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NEW REFINEMENTS OF GENERALIZED ACZÉL'S INEQUALITY AND THEIR APPLICATIONS

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Abstract. In this paper, we give several new refinements of generalized Aczél's inequality. Moreover, as applications, some new refinements of integral form of generalized Aczél-type inequality are given.

1. Introduction

In 1956, Aczél [1] established the following inequality, which is called Aczél's inequality.

THEOREM A. *If a_i, b_i $(i=1,2,\dots,n)$ are positive numbers such that $a_1^2-\sum_{i=2}^n a_i^2>0$ and $b_1^2-\sum_{i=2}^n b_i^2>0$, then*

$$\left(a_1^2-\sum_{i=2}^n a_i^2\right)\left(b_1^2-\sum_{i=2}^n b_i^2\right)\leqslant\left(a_1b_1-\sum_{i=2}^n a_ib_i\right)^2. \tag{1}$$

As we all know, Aczél's inequality has many applications in the theory of functional equations in non-Euclidean geometry, and many authors (see [2, 4-15] and references therein) have given considerable attention to this inequality and its refinements.

In 1959, Popoviciu [3] derived an exponential generalization of the Aczél inequality, which is stated in the following theorem.

THEOREM B. *Let $p\geqslant q>1$, $\frac{1}{p}+\frac{1}{q}=1$, and let a_i, b_i $(i=1,2,\dots,n)$ be positive numbers such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{\frac{1}{q}}\leqslant a_1b_1-\sum_{i=2}^n a_ib_i. \tag{2}$$

Later, in 1982, Vasić and Pečarić [5] presented a reversed version of inequality (2) as follows.

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THEOREM C. *Let* $q < 0,\ p > 0,\ \frac{1}{p} + \frac{1}{q} = 1$, *and let* $a_i,\ b_i\ \ (i = 1,2,\ldots,n)$ *be positive numbers such that* $a_1^p - \sum_{i=2}^n a_i^p > 0$ *and* $b_1^q - \sum_{i=2}^n b_i^q > 0$. *Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \geqslant a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{3}$$

In another paper, Vasić and Pečarić [6] presented a further extension of inequality (2).

THEOREM D. *Let* $a_{rj} > 0,\ \lambda_j > 0,\ a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0,\ r = 1,2,\ldots,n,\ j = 1,2,\ldots,m$, *and let* $\sum_{j=1}^m \frac{1}{\lambda_j} \geqslant 1$. *Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leqslant \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \tag{4}$$

Recently, Tian in [4] gave the reversed version of inequality (4) as follows.

THEOREM E. *Let* $a_{rj} > 0\ (r = 1,2,\ldots,n,\ j = 1,2,\ldots,m),\ \lambda_1 \neq 0,\ \lambda_j < 0\ \ (j = 2,3,\ldots,m),\ \sum_{j=1}^m \frac{1}{\lambda_j} \leqslant 1,\ a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0\ (j = 1,2,\ldots,m)$. *Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geqslant \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \tag{5}$$

Moreover, in [4] Tian established an integral type of inequality (5).

THEOREM F. *Let* $\lambda_1 > 0,\ \lambda_j < 0\ \ (j = 2,3,\cdots,m),\ \sum_{j=1}^m \frac{1}{\lambda_j} = 1$, *let* $M_j > 0$ $(j = 1,2,\cdots,m)$, *and let* $f_j\ \ (j = 1,2,\cdots,m)$ *be positive Riemann integrable functions on* $[a,b]$ *such that* $M_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x > 0$. *Then*

$$\prod_{j=1}^m \left(M_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_j}} \geqslant \prod_{j=1}^m M_j - \int_a^b \prod_{j=1}^m f_j(x)\mathrm{d}x. \tag{6}$$

Obviously, the integral form of inequality (4) under the assumption $\sum_{j=1}^m \frac{1}{\lambda_j} = 1$ is also valid, i.e.,

THEOREM G. *Let* $\lambda_j > 0\ \ (j = 1,2,\cdots,m),\ \sum_{j=1}^m \frac{1}{\lambda_j} = 1$, *let* $M_j > 0\ \ (j = 1,2,\cdots,m)$, *and let* $f_j\ \ (j = 1,2,\cdots,m)$ *be positive Riemann integrable functions on* $[a,b]$ *such that* $M_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x > 0$. *Then*

$$\prod_{j=1}^m \left(M_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_j}} \leqslant \prod_{j=1}^m M_j - \int_a^b \prod_{j=1}^m f_j(x)\mathrm{d}x. \tag{7}$$

The main purpose of this work is to give new refinements of inequality (4) and (5). As applications, refinements of inequalities (6) and (7) are also given.

2. New refinements of generalized Aczél's inequality

We need the following lemmas in our deduction.

LEMMA 2.1. [4] *Let* $a_{rj} > 0$ $(r = 1,2,\dots,n,\ j = 1,2,\dots,m)$, *let* λ_m *be a real number,* $\lambda_j \leqslant 0$ $(j = 1,2,\dots,m-1)$, *and let* $\beta = \max\{\sum_{j=1}^{m}\lambda_j, 1\}$. *Then*

$$\sum_{r=1}^{n}\prod_{j=1}^{m}a_{rj}^{\lambda_j} \geqslant n^{1-\beta}\prod_{j=1}^{m}\left(\sum_{r=1}^{n}a_{rj}\right)^{\lambda_j}. \tag{8}$$

LEMMA 2.2. [8] *Let* $a_{rj} > 0$ $(r = 1,2,\dots,n,\ j = 1,2,\dots,m)$, *let* $\lambda_j \geqslant 0$ $(j = 1,2,\dots,m)$, *and let* $\gamma = \min\{\sum_{j=1}^{m}\lambda_j, 1\}$. *Then*

$$\sum_{r=1}^{n}\prod_{j=1}^{m}a_{rj}^{\lambda_j} \leqslant n^{1-\gamma}\prod_{j=1}^{m}\left(\sum_{r=1}^{n}a_{rj}\right)^{\lambda_j}. \tag{9}$$

LEMMA 2.3. [7] *Let* $0 \leqslant x < 1$, $\alpha > 0$. *Then*

$$(1-x)^{\frac{1}{\alpha}} \leqslant 1 - \frac{x}{\max\{\alpha, 1\}}. \tag{10}$$

LEMMA 2.4. [2] *Let* a_i, x_i $(i = 1,2,\dots,n)$ *be real numbers such that* $a_i \geqslant 0$ *and* $x_i > -1$. *If* $A_n = \sum_{i=1}^{n}a_i \leqslant 1$, *then*

$$\prod_{i=1}^{n}(1+x_i)^{a_i} \leqslant 1 + \sum_{i=1}^{n}a_i x_i. \tag{11}$$

If either $a_i \geqslant 1$ $(i = 1,2,\dots,n)$ *or* $a_i \leqslant 0$ $(i = 1,2,\dots,n)$, *and if all* x_i *are positive or negative with* $x_i > -1$, *then the reverse inequality of (11) holds.*

LEMMA 2.5. *Let* $\lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_m < 0$, *let* $X_j > 1$ $(j = 1,2,\dots,m)$, *and let* $m \geqslant 2$. *Then*

$$\prod_{j=1}^{m}\left(1 - X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{m}X_j \geqslant 1 - \frac{1}{2\lambda_1}\sum_{j=1}^{m-1}\left(X_j^{\lambda_j} - X_{j+1}^{\lambda_{j+1}}\right)^2. \tag{12}$$

Proof. From the hypotheses in Lemma 2.5, it is easy to verify that

$$0 > \frac{1}{\lambda_1} \geqslant \frac{1}{\lambda_2} \geqslant \dots \geqslant \frac{1}{\lambda_{m-1}} \geqslant \frac{1}{\lambda_m},$$

and

$$\frac{1}{2\lambda_j} - \frac{1}{2\lambda_{j-1}} \leqslant 0\ (j = 2,3,\dots,m-1), \frac{1}{\lambda_m} - \frac{1}{2\lambda_{m-1}} < 0.$$

Consequently, according to $\frac{1}{2\lambda_1}+\frac{1}{2\lambda_1}+\left(\frac{1}{2\lambda_2}-\frac{1}{2\lambda_1}\right)+\frac{1}{2\lambda_2}+\frac{1}{2\lambda_2}+\left(\frac{1}{2\lambda_3}-\frac{1}{2\lambda_2}\right)+\ldots+\frac{1}{2\lambda_{m-2}}+\frac{1}{2\lambda_{m-2}}+\left(\frac{1}{2\lambda_{m-1}}-\frac{1}{2\lambda_{m-2}}\right)+\frac{1}{2\lambda_{m-1}}+\frac{1}{2\lambda_{m-1}}+\left(\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}\right)+\frac{1}{2\lambda_1}=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\ldots+\frac{1}{\lambda_m}<0$, by using Lemma 2.1 we have

$$\begin{aligned}
&\left\{\prod_{j=1}^{m-1}\left[X_j^{\lambda_j}+\left(1-X_{j+1}^{\lambda_{j+1}}\right)\right]^{\frac{1}{2\lambda_j}}\right\}\left\{\prod_{j=1}^{m-1}\left[X_{j+1}^{\lambda_{j+1}}+\left(1-X_j^{\lambda_j}\right)\right]^{\frac{1}{2\lambda_j}}\right\}\\
&\quad\times\left\{\prod_{j=1}^{m-2}\left[X_{j+1}^{\lambda_{j+1}}+\left(1-X_{j+1}^{\lambda_{j+1}}\right)\right]^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right\}\\
&\quad\times\left[X_m^{\lambda_m}+\left(1-X_m^{\lambda_m}\right)\right]^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\times\left[X_1^{\lambda_1}+\left(1-X_1^{\lambda_1}\right)\right]^{\frac{1}{2\lambda_1}}\\
&\leqslant\left\{\prod_{j=1}^{m-2}\left[\left(X_j^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(X_{j+1}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(X_{j+1}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\right\}\\
&\quad\times\left(X_{m-1}^{\lambda_{m-1}}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_m^{\lambda_m}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_m^{\lambda_m}\right)^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left(X_1^{\lambda_1}\right)^{\frac{1}{2\lambda_1}}\\
&\quad+\left\{\prod_{j=1}^{m-2}\left[\left(1-X_{j+1}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(1-X_{j+1}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\right\}\\
&\quad\times\left(1-X_m^{\lambda_m}\right)^{\frac{1}{2\lambda_{m-1}}}\left(1-X_{m-1}^{\lambda_{m-1}}\right)^{\frac{1}{2\lambda_{m-1}}}\left(1-X_m^{\lambda_m}\right)^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left(1-X_1^{\lambda_1}\right)^{\frac{1}{2\lambda_1}},
\end{aligned}\tag{13}$$

which is equivalent to the following inequality:

$$\prod_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}}\leqslant\prod_{j=1}^{m}X_j+\prod_{j=1}^{m}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}.\tag{14}$$

On the other hand, applying Lemma 2.4 we have

$$\begin{aligned}
\prod_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}}&\geqslant\prod_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_1}}\\
&\geqslant 1-\sum_{j=1}^{m-1}\frac{1}{2\lambda_1}\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2.
\end{aligned}\tag{15}$$

Combining inequalities (14) and (15) we can get inequality (12). The proof of Lemma 2.5 is completed. □

LEMMA 2.6. *Let* $\lambda_m>0,\ \lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_{m-1}<0$, *let* $0<X_m<1,\ X_j>1$ $(j=1,2,\ldots,m-1)$, *and let* $m\geqslant 2,\ \beta=\max\{\sum_{j=1}^{m}\frac{1}{\lambda_j},1\}$. *Then*

$$\prod_{j=1}^{m}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\prod_{j=1}^{m}X_j\geqslant n^{1-\beta}\left[1-\frac{1}{2\lambda_1}\sum_{j=1}^{m-1}\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right].\tag{16}$$

Proof. The proof of Lemma 2.6 is similar to the one of Lemma 2.5, and we omit it. □

LEMMA 2.7. *Let* $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0$, *let* $0 < X_j < 1\ \ (j = 1,2,\ldots,m)$, *and let* $m \geqslant 2,\ \gamma = \min\{\sum_{j=1}^m \frac{1}{\lambda_j}, 1\}$. *Then*

$$\prod_{j=1}^{m}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\prod_{j=1}^{m}X_j \leqslant n^{1-\gamma}\left[1-\frac{1}{2\max\{\lambda_1,\frac{m-1}{2}\}}\sum_{j=1}^{m-1}\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]. \tag{17}$$

Proof. From the hypotheses in Lemma 2.7, it is easy to verify that

$$0<\frac{1}{\lambda_1}\leqslant\frac{1}{\lambda_2}\leqslant\ldots\leqslant\frac{1}{\lambda_{m-1}}\leqslant\frac{1}{\lambda_m},$$

and

$$\frac{1}{2\lambda_j}-\frac{1}{2\lambda_{j-1}}\geqslant 0\ (j=2,3,\ldots,m-1),\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}>0.$$

Consequently, according to $\frac{1}{2\lambda_1}+\frac{1}{2\lambda_1}+\left(\frac{1}{2\lambda_2}-\frac{1}{2\lambda_1}\right)+\frac{1}{2\lambda_2}+\frac{1}{2\lambda_2}+\left(\frac{1}{2\lambda_3}-\frac{1}{2\lambda_2}\right)+\ldots+\frac{1}{2\lambda_{m-2}}+\frac{1}{2\lambda_{m-2}}+\left(\frac{1}{2\lambda_{m-1}}-\frac{1}{2\lambda_{m-2}}\right)+\frac{1}{2\lambda_{m-1}}+\frac{1}{2\lambda_{m-1}}+\left(\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}\right)+\frac{1}{2\lambda_1}=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\ldots+\frac{1}{\lambda_m}>0$, by using Lemma 2.2 we have

$$\begin{aligned}
&\left\{\prod_{j=1}^{m-1}\left[X_j^{\lambda_j}+\left(1-X_{j+1}^{\lambda_{j+1}}\right)\right]^{\frac{1}{2\lambda_j}}\right\}\left\{\prod_{j=1}^{m-1}\left[X_{j+1}^{\lambda_{j+1}}+\left(1-X_j^{\lambda_j}\right)\right]^{\frac{1}{2\lambda_j}}\right\}\\
&\quad\times\left\{\prod_{j=1}^{m-2}\left[X_{j+1}^{\lambda_{j+1}}+\left(1-X_{j+1}^{\lambda_{j+1}}\right)\right]^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right\}\\
&\quad\times\left[X_m^{\lambda_m}+\left(1-X_m^{\lambda_m}\right)\right]^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\times\left[X_1^{\lambda_1}+\left(1-X_1^{\lambda_1}\right)\right]^{\frac{1}{2\lambda_1}}\\
&\geqslant n^{\gamma-1}\left\{\prod_{j=1}^{m-2}\left[\left(X_j^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(X_{j+1}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(X_{j+1}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\right\}\\
&\quad\times\left(X_{m-1}^{\lambda_{m-1}}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_m^{\lambda_m}\right)^{\frac{1}{2\lambda_{m-1}}}\left(X_m^{\lambda_m}\right)^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left(X_1^{\lambda_1}\right)^{\frac{1}{2\lambda_1}}\\
&\quad+n^{\gamma-1}\left\{\prod_{j=1}^{m-2}\left[\left(1-X_{j+1}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_j}}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{2\lambda_j}}\left(1-X_{j+1}^{\lambda_{j+1}}\right)^{\frac{1}{2\lambda_{j+1}}-\frac{1}{2\lambda_j}}\right]\right\}\\
&\quad\times\left(1-X_m^{\lambda_m}\right)^{\frac{1}{2\lambda_{m-1}}}\left(1-X_{m-1}^{\lambda_{m-1}}\right)^{\frac{1}{2\lambda_{m-1}}}\left(1-X_m^{\lambda_m}\right)^{\frac{1}{\lambda_m}-\frac{1}{2\lambda_{m-1}}}\left(1-X_1^{\lambda_1}\right)^{\frac{1}{2\lambda_1}},
\end{aligned}\tag{18}$$

which is equivalent to the following inequality:

$$\prod_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}}\geqslant n^{\gamma-1}\left[\prod_{j=1}^{m}X_j+\prod_{j=1}^{m}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]. \tag{19}$$

On the other hand, applying the arithmetic-geometric means inequality we obtain

$$\prod_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right] \leqslant \left\{\frac{1}{m-1}\sum_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]\right\}^{m-1}$$
$$=\left[1-\frac{1}{m-1}\sum_{j=1}^{m-1}\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]^{m-1}. \tag{20}$$

Hence, by using the above inequality and Lemma 2.3 we have

$$\prod_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_j}} \leqslant \prod_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]^{\frac{1}{2\lambda_1}}$$
$$=\left\{\prod_{j=1}^{m-1}\left[1-\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]\right\}^{\frac{1}{2\lambda_1}}$$
$$\leqslant\left[1-\frac{1}{m-1}\sum_{j=1}^{m-1}\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2\right]^{\frac{m-1}{2\lambda_1}}$$
$$\leqslant 1-\frac{1}{(m-1)\max\{\frac{2\lambda_1}{m-1},1\}}\sum_{j=1}^{m-1}\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2$$
$$=1-\frac{1}{2\max\{\lambda_1,\frac{m-1}{2}\}}\sum_{j=1}^{m-1}\left(X_j^{\lambda_j}-X_{j+1}^{\lambda_{j+1}}\right)^2. \tag{21}$$

Combining inequalities (19) and (21) we can get inequality (17). The proof of Lemma 2.7 is completed. □

THEOREM 2.8. *Let* $a_{rj}>0$ $(r=1,2,\ldots,n,\ j=1,2,\ldots,m)$, $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_m$, $a_{1j}^{\lambda_j}-\sum_{r=2}^n a_{rj}^{\lambda_j}>0$ $(j=1,2,\ldots,m)$, *let* $m\geqslant 2, n\geqslant 2$, *and let* $\tau=\max\{\sum_{j=1}^m\frac{1}{\lambda_j},1\}$. *Then*

$$\prod_{j=1}^m\left(a_{1j}^{\lambda_j}-\sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geqslant n^{1-\tau}\prod_{j=1}^m a_{1j}-\sum_{r=2}^n\prod_{j=1}^m a_{rj}$$
$$-\frac{a_{11}a_{12}\ldots a_{1m}}{2\lambda_1}\sum_{j=1}^{m-1}\left[\sum_{r=2}^n\left(\frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}-\frac{a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2. \tag{22}$$

The inequality (22) is also valid for $\lambda_m>0,\ \lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_{m-1}<0$.

Proof. Case (I). When $\lambda_1,\lambda_2,\ldots,\lambda_m<0$, then $\tau=1$. On the one hand, from the hypotheses of Theorem 2.8 we have

$$\frac{\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}{\left(a_{1j}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}>1 \quad (j=1,2,\ldots,m). \tag{23}$$

Consequently, by using Lemma 2.5 with a substitution $X_j=\left(\frac{a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}}$ in (12), we have

$$\begin{aligned}
&\prod_{j=1}^{m}\left(\frac{\sum_{r=2}^{n}a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}}+\prod_{j=1}^{m}\left(\frac{a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}}\\
&\geqslant 1-\frac{1}{2\lambda_1}\sum_{j=1}^{m-1}\left[\left(1-\frac{\sum_{r=2}^{n}a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)-\left(1-\frac{\sum_{r=2}^{n}a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2\\
&=1-\frac{1}{2\lambda_1}\sum_{j=1}^{m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}-\frac{a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2,
\end{aligned} \tag{24}$$

which implies

$$\begin{aligned}
&\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\\
&\geqslant\prod_{j=1}^{m}a_{1j}-\prod_{j=1}^{m}\left(\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}-\frac{a_{11}a_{12}\ldots a_{1m}}{2\lambda_1}\sum_{j=1}^{m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}-\frac{a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2.
\end{aligned} \tag{25}$$

On the other hand, applying Lemma 2.1, we obtain

$$\prod_{j=1}^{m}\left(\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\leqslant\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}. \tag{26}$$

Combining inequalities (25) and (26) we can get inequality (22).

Case (II). When $\lambda_m>0,\lambda_1,\ldots,\lambda_{m-1}<0$. From the hypotheses of Theorem 2.8, it is easy to verify that

$$0<\frac{\left(a_{1m}^{\lambda_m}-\sum_{r=2}^{n}a_{rm}^{\lambda_m}\right)^{\frac{1}{\lambda_m}}}{\left(a_{1m}^{\lambda_m}\right)^{\frac{1}{\lambda_m}}}<1, \tag{27}$$

and

$$\frac{\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}{\left(a_{1j}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}>1 \quad (j=1,\ldots,m-1). \tag{28}$$

Consequently, by the same method as in Case (I), and using Lemma 2.6 with a substitution $X_m \to \frac{\left(a_{1m}^{\lambda_m}-\sum_{r=2}^{n}a_{rm}^{\lambda_m}\right)^{\frac{1}{\lambda_m}}}{\left(a_{1m}^{\lambda_m}\right)^{\frac{1}{\lambda_m}}}$, $X_j \to \left(\frac{a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}}$ $(j=1,\ldots,m-1)$ in (16), we obtain the desired inequality.

The proof of Theorem 2.8 is completed. □

If we set $\sum_{j=1}^{m}\frac{1}{\lambda_j}\leqslant 1$, then from Theorem 2.8 we obtain the following refinement of inequality (5) under the assumption $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_m$, or $\lambda_m>0$, $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_{m-1}<0$.

COROLLARY 2.9. *Let* $a_{rj}>0$ $(r=1,2,\ldots,n,\ j=1,2,\ldots,m)$, $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_m$, $\sum_{j=1}^{m}\frac{1}{\lambda_j}\leqslant 1$, $a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}>0$ $(j=1,2,\ldots,m)$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j}-\sum_{r=2}^{n}a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\geqslant\prod_{j=1}^{m}a_{1j}-\sum_{r=2}^{n}\prod_{j=1}^{m}a_{rj}$$
$$-\frac{a_{11}a_{12}\ldots a_{1m}}{2\lambda_1}\sum_{j=1}^{m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}-\frac{a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2. \tag{29}$$

The inequality (29) is also valid for $\lambda_m>0$, $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_{m-1}<0$.

In particular, putting $m=2$, $\lambda_1=p>0$, $\lambda_2=q<0$, $a_{r1}=a_r$, $a_{r2}=b_r$ $(r=1,2,\ldots,n)$ in Theorem 2.8, we obtain a new refinement and generalization of inequality (3).

COROLLARY 2.10. *Let* $a_r>0$, $b_r>0$ $(r=1,2,\ldots,n)$, $a_1^p-\sum_{r=2}^{n}a_r^p>0$, $b_1^q-\sum_{r=2}^{n}b_r^q>0$, $p>0$, $q<0$, $\rho=\max\{\frac{1}{p}+\frac{1}{q},1\}$. *Then, the following inequality holds:*

$$\left(a_1^p-\sum_{r=2}^{n}a_r^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{r=2}^{n}b_r^q\right)^{\frac{1}{q}}$$
$$\geqslant n^{1-\rho}a_1b_1-\sum_{r=2}^{n}a_rb_r-\frac{a_1b_1}{2q}\left[\sum_{r=2}^{n}\left(\frac{a_r^p}{a_1^p}-\frac{b_r^q}{b_1^q}\right)\right]^2. \tag{30}$$

If we set $\frac{1}{p}+\frac{1}{q}=1$ in Corollary 2.10, then the following refinement of inequality (3) holds.

COROLLARY 2.11. *Let* $a_r>0$, $b_r>0$ $(r=1,2,\ldots,n)$, *let* $p>0$, $q<0$, $\frac{1}{p}+\frac{1}{q}=1$, *and let* $a_1^p-\sum_{r=2}^{n}a_r^p>0$, $b_1^q-\sum_{r=2}^{n}b_r^q>0$. *Then*

$$\left(a_1^p-\sum_{r=2}^{n}a_r^p\right)^{\frac{1}{p}}\left(b_1^q-\sum_{r=2}^{n}b_r^q\right)^{\frac{1}{q}}$$
$$\geqslant a_1b_1-\sum_{r=2}^{n}a_rb_r-\frac{a_1b_1}{2q}\left[\sum_{r=2}^{n}\left(\frac{a_r^p}{a_1^p}-\frac{b_r^q}{b_1^q}\right)\right]^2. \tag{31}$$

THEOREM 2.12. *Let* $a_{rj} > 0$ $(r = 1,2,\ldots,n,\ j = 1,2,\ldots,m)$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j = 1,2,\ldots,m)$, *let* $m \geqslant 2, n \geqslant 2$, *and let* $\rho = \min\{\sum_{j=1}^{m} \frac{1}{\lambda_j}, 1\}$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leqslant n^{1-\rho}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj} - \frac{a_{11}a_{12}\ldots a_{1m}}{2\max\{\lambda_1, \frac{m-1}{2}\}}\sum_{j=1}^{m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}} - \frac{a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2. \tag{32}$$

Proof.
From the hypotheses of Theorem 2.12, it is easy to verify that

$$0 < \frac{\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}{\left(a_{1j}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}} < 1 \quad (j = 1,2,\ldots,m). \tag{33}$$

Consequently, by the same method as in Theorem 2.8, and using Lemma 2.7 with a substitution $X_j \to \left(\frac{a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}}$ $(j = 1,2,\ldots,m)$ in (17), we obtain the desired inequality. □

If we set $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, then from Theorem 2.12 we obtain the following refinement of inequality (4) under the assumption $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0$.

COROLLARY 2.13. *Let* $a_{rj} > 0$ $(r = 1,2,\ldots,n,\ j = 1,2,\ldots,m)$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ $(j = 1,2,\ldots,m)$, *and let* $m \geqslant 2,\ n \geqslant 2$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leqslant \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj} - \frac{a_{11}a_{12}\ldots a_{1m}}{2\max\{\lambda_1, \frac{m-1}{2}\}}\sum_{j=1}^{m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}} - \frac{a_{r(j+1)}^{\lambda_{j+1}}}{a_{1(j+1)}^{\lambda_{j+1}}}\right)\right]^2. \tag{34}$$

In particular, putting $m = 2,\ \lambda_1 = p \geqslant \lambda_2 = q > 0,\ a_{r1} = a_r,\ a_{r2} = b_r$ $(r = 1,2,\ldots,n)$ in Theorem 2.12, we obtain a new refinement and generalization of inequality (2).

COROLLARY 2.14. *Let* $a_r > 0,\ b_r > 0\ (r = 1,2,\ldots,n)$, *let* $p \geqslant q > 0,\ \rho = \min\{\frac{1}{p}+\frac{1}{q},1\}$, *and let* $a_1^p - \sum_{r=2}^n a_r^p > 0,\ b_1^q - \sum_{r=2}^n b_r^q > 0$. *Then*

$$\left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{r=2}^n b_r^q\right)^{\frac{1}{q}} \leqslant n^{1-\rho} a_1 b_1 - \sum_{r=2}^n a_r b_r - \frac{a_1 b_1}{2\max\{p,\frac{1}{2}\}} \left[\sum_{r=2}^n \left(\frac{a_r^p}{a_1^p} - \frac{b_r^q}{b_1^q}\right)\right]^2. \tag{35}$$

If we set $\frac{1}{p}+\frac{1}{q}=1$ in Corollary 2.14, then the following refinement of inequality (2) holds.

COROLLARY 2.15. *Let* $a_r > 0,\ b_r > 0\ (r = 1,2,\ldots,n)$, *let* $p,q > 0,\ \frac{1}{p}+\frac{1}{q}=1$, *and let* $a_1^p - \sum_{r=2}^n a_r^p > 0,\ b_1^q - \sum_{r=2}^n b_r^q > 0$. *Then*

$$\left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{r=2}^n b_r^q\right)^{\frac{1}{q}} \leqslant a_1 b_1 - \sum_{r=2}^n a_r b_r - \frac{a_1 b_1}{2\max\{p,q\}} \left[\sum_{r=2}^n \left(\frac{a_r^p}{a_1^p} - \frac{b_r^q}{b_1^q}\right)\right]^2. \tag{36}$$

3. Applications

In this section, we show two applications of the inequalities newly-obtained in Section 2.

Firstly, we present a new refinement of inequality (6).

THEOREM 3.1. *Let* $B_j > 0\ (j = 1,2,\ldots,m)$, *let* $\lambda_m > 0,\ \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_{m-1} < 0,\ \sum_{j=1}^m \frac{1}{\lambda_j} = 1$, *and let* $f_j\ (j = 1,2,\ldots,m)$ *be positive integrable functions defined on* $[a,b]$ *with* $B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x > 0$. *Then*

$$\prod_{j=1}^m \left(B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_j}} \geqslant \prod_{j=1}^m B_j - \int_a^b \prod_{j=1}^m f_j(x)\mathrm{d}x - \frac{B_1 B_2 \ldots B_m}{2\lambda_1} \sum_{j=1}^{m-1} \left[\int_a^b \left(\frac{f_j^{\lambda_j}(x)}{B_j^{\lambda_j}} - \frac{f_{j+1}^{\lambda_{j+1}}(x)}{B_{j+1}^{\lambda_{j+1}}}\right)\mathrm{d}x\right]^2. \tag{37}$$

Proof. For any positive integers n, we choose an equidistant partition of $[a,b]$ as

$$a < a + \frac{b-a}{n} < \cdots < a + \frac{b-a}{n}k < \cdots < a + \frac{b-a}{n}(n-1) < b,$$

$$x_i = a + \frac{b-a}{n}i, \quad i = 0,1,\cdots,n, \quad \Delta x_k = \frac{b-a}{n}, \quad k = 1,2,\cdots,n.$$

In view of $B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x > 0 \ (j = 1,2,\ldots,m)$, it follows

$$B_j^{\lambda_j} - \lim_{n\to\infty}\sum_{k=1}^{n} f_j^{\lambda_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n} > 0\ (j=1,2,\ldots,m).$$

Consequently, there exists a positive integer N, such that for all $n,l > N$ and $j = 1,2,\ldots,m$,

$$B_j^{\lambda_j} - \sum_{k=1}^{n} f_j^{\lambda_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n} > 0.$$

And then, by using Theorem 2.8, for any $n > N$, we find

$$\begin{aligned}
&\prod_{j=1}^{m}\left[B_j^{\lambda_j} - \sum_{k=1}^{n} f_j^{\lambda_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{\lambda_j}} \\
&\geqslant \prod_{j=1}^{m} B_j^{\lambda_j} - \sum_{k=1}^{n}\left(\prod_{j=1}^{m} f_j\left(a+\frac{k(b-a)}{n}\right)\right)\left(\frac{b-a}{n}\right)^{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\ldots+\frac{1}{\lambda_m}} \\
&\quad -\frac{B_1B_2\ldots B_m}{2\lambda_1}\sum_{j=1}^{m-1}\left[\sum_{k=1}^{n}\left(\frac{1}{B_j^{\lambda_j}} f_j^{\lambda_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right.\right. \\
&\quad \left.\left. -\frac{1}{B_{j+1}^{\lambda_{j+1}}} f_{j+1}^{\lambda_{j+1}}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right)\right]^2.
\end{aligned} \tag{38}$$

Since

$$\sum_{j=1}^{m}\frac{1}{\lambda_j} = 1,$$

we have

$$\begin{aligned}
&\prod_{j=1}^{m}\left[B_j^{\lambda_j} - \sum_{k=1}^{n} f_j^{\lambda_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{\lambda_j}} \\
&\geqslant \prod_{j=1}^{m} B_j^{\lambda_j} - \sum_{k=1}^{n}\left(\prod_{j=1}^{m} f_j\left(a+\frac{k(b-a)}{n}\right)\right)\left(\frac{b-a}{n}\right) \\
&\quad -\frac{B_1B_2\ldots B_m}{2\lambda_1}\sum_{j=1}^{m-1}\left[\sum_{k=1}^{n}\left(\frac{1}{B_j^{\lambda_j}} f_j^{\lambda_j}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right.\right. \\
&\quad \left.\left. -\frac{1}{B_{j+1}^{\lambda_{j+1}}} f_{j+1}^{\lambda_{j+1}}\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right)\right]^2.
\end{aligned} \tag{39}$$

Noting that $f_j(x)\ \ (j = 1,2,\ldots,m)$ are positive Riemann integrable functions on $[a,b]$, we know that $\prod_{j=1}^{m} f_j(x)$ and $f_j^{\lambda_j}(x)$ are also integrable on $[a,b]$. Letting $n \to$

∞ on both sides of inequality (39), we get the desired inequality (37). The proof of Theorem 3.1 is completed. □

We give here a direct consequence from Theorem 3.1. Putting $m=2, \lambda_1 = q < 0, \lambda_2 = p > 0, B_1 = a_1, B_2 = b_1, f_1 = f, f_2 = g$ in (37), we obtain a special important case follows.

COROLLARY 3.2. *Let p and q be real numbers such that $p>0, q<0$ and $\frac{1}{p}+\frac{1}{q}=1$, let $a_1, b_1 > 0$, and let f, g be positive integrable functions defined on $[a,b]$ with $a_1^p - \int_a^b f^p(x)\mathrm{d}x > 0$ and $b_1^q - \int_a^b g^q(x)\mathrm{d}x > 0$. Then*

$$\left(a_1^p - \int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}} \left(b_1^q - \int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{q}} \geqslant a_1 b_1 - \int_a^b f(x)g(x)\mathrm{d}x - \frac{a_1 b_1}{2q}\left[\int_a^b \left(\frac{f^p(x)}{a_1^p} - \frac{g^q(x)}{b_1^q}\right)\mathrm{d}x\right]^2. \tag{40}$$

Nextly, we give a new refinement of inequality (7).

THEOREM 3.3. *Let $B_j > 0$ $(j=1,2,\ldots,m)$, let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0$, $\sum_{j=1}^m \frac{1}{\lambda_j} = 1$, $m \geqslant 2$, and let f_j $(j=1,2,\ldots,m)$ be positive integrable functions defined on $[a,b]$ with $B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x > 0$. Then*

$$\prod_{j=1}^m \left(B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x\right)^{\frac{1}{\lambda_j}} \leqslant \prod_{j=1}^m B_j - \int_a^b \prod_{j=1}^m f_j(x)\mathrm{d}x - \frac{B_1 B_2 \ldots B_m}{2\max\{\lambda_1, \frac{m-1}{2}\}} \sum_{j=1}^{m-1} \left[\int_a^b \left(\frac{f_j^{\lambda_j}(x)}{B_j^{\lambda_j}} - \frac{f_{j+1}^{\lambda_{j+1}}(x)}{B_{j+1}^{\lambda_{j+1}}}\right)\mathrm{d}x\right]^2. \tag{41}$$

Proof. The proof of Theorem 3.3 is similar to the one of Theorem 3.1, and we omit it. □

Putting $m=2, \lambda_1 = p, \lambda_2 = q, B_1 = a_1, B_2 = b_1, f_1 = f, f_2 = g$ in (41), a special case to the last theorem follows.

COROLLARY 3.4. *Let p and q be real numbers such that $p \geqslant q > 0$ and $\frac{1}{p}+\frac{1}{q}=1$, let $a_1, b_1 > 0$, and let f, g be positive integrable functions defined on $[a,b]$ with $a_1^p - \int_a^b f^p(x)\mathrm{d}x > 0$ and $b_1^q - \int_a^b g^q(x)\mathrm{d}x > 0$. Then*

$$\left(a_1^p - \int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}} \left(b_1^q - \int_a^b g^q(x)\mathrm{d}x\right)^{\frac{1}{q}} \leqslant a_1 b_1 - \int_a^b f(x)g(x)\mathrm{d}x - \frac{a_1 b_1}{2p}\left[\int_a^b \left(\frac{f^p(x)}{a_1^p} - \frac{g^q(x)}{b_1^q}\right)\mathrm{d}x\right]^2. \tag{42}$$

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