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NONLINEAR RETARDED INTEGRAL INEQUALITIES OF GRONWALL–BELLMAN TYPE AND APPLICATIONS

A. ABDELDAIM

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Abstract. The aim of this paper is to establish some new nonlinear retarded integral inequalities of Gronwall-Bellman type. More accurately we extend certain results which have been proved in El-Owaidy et. al, [12] and Abdeldaim and Yakout [2] to some nonlinear retarded integral inequalities. We also give some applications to estimate the solutions of some nonlinear retarded differential equations to illustrate the effectiveness of some from our results.

1. Introduction

Differential and integral inequalities have become major tools in the analysis of the differential and integral equations that occur in nature or are constructed by many mathematicians. Integral inequalities that give explicit bounds on unknown functions provide a very useful and important device in the study of many qualitative as well as quantitative properties of solutions of nonlinear differential equations see for instance [1, 3–10, 13, 15, 16, 19, 21–24].

Throughout this paper, $\mathbb{R}$ denoted the set of real numbers, $I=[0,\infty)$ is the subset of $\mathbb{R}$, $\prime$ denotes the derivative. $\mathscr{C}(I,I)$ denotes the set of all continuous functions from I into I and $\mathscr{C}^1(I,I)$ denotes the set of all continuously differentiable functions from I into I.

One of the best known and widely used inequalities in the study of differential equations can be stated as follows:

THEOREM 1.1. (Gronwall inequality [13]) *Let $u(t)$ be a continuous function defined on the interval $D=[\alpha,\alpha+h]$ and*

$$0\leqslant u(t)\leqslant \int_{\alpha}^{t}[bu(s)+a]ds,\ \forall t\in D,$$

where α,h,a and b are nonnegative constants. Then, $0\leqslant u(t)\leqslant ahe^{bh}$, $\forall t\in D$.

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The Gronwall type (Theorem 1.1) integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of the various type. After the discovery of this integral inequality which resulting from Gronwall [13], a number of mathematicians have shown their considerable interest to generalize the original form of this inequality which were partly inspired by the Gronwall's inequality for example in [2, 7–9, 11, 12, 14, 16, 17, 25]. Closely related to the Gronwall inequality is the following theorem

THEOREM 1.2. (Gronwall-Bellman inequality [6]) *Let $f(t)$ and $u(t)$ be real-valued nonnegative continuous functions defined on $D_1 = [0,h]$, and let u_0 and h are positive constants for which the inequality*

$$u(t) \leqslant u_0 + \int_0^t f(s)u(s)ds, \forall t \in D_1.$$

Then, $u(t) \leqslant u_0 \exp(\int_0^t f(s)ds)$, $\forall t \in D_1$.

In [12], El-Owaidy et al introduced the following theorems

THEOREM 1.3. *Let $u(t)$, $g(t)$, $f(t)$ $\in \mathscr{C}(I,I)$ and satisfy the inequality*

$$u(t) \leqslant u_0 + \int_0^t f(s)\left[u^{(2-p)}(s) + \int_0^s g(\lambda)u^q(\lambda)d\lambda\right]^p ds, \ \forall t \in I,$$

where $u_0 > 0$, and $0 < p \leqslant 1$, $0 \leqslant q < 1$, are constants. Then

$$u(t) \leqslant u_0 + \int_0^t f(s)k(s)\exp\left(p(2-p)\int_0^s f(\lambda)d\lambda\right)ds, \ \forall t \in I,$$

where

$$k(t) = \left[u_0^{(1-q)(2-p)} + (1-q)\int_0^t g(s)\exp\left(-(1-q)(2-p)\int_0^s f(\lambda)d\lambda\right)ds\right]^{[\frac{p}{(1-q)}]}, \ \forall t \in I.$$

THEOREM 1.4. *Let $u(t)$ be a real valued positive continuous function and $g(t)$, $f(t)$ $\in \mathscr{C}(I,I), n(t)$ be a positive monotonic nondecreasing continuous function defined on I and satisfy the inequality*

$$u(t) \leqslant n(t) + \int_0^t f(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\right]^p ds, \ \forall t \in I,$$

where p a constant such that $p \in (0,1)$. Then

$$u(t) \leqslant n(t)\left[1 + \int_0^t f(s)k_0(s)n^{(p-1)}(s)\exp\left(p(1-p)\int_0^s g(\lambda)d\lambda\right)ds\right], \ \forall t \in I,$$

where

$$k_0(t) = \left[1 + (1-q)\int_0^t f(s)n^{(p-1)}(s)\exp\left(-(1-p)\int_0^s g(\lambda)d\lambda\right)ds\right]^{[\frac{p}{(1-p)}]}, \ \forall t \in I.$$

Some applications of this results can be used to the study of existence, uniqueness theory of differential equations and the stability of the solution of linear and nonlinear differential equations (see [6–9, 16]). This century has seen considerable and fruitful research in the field of inequalities and their applications in various branches of mathematics.

However many real-life problems that have in the past sometimes been modeled by initial-value problems for differential equations. Actually involve a significant memory effect that can be represented in a more refined model using a differential equation incorporating retarded or delayed arguments. In this situations, we need to discuss some retarded nonlinear integral inequalities, where non retarded argument t is changed into retarded argument $\alpha(t)$. In order to study retarded differential and integral equations we consider some inequalities with retarded argument. Closely related to the this type from integral inequalities is the following theorem

THEOREM 1.5. (Lipovan [15]) *Let $u(t), f(t) \in \mathscr{C}([t_0,T_0],\mathbb{R}_+)$, $\alpha(t) \in \mathscr{C}([t_0,T_0],[t_0,T_0])$ be nondecreasing with $\alpha(t) \leqslant t$ on $[t_0,T_0]$ and let u_0 be a nonnegative constant, then the inequality*

$$u(t) \leqslant u_0 + \int_{\alpha(t_0)}^{\alpha(t)} f(s)u(s)ds, \quad t_0 < t < T_0,$$

implies that,

$$u(t) \leqslant u_0 \exp\Big(\int_{\alpha(t_0)}^{\alpha(t)} f(s)ds\Big), \quad t_0 < t < T_0.$$

Recently, many new results on the nonlinear retarded inequalities can be found, see for instance in [1, 4, 5, 10, 15, 18, 20–24]. The main objective of this paper is investigate explicit bounds on retarded integral inequalities of Gronwall-Bellman type which can be used as handy tools to study the qualitative behavior of certain retarded differential and integral equations.

Pachpatte in [17] investigated the following retarded inequality

$$u(t) \leqslant u_0 + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds, \ \forall\, t \in J, \tag{1.1}$$

where u_0 is a constant. Replacing u_0 by a nondecreasing continuous function $n(t)$ in the above inequality El-Owaidy et al in [10] studied the following nonlinear retarded inequality

$$u^p(t) \leqslant n^p(t) + \int_0^t g(s)u^p(s)ds + \int_0^{\alpha(t)} h(s)u^q(s)ds, \ \forall t \in J. \tag{1.2}$$

Also, Abdeldaim and El-Deeb in [3] studied the following nonlinear retarded inequality

$$u(t) \leqslant n(t) + \int_0^{\alpha(t)} f(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\right]^p ds, \ \forall\, t \in J. \tag{1.3}$$

However, sometimes we need to study such inequalities with differentiable function in place of nondecreasing continuous function $n(t)$ term outside the integrals. In this paper, our results concern with integral inequalities with such a differentiable function $n(t)$ term outside the integrals, which gives us another analysis techniques in the proof and different form in the solution of the integral inequality as we will see in the Theorem 2.3 and the Theorem 2.4 in Section 2.

2. Main results

In this section, we state and prove some new nonlinear retarded integral inequalities of Gronwall-Bellman type, which are further generalizations for some known results and can be used as ready and powerful tools in developing the theory of nonlinear retarded differential and integral equations.

THEOREM 2.1. *Let $u(t)$, $g(t)$, $f(t) \in \mathscr{C}(I,I)$, $\alpha(t) \in \mathscr{C}^1(I,I)$ be nondecreasing with $\alpha(t) \leqslant t$ on I. If the inequality*

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)\left[u^{(2-p)}(s) + \int_0^s g(\lambda)u^q(\lambda)d\lambda\right]^p ds, \ \forall t \in I, \tag{2.1}$$

where $q, u_0 > 0$ and $0 < p \leqslant 1$ are constants.

A: *If $0 \leqslant q < 1$, then*

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)k_1(\alpha^{-1}(s))\exp\left(p(2-p)\int_0^s f(\lambda)d\lambda\right)ds, \ \forall t \in I, \tag{2.2}$$

where for all $t \in I$

$$k_1(t) = \left[u_0^{(1-q)(2-p)} + (1-q)\int_0^{\alpha(t)} g(s)\exp\left(-(1-q)(2-p)\int_0^s f(\lambda)d\lambda\right)ds\right]^{\left[\frac{p}{(1-q)}\right]}. \tag{2.3}$$

B: *If $q > 1$, then*

$$u(t) \leqslant u_0 + u_0^{p(2-p)}\int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(s))\exp\left(p(2-p)\int_0^s f(\lambda)d\lambda\right)ds, \ \forall t \in I, \tag{2.4}$$

where

$$k_2(t) = \left[1-(q-1)u_0^{(2-p)(q-1)}\int_0^{\alpha(t)} g(s)\exp\left((2-p)(q-1)\int_0^s f(\lambda)d\lambda\right)ds\right]^{\left[\frac{-p}{(q-1)}\right]}, \tag{2.5}$$

for all $t \in I$, and

$$1-(q-1)u_0^{(2-p)(q-1)}\int_0^{\alpha(t)} g(s)\exp\left((2-p)(q-1)\int_0^s f(\lambda)d\lambda\right)ds > 0, \ \forall t \in I.$$

Proof. **A:** Let $J(t)$ equal the right hand side in (2.1), we have $J(0) = u_0$ and

$$u(t) \leqslant J(t), \ \forall t \in I. \tag{2.6}$$

Differentiating $J(t)$, with respect to t and using (2.6) leads to

$$\frac{dJ(t)}{dt} \leqslant \alpha'(t) f(\alpha(t)) Y^p(t), \ \forall t \in I, \tag{2.7}$$

where $Y(t) = J^{(2-p)}(t) + \int_0^{\alpha(t)} g(s) J^q(s) ds$, thus we have $Y(0) = u_0^{(2-p)}$, but $p \leqslant 1 \rightarrow 1-p \geqslant 0 \rightarrow 2-p \geqslant 1$ and $J^{(2-p)}(t) \leqslant Y(t)$ thus we have

$$J(t) \leqslant Y(t), \ \forall t \in I. \tag{2.8}$$

Differentiating $Y(t)$ with respect to t and using (2.7) and (2.8), leads to

$$\frac{dY(t)}{dt} \leqslant (2-p)\alpha'(t) f(\alpha(t)) Y(t) + \alpha'(t) g(\alpha(t)) Y^q(t), \ \forall t \in I, \tag{2.9}$$

but $Y(t) > 0$ then we can write the inequality (2.9) in the following form

$$Y^{-q}(t)\frac{dY(t)}{dt} - (2-p)\alpha'(t) f(\alpha(t)) Y^{(1-q)}(t) \leqslant \alpha'(t) g(\alpha(t)), \ \forall t \in I. \tag{2.10}$$

If we let $Y^{(1-q)}(t) = Z(t)$, we have $Z(0) = u_0^{(1-q)(2-p)}$ and $Y^{-q}(t)\frac{dY(t)}{dt} = \frac{1}{(1-q)}\frac{dZ(t)}{dt}$, then we can write the inequality (2.10) as follows

$$\frac{dZ(t)}{dt} - (1-q)(2-p)\alpha'(t) f(\alpha(t)) Z(t) \leqslant (1-q)\alpha'(t) g(\alpha(t)), \ \forall t \in I. \tag{2.11}$$

The inequality (2.11) implies an estimation for $Z(t)$ as in the following

$$\begin{aligned} Z(t) \leqslant & \left[u_0^{(1-q)(2-p)} + (1-q) \int_0^{\alpha(t)} g(s) \exp\left(-(1-q)(2-p) \int_0^s f(\lambda) d\lambda \right) ds \right] \\ & \times \exp\left((1-q)(2-p) \int_0^{\alpha(t)} f(s) ds \right), \forall t \in I. \end{aligned} \tag{2.12}$$

But $Y^{(1-q)}(t) = Z(t)$, then from (2.12), we have

$$Y^p(t) \leqslant k_1(t) \exp\left(p(2-p) \int_0^{\alpha(t)} f(s) ds \right), \ \forall t \in I, \tag{2.13}$$

where $k_1(t)$ as defined in (2.3), and from (2.13) in (2.7), we obtain

$$\frac{dJ(t)}{dt} \leqslant \alpha'(t) f(\alpha(t)) k_1(t) \exp\left(p(2-p) \int_0^{\alpha(t)} f(s) ds \right), \ \forall t \in I.$$

The above inequality implies an estimation for $J(t)$ as in the following

$$J(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)k_1(\alpha^{-1}(s))\exp\left(p(2-p)\int_0^s f(\lambda)d\lambda\right)ds,\ \forall t \in I. \tag{2.14}$$

Using (2.14) in (2.6), we get the required inequality in (2.2).

B: Let $J_1(t)$ equal the right hand side in (2.1), we have $J_1(0) = u_0$ and

$$u(t) \leqslant J_1(t),\ \forall t \in I. \tag{2.15}$$

Differentiating $J_1(t)$, with respect to t and using (2.15) leads to

$$\frac{dJ_1(t)}{dt} \leqslant \alpha'(t)f(\alpha(t))Y_1^p(t),\ \forall t \in I, \tag{2.16}$$

where $Y_1(t) = J_1^{(2-p)}(t) + \int_0^{\alpha(t)} g(s)J_1^q(s)ds$, thus we have $Y_1(0) = u_0^{(2-p)}$, but $p \leqslant 1 \longrightarrow 1-p \geqslant 0 \longrightarrow 2-p \geqslant 1$ and $J_1^{(2-p)}(t) \leqslant Y_1(t)$ thus we have

$$J_1(t) \leqslant Y_1(t),\ \forall t \in I. \tag{2.17}$$

Differentiating $Y_1(t)$ with respect to t and using (2.16) and (2.17), leads to

$$\frac{dY_1(t)}{dt} \leqslant (2-p)\alpha'(t)f(\alpha(t))Y_1(t) + \alpha'(t)g(\alpha(t))Y_1^q(t),\ \forall t \in I, \tag{2.18}$$

but $Y_1(t) > 0$ then we can write the inequality (2.18) in the following form

$$Y_1^{-q}(t)\frac{dY_1(t)}{dt} - (2-p)\alpha'(t)f(\alpha(t))Y_1^{-(q-1)}(t) \leqslant \alpha'(t)g(\alpha(t)),\ \forall t \in I. \tag{2.19}$$

If we let, $Y_1^{-(q-1)}(t) = Z_1(t)$, we have $Z_1(0) = u_0^{-(2-p)(q-1)}$, and $Y_1^{-q}(t)\frac{dY_1(t)}{dt} = \frac{-1}{(q-1)}\frac{dZ_1(t)}{dt}$, then we can write the inequality (2.19) as follows

$$\frac{dZ_1(t)}{dt} + (2-p)(q-1)\alpha'(t)f(\alpha(t))Z_1(t) \geqslant -(q-1)\alpha'(t)g(\alpha(t)),\ \forall t \in I. \tag{2.20}$$

The inequality (2.20) implies an estimation for $Z_1(t)$ as in the following

$$Z_1(t) \geqslant \frac{\left[1-(q-1)u_0^{(2-p)(q-1)}\int_0^{\alpha(t)} g(s)\exp\left((2-p)(q-1)\int_0^s f(\lambda)d\lambda\right)ds\right]}{u_0^{(2-p)(q-1)}\exp\left((2-p)(q-1)\int_0^{\alpha(t)} f(s)ds\right)}, \tag{2.21}$$

for all $t \in I$. But $Y_1(t) = \left[\frac{1}{Z_1(t)}\right]^{(q-1)}$, then from (2.21), we have

$$Y_1^p(t) \leqslant u_0^{p(2-p)}k_2(t)\exp\left(p(2-p)\int_0^{\alpha(t)} f(s)ds\right),\ \forall t \in I, \tag{2.22}$$

where $k_2(t)$ as defined in (2.5), and from (2.22) in (2.16), we obtain

$$\frac{dJ_1(t)}{dt} \leqslant u_0^{p(2-p)}\alpha'(t)f(\alpha(t))k_2(t)\exp\left(p(2-p)\int_0^{\alpha(t)} f(s)ds\right),\ \forall t \in I.$$

The above inequality implies an estimation for $J_1(t)$ as in the following

$$J_1(t) \leqslant u_0 + u_0^{p(2-p)}\int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(s))\exp\left(p(2-p)\int_0^{s} f(\lambda)d\lambda\right)ds,\ \forall t \in I. \tag{2.23}$$

Using (2.23) in (2.15), we get the required inequality in (2.4). This completes the proof. □

REMARK 2.1. If we put $\alpha(t) = t$, then the part **A** in Theorem 2.1 reduces to the Theorem 1.3.

THEOREM 2.2. *Let $u(t)$, $g(t)$, $h(t)$ $\in \mathscr{C}(I,I)$, be nonnegative functions. We suppose that $\varphi_1(t)$, $\varphi_2(t)$, $\alpha(t) \in \mathscr{C}^1(I,I)$ are increasing functions with $\varphi_1'(t) = \varphi_2(t)$, $\alpha(t) \leqslant t$, $\alpha(0) = 0$, $\varphi_i > 0$; $i = 1,2$, for all $t \in I$ and u_0 be a positive constant. If the inequality*

$$\varphi_1(u(t)) \leqslant u_0 + \int_0^t g(s)\varphi_1(u(s))ds + \int_0^{\alpha(t)} h(s)\varphi_2(u(s))ds, \tag{2.24}$$

holds for all $t \in I$. Then

$$u(t) \leqslant \Phi^{-1}\left(\Phi\left(\varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} h(s)ds\right) + \int_0^t g(s)ds,\right), \forall t \leqslant T_1, \tag{2.25}$$

where

$$\Phi(r) = \int_{r_0}^{r} \frac{\varphi_2(s)}{\varphi_1(s)}ds,\quad r > 0, \tag{2.26}$$

where $\Phi^{-1}, \varphi_1^{-1}$ are the inverse functions of Φ, φ_1 respectively, and $T_1 \in I$ is the largest number such that

$$\Phi\left(\varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} h(s)ds\right) + \int_0^t g(s)ds \in Dom(\Phi^{-1}), \tag{2.27}$$

for all $t \in I$ lying in the interval $0 \leqslant t \leqslant T_1$.

Proof. Let $\varphi_1(J_2(t))$ denotes the function on the right-hand side of (2.24), which is a nonnegative and nondecreasing function on I with $J_2(0) = \varphi_1^{-1}(u_0)$. Then (2.24) is equivalent to

$$u(t) \leqslant J_2(t),\quad u(\alpha(t)) \leqslant J_2(\alpha(t)) \leqslant J_2(t),\ \forall t \in I. \tag{2.28}$$

Differentiating $\varphi_1(J_2(t))$, with respect to t, we get

$$\varphi_1'(J_2(t))\frac{dJ_2}{dt} = g(t)\varphi_1(u(t)) + \alpha'(t)h(\alpha(t))\varphi_2(u(\alpha(t))),\ \forall t \in I. \tag{2.29}$$

Using (2.28) and the relation $\varphi_1'(t) = \varphi_2(t)$, from (2.29) we have

$$\frac{dJ_2}{dt} \leqslant g(t)\frac{\varphi_1(J_2(t))}{\varphi_2(J_2(t))} + \alpha'(t)h(\alpha(t)),\ \forall t \in I, \tag{2.30}$$

by taking $t = s$ in the inequality (2.30), and integrating it from 0 to t, we obtain

$$J_2(t) \leqslant \varphi_1^{-1}(u_0) + \int_0^t g(s)\frac{\varphi_1(J_2(s))}{\varphi_2(J_2(s))}ds + \int_0^{\alpha(t)} h(s)ds,\ \forall t \in I, \tag{2.31}$$

from (2.31) we have

$$J_2(t) \leqslant \varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds + \int_0^t g(s)\frac{\varphi_1(J_2(s))}{\varphi_2(J_2(s))}ds, \tag{2.32}$$

for all $t \leqslant T$, where $0 \leqslant T < T_1$ is chosen arbitrarily, T_1 is defined by (2.27). Let $J_3(t)$ denote the function on the right hand side of (2.32), which is a positive and nondecreasing function on I with $J_3(0) = \varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds$ and

$$J_2(t) \leqslant J_3(t),\ \forall t \leqslant T. \tag{2.33}$$

Differentiating $J_3(t)$ with respect to t and using (2.33), we get

$$\frac{dJ_3(t)}{dt} = g(t)\frac{\varphi_1(J_2(t))}{\varphi_2(J_2(t))} \leqslant g(t)\frac{\varphi_1(J_3(t))}{\varphi_2(J_3(t))} \Rightarrow \frac{\varphi_2(J_3(t))dJ_3(t)}{\varphi_1(J_3(t))} \leqslant g(t)dt,\ \forall t < T, \tag{2.34}$$

by the definition of Φ in (2.26), from (2.34) we obtain

$$\begin{aligned}\Phi(J_3(t)) &\leqslant \Phi(J_3(0)) + \int_0^t g(s)ds,\\ &\leqslant \Phi\left(\varphi_1^{-1}(u_0) + \int_0^{\alpha(T)} h(s)ds\right) + \int_0^t g(s)ds,\end{aligned} \tag{2.35}$$

for all $t \leqslant T$, Since $0 < T \leqslant T_1$ is chosen arbitrary,thous if we let $t = T$, from (2.35) we have

$$\Phi(J_3(T)) \leqslant \Phi\left(\varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} h(s)ds\right) + \int_0^t g(s)ds,\ \forall t \leqslant T_1. \tag{2.36}$$

From (2.28), (2.33) and (2.36), we get the required inequality in (2.25). This completes the proof. □

REMARK 2.2. It is interesting to note that in the special case when $\varphi_1(t) = t$, $\varphi_2(t) = 1$, $h(t) = 0$ and $g(t) = b$ (constant), then inequality given in Theorem 2.2 reduces to well known Gronwall inequality (see Theorem 1.1).

REMARK 2.3. It is interesting to note that in the special case when $\varphi_1(t) = t$, $\varphi_2(t) = 1$ and $h(t) = 0$, then inequality given in Theorem 2.2 reduces to well known Gronwall-Bellman inequality (see Theorem 1.2).

THEOREM 2.3. *Let* $u(t),\ f(t),\ h(t) \in \mathscr{C}(I,I)$, *and* $n(t),\ \alpha(t) \in \mathscr{C}^1(I,I)$ *with* $n(t) \geqslant 1,\ \alpha(0) = 0$, *and* $\alpha(t) \leqslant t,\ \forall t \in I$. *If the inequality*

$$u^p(t) \leqslant n(t) + \int_0^t f(s)u^p(s)ds + \int_0^{\alpha(t)} h(s)u^q(s)ds,\ \forall\, t \in I, \tag{2.37}$$

holds, where $p > q \geqslant 0$, *are constants. Then*

$$u(t) \leqslant k_3(t)\exp\left(\frac{1}{p}\int_0^t f(s)ds\right),\ \forall t \in I, \tag{2.38}$$

where

$$k_3(t) = \left[n^{p_1}(0)+p_1\int_0^t \Big(n'(s)+\alpha'(s)h(\alpha(s))\Big)\exp\left(-p_1\int_0^s f(\lambda)d\lambda\right)ds\right]^{\frac{1}{(p-q)}}, \tag{2.39}$$

for all $t \in I$, *and* $p_1 = \frac{(p-q)}{p}$.

Proof. Let $J_4^p(t)$ equal the right hand side in (2.37), we have $J_4(0) = n^{\frac{1}{p}}(0)$ and

$$u(t) \leqslant J_4(t), \quad u(\alpha(t)) \leqslant J_4(\alpha(t)) \leqslant J_4(t),\ \forall t \in I. \tag{2.40}$$

Differentiating $J_4^p(t)$, with respect to t, and using (2.40) gives

$$pJ_4^{(p-1)}(t)J_4'(t) \leqslant n'(t) + f(t)J_4^p(s) + \alpha'(t)h(\alpha(t))J_4^q(s), \tag{2.41}$$

since $J_4(t) > 0$, we get

$$pJ_4^{(p-q-1)}J_4'(t) \leqslant \frac{n'(t)}{J_4^q(t)} + f(t)J_4^{(p-q)}(t) + \alpha'(t)h(\alpha(t)), \tag{2.42}$$

but $n(t) \geqslant 1 \Rightarrow J_4^q(t) \geqslant 1 \Rightarrow \frac{n'(t)}{J_4^q(t)} \leqslant n'(t)$, thus from the above inequality we get

$$pJ_4^{(p-q-1)}J_4'(t) - f(t)J_4^{(p-q)}(t)) \leqslant \Big(n'(t) + \alpha'(t)h(\alpha(t))\Big), \tag{2.43}$$

if we let

$$J_4^{(p-q)}(t) = z_3(t), \forall t \in I, \tag{2.44}$$

then we have $z_3(0) = n^{p_1}(0)$, and $pJ_4^{(p-q-1)}(t)\frac{dJ_4(t)}{dt} = [\frac{1}{p_1}]\frac{dz_3(t)}{dt}$, thus from (2.43) we obtain

$$\frac{dz_3(t)}{dt} - p_1 f(t)z_3(t) \leqslant p_1\Big(n'(t) + \alpha'(t)h(\alpha(t))\Big),\ \forall t \in I. \tag{2.45}$$

The inequality (2.45) implies the estimation for $z_3(t)$, as

$$\begin{aligned} z_3(t) \leqslant & \left[n^{p_1}(0) + p_1 \int_0^t \left(n'(s) + \alpha'(s) h(\alpha(s)) \right) \exp\left(-p_1 \int_0^s f(\lambda) d\lambda \right) ds \right] \\ & \times \exp\left(p_1 \int_0^t f(s) ds \right), \ \forall t \in I. \end{aligned} \tag{2.46}$$

Then from (2.46) in (2.44), we have

$$J_4(t) \leqslant k_3(t) \exp\left(\frac{1}{p} \int_0^t f(s) ds \right), \forall t \in I, \tag{2.47}$$

where $k_3(t)$ as defined in (2.39). Using (2.47) in (2.40), we get the required inequality in (2.38). This completes the proof. □

REMARK 2.4. It is interesting to note that in the special case when $n(t) = u_0$ (positive constant), $h(t) = 0$, $q = 0$ and $p = 1$ then inequality given in Theorem 2.3 reduces to well known Gronwall-Bellman inequality (see Theorem 1.2).

REMARK 2.5. It is interesting to note that in the special case when $n(t) = u_0^2$ (positive constant), $f(t) = 0$, $\alpha(t) = t$, $p = 2$ and $q = 1$ then inequality given in Theorem 2.3 reduces to well known Ou-Iang inequality [16].

REMARK 2.6. If we put $n(t) = u_0$ (positive constant) and $\alpha(t) = t$ then inequality given in Theorem 2.3 reduces to the Abdeldaim and Yakout result in the Theorem 3.1 in [2].

REMARK 2.7. If we put $n(t) = u_0$ (positive constant), $\alpha(t) = t$, $f(t) = 0$ and $q = 1$ then inequality given in Theorem 2.3 reduces to the El-Owaidy, Ragab and Abdeldaim result in the Theorem 1 in [11].

REMARK 2.8. If we replaced the function $n(t)$ by the function $n(t)$ be a positive monotonic nondecreasing continuous function defined in I, $\alpha(t) = t$, and $p = 1$ then inequality given in Theorem 2.3 reduces to the El-Owaidy, Ragab and Abdeldaim result in the Theorem 7 in [11].

REMARK 2.9. If we replaced $n(t)$ by $n^p(t)$ such that $n(t) \in \mathscr{C}(I,I)$ be a monotonic nondecreasing function in the Theorem 2.3 then we get the El-Owaidy et al result in the Inequality (1.2).

THEOREM 2.4. *Let $u(t), g(t), f(t) \in \mathscr{C}(I,I)$, and $n(t), \alpha(t) \in \mathscr{C}^1(I,I)$ with $n(t) \geqslant 1$, $\alpha(0) = 0$, and $\alpha(t) \leqslant t$ on I. If the inequality*

$$u(t) \leqslant n(t) + \int_0^{\alpha(t)} f(s) \left[u(s) + \int_0^s g(\lambda) u(\lambda) d\lambda \right]^p ds, \ \forall t \in I, \tag{2.48}$$

where $p \in [0,1)$. *Then*

$$u(t) \leqslant n(t) + \int_0^{\alpha(t)} k_4(\alpha^{-1}(s)) f(s) \exp\left(p \int_0^s g(\lambda) d\lambda\right) ds, \ \forall t \in I, \tag{2.49}$$

where

$$k_4(t) = \left[n^{(1-p)}(0) + (1-p)\int_0^t [n'(s) + \alpha'(s) f(\alpha(s))] \exp\left(-(1-p)\int_0^s g(\lambda) d\lambda\right) ds\right]^{\frac{p}{(1-p)}}, \tag{2.50}$$

for all $t \in I$.

Proof. Let $J_5(t)$ equal the right hand side of (2.48), we have $J_5(0) = n(0)$ and

$$u(t) \leqslant J_5(t), \quad u(\alpha(t)) \leqslant J_5(\alpha(t)) \leqslant J_5(t), \ \forall t \in I. \tag{2.51}$$

Differentiating $J_5(t)$, with respect to t, and using (2.51) gives

$$J_5'(t) \leqslant n'(t) + \alpha'(t) f(\alpha(t)) Y_4^p(t) \tag{2.52}$$

for all $t \in I$, where $Y_4(t) = J_5(\alpha(t)) + \int_0^{\alpha(t)} g(s) J_5(s) ds$, thus we have $Y_4(0) = n(0)$ and

$$J_5(t) \leqslant Y_4(t), \ \forall t \in I. \tag{2.53}$$

Differentiating $Y_4(t)$, with respect to t, and using (2.52) gives

$$Y_4'(t) \leqslant n'(t) + \alpha'(t) f(\alpha(t)) Y_4^p(t) + \alpha'(t) g(\alpha(t)) Y_4(s), \ \forall t \in I, \tag{2.54}$$

since $Y_4(s) > 0$, then

$$Y^{-p}(t) Y_4'(t) - \alpha'(t) g(\alpha(t)) Y_4^{(1-p)}(s) \leqslant \left(\frac{n'(t)}{Y_4^p(t)} + \alpha'(t) f(\alpha(t))\right), \ \forall t \in I, \tag{2.55}$$

but $n(t) \geqslant 1 \Rightarrow Y_4(t) \geqslant 1 \Rightarrow Y_4^p(t) \geqslant 1 \Rightarrow \frac{n'(t)}{Y_4^p(t)} \leqslant n'(t)$, thus from the above inequality we get

$$Y^{-p}(t) Y_4'(t) - \alpha'(t) g(\alpha(t)) Y_4^{(1-p)}(s) \leqslant n'(t) + \alpha'(t) f(\alpha(t))), \ \forall t \in I, \tag{2.56}$$

if we let

$$Y_4^{(1-p)}(t) = z_4(t), \ \forall t \in I, \tag{2.57}$$

then we have $z_4(0) = Y_4^{(1-p)}(0) = n^{(1-p)}(0)$, and $Y_4^{-p}(t)\frac{dY_4(t)}{dt} = [\frac{1}{(1-p)}]\frac{dz_4(t)}{dt}$, thus from (2.56) we obtain

$$\frac{dz_4(t)}{dt} - (1-p)\alpha'(t) g(\alpha(t)) z_4(t) \leqslant (1-p)(n'(t) + \alpha'(t) f(\alpha(t))), \ \forall t \in I. \tag{2.58}$$

The inequality (2.58) implies the estimation for $z_4(t)$, as

$$z_4(t) \leqslant \left[k_4(t)\exp\left(p\int_0^{\alpha(t)} g(s)ds\right)\right]^{\frac{(1-p)}{p}}, \ \forall t \in I, \tag{2.59}$$

where $k_4(t)$ as defined in (2.50). Then from (2.59) in (2.57), we have

$$Y_4^p(t) \leqslant k_4(t)\exp\left(p\int_0^{\alpha(t)} g(s)ds\right), \ \forall t \in I. \tag{2.60}$$

From (2.60) in (2.52), we obtain

$$J_5'(t) \leqslant n'(t) + \alpha'(t)f(\alpha(t))k_4(t)\exp\left(p\int_0^{\alpha(t)} g(s)ds\right), \ \forall t \in I.$$

The above inequality implies an estimation for $J_5(t)$ as in the following

$$J_5(t) \leqslant n(t) + \int_0^{\alpha(t)} k_4(\alpha^{-1}(s))f(s)\exp\left(p\int_0^s g(\lambda)d\lambda\right)ds, \ \forall t \in I. \tag{2.61}$$

Using (2.61) in (2.51)we get the required inequality in (2.49). This completes the proof. □

REMARK 2.10. If we replaced the function $n(t)$ by the function $n(t)$ be a positive monotonic nondecreasing continuous function then inequality given in the Theorem 2.4 reduces to the Abdeldaim and El-Deeb result in the Inequality (1.3).

REMARK 2.11. If we put $\alpha(t) = t$ and replaced the function $n(t)$ by the function $n(t)$ be a positive monotonic nondecreasing continuous function then inequality given in Theorem 2.4 reduces to the inequality given in the Theorem 1.4.

3. Application

In this section, we present an application for some of our results such as the inequality given in Theorem 2.1, to study the boundedness of the solutions of the following nonlinear retarded differential equation with the initial condition.

$$\begin{cases} \frac{du(t)}{dt} = M(t,u(\alpha(t)),H(t,u(\alpha(t)))), \ \forall t \in I, \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where u_0 is a positive constant, $M \in \mathscr{C}(I^3,\mathbb{R})$, $H \in \mathscr{C}(I\times I,\mathbb{R})$, satisfy the following conditions:

$$|M(t,u,H)| \leqslant f(\alpha(t))\left[|u^{(2-p)}(\alpha(t))| + \int_0^t |K(s,u(\alpha(s)))|ds\right]^p \tag{3.2}$$

$$|K(t,u(\alpha(t)))| \leqslant g(\alpha(t))u^q(\alpha(t)). \tag{3.3}$$

where $f(t)$, $g(t)$, $\alpha(t)$, p and q as defined in Theorem 2.1.

COROLLARY 3.1. *Consider nonlinear system (3.1) and suppose that M,H satisfy the conditions (3.2) and (3.3), then all solutions of Equation (3.1) exist on I and satisfy the following estimation:*

A: *If $0 \leqslant q < 1$, then*

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))} k_5(\alpha^{-1}(s)) \exp\left(p(2-p)\int_0^s \frac{f(\lambda)}{\alpha'(\alpha^{-1}(\lambda))} d\lambda\right) ds, \ \forall t \in I, \tag{3.4}$$

where for all $t \in I$

$$k_5(t) = \left[u_0^{(1-q)(2-p)} + (1-q)\int_0^{\alpha(t)} \frac{g(s)}{\alpha'(\alpha^{-1}(s))} \times \exp\left((1-q)(2-p)\int_0^s \frac{f(\lambda)}{\alpha'(\alpha^{-1}(\lambda))} d\lambda\right) ds\right]^{\left[\frac{p}{(1-q)}\right]}.$$

B: *If $q > 1$ and*

$$\left[1-(q-1)u_0^{(2-p)(q-1)} \int_0^{\alpha(t)} \frac{g(s)}{\alpha'(\alpha^{-1}(s))} \exp\left((2-p)(q-1)\int_0^s \frac{f(\lambda)}{\alpha'(\alpha^{-1}(\lambda))} d\lambda\right) ds\right] > 0,$$

for all $t \in I$ then

$$u(t) \leqslant u_0 + u_0^{p(2-p)} \int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))} k_6(\alpha^{-1}(s)) \times \exp\left(p(2-p)\int_0^s \frac{f(\lambda)}{\alpha'(\alpha^{-1}(\lambda))} d\lambda\right) ds, \forall t \in I, \tag{3.5}$$

where for all $t \in I$

$$k_6(t) = \left[1-(q-1)u_0^{(2-p)(q-1)} \int_0^{\alpha(t)} \frac{g(s)}{\alpha'(\alpha^{-1}(s))} \times \exp\left((2-p)(q-1)\int_0^s \frac{f(\lambda)}{\alpha'(\alpha^{-1}(\lambda))} d\lambda\right) ds\right]^{\left[\frac{-p}{(q-1)}\right]}.$$

Proof. Integrating both sides of the Equation (3.1) from 0 to t, we get

$$u(t) = u_0 + \int_0^t M(s, u(\alpha(s)), H(s, u(\alpha(s))))ds, \ \forall t \in I.$$

from (3.2), (3.3) and the above inequality we obtain

$$|u(t)| \leqslant |u_0| + \int_0^t f(\alpha(s))\left[|u^{(2-p)}(\alpha(s))| + \int_0^s g(\lambda)|u^q(\alpha(\lambda))|d\lambda\right]^p ds,$$

thus

$$|u(t)| \leqslant |u_0| + \int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))}\left[|u^{(2-p)}(s)| + \int_0^s \frac{g(\lambda)|u^q(\lambda)|}{\alpha'(\alpha^{-1}(\lambda))} d\lambda\right]^p ds,$$

holds for all $t \in I$. Applying Theorem 2.1 part **A** and part **B** to the above inequality we obtain the estimations (3.4) and (3.5) respectively. The right-hand side of (3.4) and (3.5) gives us the bound on the solution $u(t)$ of (3.1) in terms of the known functions. Thus, if the right-hand side of (3.4) and (3.5) are bounded, then we assert that the solution of (3.1) is buonded for all $t \in I$. This completes the proof. □

REMARK 3.1. Our results also can be used to prove the global existence, uniqueness, stability, and other properties of the solutions of various nonlinear retarded differential and integral equations. The importance of these inequalities stem from the fact that it is applicable in certain situations in which other available inequalities do not apply directly.

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A. Abdeldaim
Department of Mathematics and Computer Sciense
Faculty of Science, Port Said University
Port Said, Egypt
and
Department of Mathematics
Faculty of Science and Humanities
Shaqra University
Dawadmi, Saudi Arabia
e-mail: ahassen@su.edu.sa