\eta_k) \right\} \\
&- \limsup_{n\to\infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left\{ \bigcup_{l=1}^{k} (|X_l| > \eta_k) \right\}.
\end{aligned} \tag{3.11}$$

The remaining proof is similar to that of (3.1), so we omit it here. Therefore, we complete the proof of Theorem 1.1.

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