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ON A REFINED HÖLDER'S INEQUALITY

E. G. KWON AND J. E. BAE

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Abstract. Refining Hölder's inequality, a result of S. Wu is extended to the case of multiple sequences.

1. Statements of results

Nobody could overestimate the importance and the usefulness of Hölder's inequality. It contributes wide area of pure and applied mathematics and plays a key role in resolving many problems in social science and cultural science as well as in natural science. Of course the inequality has been extensively explored and tested to a new situation by a number of scientists.

Among a lot of generalizations and extensions of the classical Hölder's inequality (see [1–6] and references therein), there is the existence of a function $g : l^p \times l^q \to [0,1]$ for which

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} g(a,b) \tag{1.1}$$

for all positive sequences $a = \{a_i\}$, $b = \{b_i\}$. K. Hu and S. Wu ([1], [2], [3]) considered this topic, and our model in this paper is the following recent result of Wu.

THEOREM A. ([3], Theorem 1) *Let* $a_i > 0,\ b_i > 0,\ e_i \geqslant 0\ (i = 1,2,\cdots,n)$, $\sum_{i=1}^{n} e_i = 1$ *and* $p \geqslant q > 0$. *Then*

$$\sum_{i=1}^{n} a_i b_i \leqslant n^{1-\min\left\{\frac{1}{p}+\frac{1}{q},1\right\}} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \left\{1 - \left(\frac{\sum_{i=1}^{n} a_i^p e_i}{\sum_{i=1}^{n} a_i^p} - \frac{\sum_{i=1}^{n} b_i^q e_i}{\sum_{i=1}^{n} b_i^q}\right)^2\right\}^{\frac{1}{p}},$$

with equality holding if and only if $a_1^p = a_2^p = \cdots = a_n^p$ *and* $b_1^q = b_2^q = \cdots = b_n^q$ *for* $\frac{1}{p} + \frac{1}{q} < 1$, *or* $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \cdots = \frac{a_n^p}{b_n^q}$ *for* $\frac{1}{p} + \frac{1}{q} = 1$.

Hölder's inequality for multiple sequences says

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ji}^{\lambda_j} \leqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji}\right)^{\lambda_j} \tag{1.2}$$

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provided $\sum_{j=1}^{m} \lambda_j = 1$, where $a_{ji} > 0,\ \lambda_j > 0\ (i = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$. Extending Theorem A to the case of multiple sequences, we establish the following theorem as our first and main result.

THEOREM 1.1. *Let* $a_{ji} > 0,\ e_i \geqslant 0,\ p_j > 0\ (i = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$, $\sum_{i=1}^{n} e_i = 1$ *and* $\{j_1, j_2\} \subset \{1,2,\cdots,m\}$. *Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji} \leqslant n^{1-\min\left\{\sum_{j=1}^{m}\frac{1}{p_j},1\right\}} \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ji}^{p_j}\right)^{\frac{1}{p_j}} G\left(j_1, j_2 : \{a_{ji}\}\right), \tag{1.3}$$

where $G\left(j_1, j_2 : \{a_{ji}\}\right) = \left\{1 - \left(\frac{\sum_{i=1}^{n} a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^{n} a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^{n} a_{j_2 i}^{p_{j_2}} e_i}{\sum_{i=1}^{n} a_{j_2 i}^{p_{j_2}}}\right)^2\right\}^{\frac{1}{\max\left\{p_{j_1}, p_{j_2}\right\}}}$. *Equality holds in* (1.3) *if and only if* $a_{j1}^{p_j} = a_{j2}^{p_j} = \cdots = a_{jn}^{p_j}\ \ (j = 1,2,\cdots,m)$ *for* $\sum_{j=1}^{m}\frac{1}{p_j} < 1$, *or* $\frac{a_{j_1 1}^{p_{j_1}}}{a_{j_2 1}^{p_{j_2}}} = \frac{a_{j_1 2}^{p_{j_1}}}{a_{j_2 2}^{p_{j_2}}} = \cdots = \frac{a_{j_1 n}^{p_{j_1}}}{a_{j_2 n}^{p_{j_2}}}\ \ (j_1, j_2 = 1,2,\cdots,m)$ *for* $\sum_{j=1}^{m}\frac{1}{p_j} = 1$.

Our proof of Theorem 1.1 is essentially the same as that of Theorem A. The point is that a modification of the methods of Wu works for the case of multiple sequences. The process will be done in Section 4 after preparing lemmas and remarks in Section 2 and Section 3. As an application of Theorem 1.1, we refine a reversed Hölder inequality in the final section.

We refer to [4] and [5] for basic inequalities in general.

2. Lemmas

Proof of Theorem 1.1 mainly depends on two variants of Hölder's inequality. The first one is a simple extension of (1.2) to the case $\sum_{j=1}^{m} \lambda_j \neq 1$. The second variant is induced by switching the ordering differently.

LEMMA 2.1. ([3], Lemma 1) *Let* $a_{ji} > 0,\ \lambda_j > 0\ (i = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$ *and* $\Lambda := \sum_{j=1}^{m} \lambda_j \neq 0$. *Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji}^{\lambda_j} \leqslant n^{1-\min\{\Lambda,1\}} \prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ji}\right)^{\lambda_j} \tag{2.1}$$

with equality holding if and only if $a_{j1} = a_{j2} = \cdots = a_{jn}\ (j = 1,2,\cdots,m)$ *for* $\Lambda < 1$, *or* $\frac{a_{1i}}{\sum_{i=1}^{n} a_{1i}} = \frac{a_{2i}}{\sum_{i=1}^{n} a_{2i}} = \cdots = \frac{a_{mi}}{\sum_{i=1}^{n} a_{mi}}\ \ (i = 1,2,\cdots,n)$ *for* $\Lambda = 1$.

LEMMA 2.2. *Let* $x_{ji} > 0\ \ (i = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$, $1 - \sum_{i=1}^{n} x_{ji}^{p_j} > 0\ \ (j =$

$1,2,\cdots,m$) and $p_j>0$ $(j=1,2,\cdots,m)$. Then

$$\begin{aligned}
&\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}x_{ji}^{p_j}\right)^{\frac{1}{p_j}}+\sum_{i=1}^{n}\prod_{j=1}^{m}x_{ji}\\
&\leqslant (n+1)^{\left\{1-\min\left(\sum_{j=1}^{m}\frac{1}{p_j},1\right)\right\}}\left\{1-\left(\sum_{i=1}^{n}x_{j_1 i}^{p_{j_1}}-\sum_{i=1}^{n}x_{j_2 i}^{p_{j_2}}\right)^2\right\}^{\frac{1}{\max\left\{p_{j_1},p_{j_2}\right\}}}
\end{aligned}\tag{2.2}$$

for $\{j_1,j_2\}\subset\{1,2,\cdots,m\}$. When $p_{j_1}\neq p_{j_2}$, equality holds in (2.2) if and only if $x_{j1}^{p_j}=x_{j2}^{p_j}=\cdots=x_{jn}^{p_j}=\frac{1}{n+1}$ $(j=1,2,\cdots,m)$ for $\sum_{j=1}^{m}\frac{1}{p_j}<1$, or $x_{1i}^{p_1}=x_{2i}^{p_2}=\cdots=x_{mi}^{p_m}$ $(i=1,2,\cdots,n)$ for $\sum_{j=1}^{m}\frac{1}{p_j}=1$.

3. Proof of Lemma 2.2

Let $\{j_1,j_2\}\subset\{1,2,\cdots,m\}$. For notational simplicity, we assume $j_1=1,\ j_2=2$ and $p_1\geqslant p_2$. Simple rearrangement gives

$$\begin{aligned}
&\sum_{i=1}^{n}\prod_{j=1}^{m}x_{ji}+\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}x_{ji}^{p_j}\right)^{\frac{1}{p_j}}\\
=&\sum_{i=1}^{n}\left(\left(x_{1i}^{p_1}\right)^{\frac{1}{p_1}}\left(x_{2i}^{p_2}\right)^{\frac{1}{p_1}}\left(x_{2i}^{p_2}\right)^{\frac{1}{p_2}-\frac{1}{p_1}}\prod_{j=3}^{m}\left(x_{ji}^{p_j}\right)^{\frac{1}{p_j}}\right)\\
&+\left(1-\sum_{i=1}^{n}x_{2i}^{p_2}\right)^{\frac{1}{p_1}}\left(1-\sum_{i=1}^{n}x_{1i}^{p_1}\right)^{\frac{1}{p_1}}\left(1-\sum_{i=1}^{n}x_{2i}^{p_2}\right)^{\frac{1}{p_2}-\frac{1}{p_1}}\prod_{j=3}^{m}\left(1-\sum_{i=1}^{n}x_{ji}^{p_j}\right)^{\frac{1}{p_j}}.
\end{aligned}$$

To the last quantity, by applying Hölder's inequality of the form

$$\sum_{i=1}^{n}\prod_{j}a_{ji}^{\lambda_j}+\prod_{j}b_j^{\lambda_j}\leqslant(n+1)^{1-\min\{\Lambda,1\}}\prod_{j}\left(b_j+\sum_{i=1}^{n}a_{ji}\right)^{\lambda_j}$$

which is a variant of (2.1) (see also [3]), we can obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\prod_{j=1}^{m}x_{ji}+\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}x_{ji}^{p_j}\right)^{\frac{1}{p_j}}\\
\leqslant\ &(n+1)^{1-\min\left\{\sum_{j=1}^{m}\frac{1}{p_j},1\right\}}\left\{1-\left(\sum_{i=1}^{n}x_{1i}^{p_1}-\sum_{i=1}^{n}x_{2i}^{p_2}\right)^2\right\}^{\frac{1}{p_1}}
\end{aligned}$$

as desired.

Assuming $p_1 \neq p_2$, the condition of equality can be checked from Lemma 2.1: when $\sum_{j=1}^{m} \frac{1}{p_j} < 1$ we should have

$$\begin{cases} x_{11}^{p_1} = x_{12}^{p_1} = \cdots = x_{1n}^{p_1} = 1 - \sum_{i=1}^{n} x_{2i}^{p_2} \\ x_{21}^{p_2} = x_{22}^{p_2} = \cdots = x_{2n}^{p_2} = 1 - \sum_{i=1}^{n} x_{1i}^{p_1} \\ x_{j1}^{p_j} = x_{j2}^{p_j} = \cdots = x_{jn}^{p_j} = 1 - \sum_{i=1}^{n} x_{ji}^{p_j} \ \ (j = 2, 3, \cdots, m) \end{cases} \tag{3.1}$$

which is equivalent to

$$x_{j1}^{p_j} = x_{j2}^{p_j} = \cdots = x_{jn}^{p_j} = \frac{1}{n+1} \quad (j = 1, 2, \cdots, m);$$

while when $\sum_{j=1}^{m} \frac{1}{p_j} = 1$ we should have

$$\begin{cases} \frac{x_{1i}^{p_1}}{X(1,2)} = \frac{x_{2i}^{p_2}}{X(2,1)} = \frac{x_{ji}^{p_j}}{X(j,j)} \quad (j = 2, 3, \cdots, m;\ i = 1, 2, \cdots, n) \\ \frac{1-\sum_{i=1}^{n} x_{2i}^{p_2}}{X(1,2)} = \frac{1-\sum_{i=1}^{n} x_{1i}^{p_1}}{X(2,1)} = \frac{1-\sum_{i=1}^{n} x_{ji}^{p_j}}{X(j,j)} \quad (j = 2, 3, \cdots, m) \end{cases} \tag{3.2}$$

which is equivalent to

$$x_{1i}^{p_1} = x_{2i}^{p_2} = \cdots = x_{mi}^{p_m} \quad (i = 1, 2, \cdots, n),$$

where we abbreviated $X(s,t) = \sum_{i=1}^{n} x_{si}^{p_s} + \left(1 - \sum_{i=1}^{n} x_{ti}^{p_t}\right)$ for $s, t = 1, 2, \cdots, m$. The proof is complete.

REMARK 3.1. When $p_{j_1} = p_{j_2}$, the condition of equality in (2.2) can also be deduced from Lemma 2.1: assuming $p_{j_1} = 1,\ p_{j_2} = 2$, we should have (3.1) or (3.2) only for $j = 3, 4, \cdots, m$.

REMARK 3.2. The case $m = 2$ of Lemma 2.2 appeared in Lemma 3 of [3]. In the sense of Remark 3.1, there is a slight mistake of the "if and only if condition of equality". Example below: Take $p = q = 2$ and

$$x_i = \frac{1}{\sqrt{3n}}, \qquad y_i = \frac{\sqrt{2}}{\sqrt{3n}}, \qquad (i = 1, 2, \cdots, n),$$

then $x_i^p \neq y_i^q$ but

$$\left(1 - \frac{1}{3}\right)^{\frac{1}{2}} \left(1 - \frac{2}{3}\right)^{\frac{1}{2}} + n \frac{1}{\sqrt{3n}} \frac{\sqrt{2}}{\sqrt{3n}} = \sqrt{1 - \left(\frac{1}{3} - \frac{2}{3}\right)^2}$$

which violates the equality condition "$x_i^p = y_i^q \ \ (i = 1, 2, \cdots, n)$ when $\frac{1}{p} + \frac{1}{q} = 1$" of the the lemma.

4. Proof of Theorem 1.1

Our proof consists of four steps.

First step. It is straightforward to see that

$$\begin{aligned}
&\frac{\sum_{i=1}^{n}\prod_{j=1}^{m}a_{ji}}{\prod_{j=1}^{m}\left(\sum_{i=1}^{n}a_{ji}^{p_j}\right)^{\frac{1}{p_j}}}\\
=\ &\frac{\prod_{j=1}^{m}a_{jk}}{\prod_{j=1}^{m}\left(\sum_{t=1}^{n}a_{jt}^{p_j}\right)^{\frac{1}{p_j}}}+\frac{\sum_{1\leqslant i\leqslant n, i\neq k}\prod_{j=1}^{m}a_{ji}}{\prod_{j=1}^{m}\left(\sum_{t=1}^{n}a_{jt}^{p_j}\right)^{\frac{1}{p_j}}} \qquad (4.1)\\
=\ &\prod_{j=1}^{m}\left\{1-\sum_{1\leqslant i\leqslant n, i\neq k}\left(\frac{a_{ji}^{p_j}}{\sum_{t=1}^{n}a_{jt}^{p_j}}\right)\right\}^{\frac{1}{p_j}}+\sum_{1\leqslant i\leqslant n, i\neq k}\prod_{j=1}^{m}\left(\frac{a_{ji}^{p_j}}{\sum_{t=1}^{n}a_{jt}^{p_j}}\right)^{\frac{1}{p_j}}
\end{aligned}$$

for $k=1,2,\cdots,n$.

Second step. Let

$$x_{ji}=\left(\frac{a_{ji}^{p_j}}{\sum_{t=1}^{n}a_{jt}^{p_j}}\right)^{\frac{1}{p_j}}, \quad (i=1,2,\cdots,n;\ j=1,2,\cdots,m)$$

and fix

$$\{j_1,j_2\}\subset\{1,2,\cdots,m\}.$$

Then, it is obvious that

$$x_{ji}>0 \quad (i=1,2,\cdots,n;\ j=1,2,\cdots,m)$$

and

$$1-\sum_{1\leqslant i\leqslant n, i\neq k}x_{ji}^{p_j}\ >\ 0 \quad (j=1,2,\cdots,m)$$

for $k=1,2,\cdots,n$. Thus, applying Lemma 2.2,

$$\begin{aligned}
&\prod_{j=1}^{m}\left\{1-\sum_{1\leqslant i\leqslant n, i\neq k}\left(\frac{a_{ji}^{p_j}}{\sum_{t=1}^{n}a_{jt}^{p_j}}\right)\right\}^{\frac{1}{p_j}}+\sum_{1\leqslant i\leqslant n, i\neq k}\prod_{j=1}^{m}\left(\frac{a_{ji}^{p_j}}{\sum_{t=1}^{n}a_{jt}^{p_j}}\right)^{\frac{1}{p_j}} \qquad (4.2)\\
\leqslant\ & n^{1-\min\left\{\sum_{j=1}^{m}\frac{1}{p_j},1\right\}}G_k,
\end{aligned}$$

where

$$G_k=\left\{1-\left(\sum_{1\leqslant i\leqslant n, i\neq k}\frac{a_{j_1 i}^{p_{j_1}}}{\sum_{t=1}^{n}a_{j_1 t}^{p_{j_1}}}-\sum_{1\leqslant i\leqslant n, i\neq k}\frac{a_{j_2 i}^{p_{j_2}}}{\sum_{t=1}^{n}a_{j_2 t}^{p_{j_2}}}\right)^2\right\}^{\frac{1}{\max\left\{p_{j_1},p_{j_2}\right\}}}.$$

But it is straightforward to see

$$G_k = \left\{1-\left(\frac{a_{j_1k}^{p_{j_1}}}{\sum_{i=1}^n a_{j_1i}^{p_{j_1}}}-\frac{a_{j_2k}^{p_{j_2}}}{\sum_{i=1}^n a_{j_2i}^{p_{j_2}}}\right)^2\right\}^{\frac{1}{\max\left\{p_{j_1},p_{j_2}\right\}}}.$$

Third step. Since $e_i \geqslant 0$ $(i=1,2,\cdots,n)$ with $\sum_{i=1}^n e_i = 1$, by (4.1) and (4.2) it follows that

$$\frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji}}{\prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j}\right)^{\frac{1}{p_j}}} = \prod_{k=1}^n \left[\frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji}}{\prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j}\right)^{\frac{1}{p_j}}}\right]^{e_k} \leqslant n^{1-\min\left\{\sum_{j=1}^m \frac{1}{p_j},1\right\}} \prod_{k=1}^n G_k^{e_k}.$$

On the other hand, applying the arithmetic-geometric mean inequality and Jensen's inequality,

$$\begin{aligned}
&\prod_{k=1}^n \left\{1-\left(\frac{a_{j_1k}^{p_{j_1}}}{\sum_{i=1}^n a_{j_1i}^{p_{j_1}}}-\frac{a_{j_2k}^{p_{j_2}}}{\sum_{i=1}^n a_{j_2i}^{p_{j_2}}}\right)^2\right\}^{e_k} \\
\leqslant\ & \sum_{k=1}^n \left\{e_k - e_k\left(\frac{a_{j_1k}^{p_{j_1}}}{\sum_{i=1}^n a_{j_1i}^{p_{j_1}}}-\frac{a_{j_2k}^{p_{j_2}}}{\sum_{i=1}^n a_{j_2i}^{p_{j_2}}}\right)^2\right\} \\
=\ & 1-\sum_{k=1}^n e_k\left(\frac{a_{j_1k}^{p_{j_1}}}{\sum_{i=1}^n a_{j_1i}^{p_{j_1}}}-\frac{a_{j_2k}^{p_{j_2}}}{\sum_{i=1}^n a_{j_2i}^{p_{j_2}}}\right)^2 \\
\leqslant\ & 1-\left(\frac{\sum_{i=1}^n a_{j_1i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1i}^{p_{j_1}}}-\frac{\sum_{i=1}^n a_{j_2i}^{p_{j_2}} e_i}{\sum_{i=1}^n a_{j_2i}^{p_{j_2}}}\right)^2.
\end{aligned} \tag{4.3}$$

Thus,

$$\frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji}}{\prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j}\right)^{\frac{1}{p_j}}} \leqslant n^{1-\min\left\{\sum_{j=1}^m \frac{1}{p_j},1\right\}} G,$$

where

$$G := G\left(j_1,j_2:\left\{a_{ji}\right\}\right) = \left\{1-\left(\frac{\sum_{i=1}^n a_{j_1i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1i}^{p_{j_1}}}-\frac{\sum_{i=1}^n a_{j_2i}^{p_{j_2}} e_i}{\sum_{i=1}^n a_{j_2i}^{p_{j_2}}}\right)^2\right\}^{\frac{1}{\max\left\{p_{j_1},p_{j_2}\right\}}}.$$

Final step. It is left to verify the condition of equality. First assume $p_{j_1} \neq p_{j_2}$. The equality in (1.3) holds if and only if equalities in (4.2) and in (4.3) hold simultaneously. It follows from Lemma 2.2 that equality in (4.2) holds if and only if

$$x_{j1}^{p_j} = x_{j2}^{p_j} = \cdots = x_{jn}^{p_j} = \frac{1}{n+1} \quad (j=1,2,\cdots,m) \tag{4.4}$$

when $\sum_{j=1}^{m} \frac{1}{p_j} < 1$, or

$$x_{1i}^{p_1} = x_{2i}^{p_2} = \cdots = x_{mi}^{p_m} \quad (i = 1,2,\cdots,n) \tag{4.5}$$

when $\sum_{j=1}^{m} \frac{1}{p_j} = 1$. It is simple to see that (4.4) and (4.5) reduce to

$$a_{j1}^{p_j} = a_{j2}^{p_j} = \cdots = a_{jn}^{p_j} \quad (j = 1,2,\cdots,m)$$

and

$$\frac{a_{j_1 1}^{p_{j_1}}}{a_{j_2 1}^{p_{j_2}}} = \frac{a_{j_1 2}^{p_{j_1}}}{a_{j_2 2}^{p_{j_2}}} = \cdots = \frac{a_{j_1 n}^{p_{j_1}}}{a_{j_2 n}^{p_{j_2}}} \quad (j_1, j_2 = 1,2,\cdots,m)$$

respectively. These conditions obviously make equality in (4.3), so became a necessary and sufficient condition for the equality in (1.3). When $p_{j_1} = p_{j_2}$, the same conclusion can be deduced following the guidance of Remark 3.1. The proof is complete.

REMARK 4.1. Taking $e_k = 1$ and $e_j = 0$ for $j \neq k$, G reduces to G_k of the second step. This fact makes the third step proud.

5. An application

Wu [2, 3] and Tian [6] gave various applications of refined Hölder's inequality of type (1.1). We believe that Theorem 1.1 might have further applications.

Concerning reversed Hölder inequality, J. Tian established the following:

THEOREM B. ([6], Theorem 2.2) *Let* $a_i > 0$, $b_i > 0$ $(i = 1,2,\cdots,n)$, $1 - e_i + e_j \geqslant 0$ $(i,j = 1,2,\cdots,n)$, $q < 0$ *and* $\frac{1}{p} + \frac{1}{q} \geqslant 0$. *Then*

$$\sum_{i=1}^{n} a_i b_i \geqslant n^{1-\max\left\{\frac{1}{p}+\frac{1}{q},1\right\}} \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$$

$$\times \left\{1 - \left(\frac{\sum_{i=1}^{n} b_i^q e_i}{\sum_{i=1}^{n} b_i^q} - \frac{\sum_{i=1}^{n} a_i b_i e_i}{\sum_{i=1}^{n} a_i b_i}\right)^2\right\}^{\frac{\max\left\{-1,\frac{1}{q}\right\}}{2}}.$$

In the same vein, we illustrate the following as an application of Theorem 1.1.

THEOREM 5.1. *Let* $a_{ji} > 0$ $(i = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$, $e_i \geqslant 0$ $(i = 1,2,\cdots,n)$, $\sum_{i=1}^{n} e_i = 1$, $p_j < 0$ $(j = 1,2,\cdots,m-1)$ *and* $p_m > 0$. *Let* $k \in \{1,2,\cdots,m-1\}$. *Then*

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ji} \geqslant n^{1-\max\left\{\sum_{j=1}^{m} \frac{1}{p_j},1\right\}} \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji}^{p_j}\right)^{\frac{1}{p_j}}$$

$$\times \left\{1 - \left(\frac{\sum_{i=1}^{n} a_{ki}^{p_k} e_i}{\sum_{i=1}^{n} a_{ki}^{p_k}} - \frac{\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ji} e_i}{\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ji}}\right)^2\right\}^{\max\left\{-1,\frac{1}{p_k}\right\}}.$$

Proof. Let $q_j = -\frac{p_j}{p_m} > 0$ ($j = 1, 2, \cdots, m-1$) and $q_m = \frac{1}{p_m} > 0$. Then

$$\sum_{i=1}^{n}\left(a_{1i}^{-p_m} \times a_{2i}^{-p_m} \times \cdots \times a_{m-1i}^{-p_m} \times \prod_{j=1}^{m} a_{ji}^{p_m}\right)$$
$$\leqslant n^{1-\min\left\{\sum_{j=1}^{m}\frac{1}{q_j},1\right\}}\left\{\prod_{j=1}^{m-1}\left(\sum_{i=1}^{n} a_{ji}^{p_j}\right)^{\frac{1}{q_j}} \times \left(\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji}\right)^{\frac{1}{q_m}}\right\}$$
$$\times\left\{1-\left(\frac{\sum_{i=1}^{n} a_{ki}^{p_k} e_i}{\sum_{i=1}^{n} a_{ki}^{p_k}} - \frac{\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji} e_i}{\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji}}\right)^2\right\}^{\frac{1}{\max\{q_m,q_k\}}},$$

by applying Theorem 1.1. That is,

$$\left(\sum_{i=1}^{n} a_{mi}^{p_m}\right)^{\frac{1}{p_m}} \leqslant n^{\max\left\{\sum_{j=1}^{m}\frac{1}{p_j},1\right\}-1}\left\{\prod_{j=1}^{m-1}\left(\sum_{i=1}^{n} a_{ji}^{p_j}\right)^{-\frac{1}{p_j}} \times \left(\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji}\right)\right\}$$
$$\times\left\{1-\left(\frac{\sum_{i=1}^{n} a_{ki}^{p_k} e_i}{\sum_{i=1}^{n} a_{ki}^{p_k}} - \frac{\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji} e_i}{\sum_{i=1}^{n}\prod_{j=1}^{m} a_{ji}}\right)^2\right\}^{\frac{1}{\max\{1,-p_k\}}}.$$

Multiplying $\prod_{j=1}^{m-1}\left(\sum_{i=1}^{n} a_{ji}^{p_j}\right)^{\frac{1}{p_j}}$ on both sides, it reduces to what we want. The proof is complete. $\square$

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E. G. Kwon
Department of Mathematics-Education
Andong National University
Andong 760–749 Korea
e-mail: egkwon@andong.ac.kr

J. E. Bae
Department of Mathematic, Graduate School
Andong National University
Andong 760–749 Korea
e-mail: ner-salanghae@nate.com