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INTEGRAL INEQUALITIES OF THE HEINZ MEANS AS CONVEX FUNCTIONS

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Abstract. Invoking the Hermite-Hadamard inequalities for convex functions, we present different weighted inequalities of the Heinz means, and any such convex function.

1. Introduction

Convex functions have played a major role in several applications, including optimization, geometry and inequalities. Of the convex functions that have attracted many researchers is the Heinz-means function

$$f(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||, \quad 0 \leqslant \nu \leqslant 1.$$

In this context, $A, B \in \mathbb{M}_n^+$, the class of positive semidefinite $n \times n$ complex matrices, viewed as a subset of $\mathbb{M}_n$ of all complex $n \times n$ matrices, $X \in \mathbb{M}_n$ and $|||\ |||$ is any unitarily invariant norm. This function is convex on $(0,1)$, symmetric about $\nu = \frac{1}{2}$, attains its minimum at $\nu = \frac{1}{2}$, and attains its maximum at $\nu = 0$ and $\nu = 1$, see [2] page 265. That is

$$f\left(\frac{1}{2}\right) \leqslant f(\nu) \leqslant f(1) = f(0), \quad 0 \leqslant \nu \leqslant 1.$$

Inequalities of $f = f(\nu)$ have been investigated thoroughly in the literature, and the reader is encouraged to see [3] and [4] and their references for a comprehensive study of these and related inequalities.

This article is motivated mainly by the work in [4] and [3], where the function $f = f(\nu)$ was studied merely as a convex function and hence, properties of convex functions were the key behind most of these studies.

Our main goal in this article is to utilize more properties of convex functions to obtain generalizations of some results obtained in [3] and [4]. However, we shall discuss most of our results in a general setting, where our treatment of the above function will be based mainly on the fact that it is a convex function, without referring to the function itself.

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The first part of our work will treat the Heinz means themselves, where we prove the monotonicity of this function by a simple idea that allows us to interpolate these means. In particular, we prove that

$$|||A^pXB^q+A^qXB^p|||\leqslant|||A^{p+r}XB^{q-r}+A^{q-r}XB^{p+r}|||$$

for all $p\geqslant q\geqslant r\geqslant 0$. This inequality will be used then to prove that the Heinz means function $f=f(\nu)$ is decreasing on $(0,\frac{1}{2})$ and increasing on $(\frac{1}{2},1)$.

The second part of the paper, which will be the main constitution, will treat the function $f=f(\nu)$ as a convex function, where we present the general form governing the inequalities of the weighted integral

$$\int f(\nu)g(\nu)d\nu.$$

In particular, we prove that

$$\int_a^b f(x)g(x)dx\leqslant\frac{f(a)+f(b)}{2}\int_a^b g(x)dx$$

for the increasing convex function f, when g is continuous, positive and decreasing on $[a,b]$, see Theorem 2.18. This result is of special interest as it serves as a counterpart of the well known Fejer inequality that states the same inequality when f is convex and g is symmetric about $\frac{a+b}{2}$.

Moreover, we present the general refinement governing the refinement

$$f(\nu)\leqslant 2r_0f\left(\frac{1}{2}\right)+(1-2r_0)f(0),\quad r_0=\min\{\nu,1-\nu\},$$

of [4] and its refinement in [3].

2. Main results

2.1. Monotonicity and convexity of the Heinz means

The following inequality interpolates the well known Heinz inequality. The inequality has been recently proved in [5], but we present it here for completeness.

THEOREM 2.1. *Let $A,B\in\mathbb{M}_n^+$ and $X\in\mathbb{M}_n$. Then, for $0\leqslant r\leqslant q\leqslant p$ and any unitarily invariant norm $|||\ |||$, we have*

$$|||A^pXB^q+A^qXB^p|||\leqslant|||A^{p+r}XB^{q-r}+A^{q-r}XB^{p+r}|||. \tag{2.1}$$

Proof. Observe that

$$\begin{aligned}|||A^pXB^q+A^qXB^p||| &= |||A^{p-q+r}(A^{q-r}XB^{q-r})B^r+A^r(A^{q-r}XB^{q-r})B^{p-q+r}|||\\ &\leqslant |||A^{p-q+2r}(A^{q-r}XB^{q-r})+(A^{q-r}XB^{q-r})B^{p-q+2r}|||\\ &\leqslant |||A^{p+r}XB^{q-r}+A^{q-r}XB^{p+r}|||,\end{aligned}$$

where we have used the Heinz inequality $|||A^pXB^q + A^qXB^p||| \leqslant |||A^{p+q}X + XB^{p+q}|||$, with p, X, q replaced by $p-q+r, A^{q-r}XB^{q-r}, r$ respectively. $\square$

In fact these interpolations interpolate increasingly from $f(0)$ to $f(r)$.

THEOREM 2.2. *Let $X \in \mathbb{M}_n, A, B \in \mathbb{M}_n^+, p \geqslant q > 0$. Then, the function*

$$f(r) = |||A^{p+r}XB^{q-r} + A^{q-r}XB^{p+r}|||$$

is increasing on $[0, q]$.

Proof.
Let $0 \leqslant r_1 < r_2 \leqslant q$, then

$$\begin{aligned} f(r_1) &= |||A^{p+r_1}XB^{q-r_1} + A^{q-r_1}XB^{p+r_1}||| \\ &\leqslant |||A^{p+r_1+(r_2-r_1)}XB^{q-r_1-(r_2-r_1)} + A^{q-r_1-(r_2-r_1)}XB^{p+r_1+(r_2-r_1)}||| \\ &= f(r_2), \end{aligned}$$

where we have used Theorem 2.1 replacing p by $p+r_1$, q by $q-r_1$ and r by $r_2 - r_1$. $\square$

Observe that $f(0) = |||A^pXB^q + A^qXB^p|||$ and $f(q) = |||A^{p+q}X + XB^{p+q}|||$. Since f is increasing, $f(0) \leqslant f(q)$, which is the well known Heins inequality. However, for $0 < r < q$ we have $f(0) \leqslant f(r) \leqslant f(q)$ giving intermediate inequalities that interpolate the Heinz inequality increasingly.

COROLLARY 2.3. *Let $A, B \in \mathbb{M}_n^+, X \in \mathbb{M}_n$. Then the function*

$$f(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$$

is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.

Proof. We treat both cases:
Case I. If $0 \leqslant \nu \leqslant \frac{1}{2}$, then

$$f(\nu) = |||A^{\frac{1}{2}+(\frac{1}{2}-\nu)}XB^{\frac{1}{2}-(\frac{1}{2}-\nu)} + A^{\frac{1}{2}-(\frac{1}{2}-\nu)}XB^{\frac{1}{2}+(\frac{1}{2}-\nu)}|||,$$

which can be viewed as $|||A^{p+r}XB^{q-r} + A^{q-r}XB^{p+r}|||$ with $p = q = \frac{1}{2}$ and $r = \frac{1}{2} - \nu$. But this function is increasing as r increases from 0 to $\frac{1}{2}$, which implies that f decreases as ν goes from 0 to $\frac{1}{2}$.

Case II. If $\frac{1}{2} \leqslant \nu \leqslant 1$, then

$$f(\nu) = |||A^{\frac{1}{2}+(\nu-\frac{1}{2})}XB^{\frac{1}{2}-(\nu-\frac{1}{2})} + A^{\frac{1}{2}-(\nu-\frac{1}{2})}XB^{\frac{1}{2}+(\nu-\frac{1}{2})}|||,$$

which can be viewed as $|||A^{p+r}XB^{q-r} + A^{q-r}XB^{p+r}|||$ with $p = q = \frac{1}{2}$ and $r = \nu - \frac{1}{2}$. But this function is increasing as r increases from 0 to $\frac{1}{2}$, which implies that f decreases as ν goes from $\frac{1}{2}$ to 1. $\square$

Consequently, f attains its minimum at $\nu=\frac{1}{2}$ and its maximum at $\nu=1$, which means

COROLLARY 2.4. *Let* $A,B\in\mathbb{M}_n^+, X\in\mathbb{M}_n$. *Then*

$$|||\sqrt{A}X\sqrt{B}|||\leqslant\frac{1}{2}|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||\leqslant|||AX+XB|||,$$

for all $\nu\in[0,1]$.

This is a well known result of the Heinz means; see [1].

PROPOSITION 2.5. *Let* $X\in\mathbb{M}_n$, $A,B\in\mathbb{M}_n^+$, $p\geqslant q>0$ *and* $0\leqslant r\leqslant q$. *Then, the function*

$$f(r)=|||A^{p+r}XB^{q-r}+A^{q-r}XB^{p+r}|||$$

is convex on $(0,q)$.

Proof. Let $C=A^{p+q}$, $D=B^{p+q}$ and $\nu=\frac{p+r}{p+q}$. Then, $f(r)=g(\nu)$, where

$$g(\nu)=|||C^{\nu}XD^{1-\nu}+C^{1-\nu}XD^{\nu}|||.$$

Since g is convex on $(0,1)$, f is convex on $(0,q)$. □

Now we utilize the convexity and monotonicity of the function $f(\nu)=|||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||$ to obtain new inequalities that refines and generalize related inequalities in [3] and [4]. However, results will be stated for general convex functions.

We shall state our results for general convex functions that satisfy similar properties of the Heinz-means function.

DEFINITION 2.6. A function f defined on $[0,1]$ will be called a Heinz function if f is convex on $[0,1]$, continuous on $[0,1]$, decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$, and symmetric about $\frac{1}{2}$.

2.2. Inequalities of the function $f=f(\nu)$

The following result has been proved in [4] using the Hermite-Hadamard inequalities

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2} \tag{2.2}$$

when f is convex on $[a,b]$.

THEOREM 2.7. *Let* $0<\mu\leqslant\frac{1}{2}$, *then for any Heinz function* f, *we have*

$$f(\mu)\leqslant f(\mu/2)\leqslant\frac{1}{\mu}\int_0^{\mu}f(\nu)d\nu\leqslant\frac{f(0)+f(\mu)}{2}\leqslant f(0), \tag{2.3}$$

and for $\frac{1}{2} \leqslant \mu \leqslant 1$, *we have*

$$f(\mu) \leqslant f\left(\frac{1+\mu}{2}\right) \leqslant \frac{1}{1-\mu}\int_{\mu}^{1} f(v)dv \leqslant \frac{f(1)+f(\mu)}{2} \leqslant f(1). \tag{2.4}$$

Our goal now is to present some inequalities of Heinz functions similar to those in Theorem 2.7, but in a more general form.

THEOREM 2.8. *Let* f *be a Heinz function and let* $\varphi : [a,b] \to [\frac{1}{2},1]$ *be strictly monotone, differentiable and convex. If* $\varphi'(t) \neq 0$ *for all* $t \in (a,b)$ *then*

$$f\left(\varphi\left(\frac{\varphi^{-1}(\mu)+b}{2}\right)\right) \leqslant \frac{1}{b-\varphi^{-1}(\mu)}\int_{\mu}^{\varphi(b)} \frac{f(v)}{\varphi'(\varphi^{-1}(v))}dv \leqslant \frac{f(\mu)+f(\varphi(b))}{2}, \tag{2.5}$$

and

$$f\left(\varphi\left(\frac{\varphi^{-1}(\mu)+a}{2}\right)\right) \leqslant \frac{1}{a-\varphi^{-1}(\mu)}\int_{\mu}^{\varphi(a)} \frac{f(v)}{\varphi'(\varphi^{-1}(v))}dv \leqslant \frac{f(\mu)+f(\varphi(a))}{2}, \tag{2.6}$$

for all $\mu \in [\frac{1}{2},1]$.

Proof. Let f and φ as stated. Without loss of generality, assume that φ is increasing, and let $\mu \in [\frac{1}{2},1)$. Since f is increasing on $[\frac{1}{2},1]$ and φ is convex, we infer that $g = f \circ \varphi$ is convex on $[a,b]$. Consequently, the Hermite-Hadamard inequalities (2.2) applied on the interval $[\varphi^{-1}(\mu),b]$ imply

$$g\left(\frac{\varphi^{-1}(\mu)+b}{2}\right) \leqslant \frac{1}{b-\varphi^{-1}(\mu)}\int_{\varphi^{-1}(\mu)}^{b} f(\varphi(t))dt \leqslant \frac{g(\varphi^{-1}(\mu))+g(b)}{2}.$$

By making the substitution $v = \varphi(t)$ in the integral, we get (2.5). A similar argument works if φ is decreasing except that $\mu \in (\frac{1}{2},1]$ instead of $\mu \in [\frac{1}{2},1)$. Then similar arguments yield (2.6). $\square$

By selecting different functions φ we obtain some interesting inequalities. For example, by letting $\varphi(t) = t$, we get (2.4).

COROLLARY 2.9. *Let* f *be a Heinz function. Then, for* $\frac{1}{2} \leqslant \mu \leqslant 1$ *we have*

$$f\left(\frac{2\mu}{2\mu+1}\right) \leqslant \frac{\mu}{2\mu-1}\int_{1/2}^{\mu} \frac{f(v)}{v^2}dv \leqslant \frac{f(\mu)+f(1/2)}{2},$$

and

$$f\left(\frac{2\mu}{\mu+1}\right) \leqslant \frac{\mu}{1-\mu}\int_{\mu}^{1} \frac{f(v)}{v^2}dv \leqslant \frac{f(\mu)+f(1)}{2}.$$

Proof. Let $\varphi:[1,2]\to[\frac{1}{2},1]$ be defined by $\varphi(t)=\frac{1}{t}$. Then the result follows by direct application of Theorem 2.8. □

Observe that when $\mu\in[0,\frac{1}{2}]$, then $1-\mu\in[\frac{1}{2},1]$. Hence, by applying the above theorem, with μ replaced by $1-\mu$, and be recalling the symmetry of f, we get the following inequality.

COROLLARY 2.10. *Let f be a Heinz function. Then, for $0\leqslant\mu\leqslant\frac{1}{2}$ we have*

$$f\left(\frac{2-2\mu}{3-2\mu}\right)\leqslant\frac{1-\mu}{1-2\mu}\int_{1/2}^{1-\mu}\frac{f(v)}{v^2}dv\leqslant\frac{f(\mu)+f(1/2)}{2},$$

and

$$f\left(\frac{2-2\mu}{2-\mu}\right)\leqslant\frac{1-\mu}{\mu}\int_{1-\mu}^{1}\frac{f(v)}{v^2}dv\leqslant\frac{f(\mu)+f(1)}{2}.$$

Then, integrating these inequalities again yields another type of integral inequalities.

COROLLARY 2.11. *Let f be a Heinz function. Then*

$$\int_0^\beta f(v)dv\leqslant\int_0^\beta vf(v)dv+(1-\beta)^2\int_{1-\beta}^1\frac{f(v)}{v^2}dv+\frac{f(1)}{2}\beta^2,\ 0\leqslant\beta\leqslant\frac{1}{2},$$

and

$$\int_\beta^1 f(v)dv\leqslant\int_0^{1-\beta} vf(v)dv+\beta^2\int_\beta^1\frac{f(v)}{v^2}dv+\frac{f(1)}{2}(1-\beta)^2,\ \frac{1}{2}\leqslant\beta\leqslant 1.$$

Proof. By Corollary 2.9, we have

$$(1-\mu)\int_{1-\mu}^1\frac{f(v)}{v^2}dv\leqslant\frac{1}{2}\mu f(\mu)+\frac{f(1)}{2}\mu,\quad 0\leqslant\mu\leqslant\frac{1}{2}.$$

By integrating both sides on the interval $\mu\in[0,\beta]$ we get

$$\int_0^\beta\left\{(1-\mu)\int_{1-\mu}^1\frac{f(v)}{v^2}dv\right\}d\mu\leqslant\int_0^\beta\left\{\frac{1}{2}\mu f(\mu)+\frac{f(1)}{2}\mu\right\}d\mu,\quad 0\leqslant\beta\leqslant\frac{1}{2}.$$

For the integral on the left, change the order of integration to get

$$\int_{1-\beta}^1\left\{\frac{f(v)}{v^2}\int_{1-v}^\beta(1-\mu)d\mu\right\}dv\leqslant\int_0^\beta\left\{\frac{1}{2}\mu f(\mu)+\frac{f(1)}{2}\mu\right\}d\mu,\quad 0\leqslant\beta\leqslant\frac{1}{2}.$$

Then direct computations give the first inequality. The second inequality is obtained from the first by replacing β with $1-\beta$. □

It should be noted that the right side of these inequalities is not so big. In fact,

$$\begin{aligned}\int_0^\beta \nu f(\nu)d\nu + (1-\beta)^2\int_{1-\beta}^1 \frac{f(\nu)}{\nu^2}d\nu + \frac{f(1)}{2}\beta^2 \\ &\leqslant f(1)\int_0^\beta \nu d\nu + (1-\beta)^2 f(1)\int_{1-\beta}^1 \frac{1}{\nu^2}d\nu + \frac{f(1)}{2}\beta^2 \\ &= \beta f(1).\end{aligned}$$

That is, when $0 \leqslant \beta \leqslant \frac{1}{2}$ we have

$$\frac{1}{\beta}\int_0^\beta f(\nu)d\nu \leqslant f(1),$$

and this is one implication of the inequalities of Theorem 2.7. However, neither the inequalities of Theorem 2.7 nor those of Corollary 2.11 are uniformly better than each other.

On the other hand, by letting $\varphi : [-\ln 2, 0] \to [\frac{1}{2}, 1]$ be the function $\varphi(t) = e^t$, we obtain the following inequalities.

COROLLARY 2.12. *Let f be a Heinz function. Then, for $\frac{1}{2} \leqslant \mu \leqslant 1$ we have*

$$f(\sqrt{\mu}) \leqslant \frac{-1}{\ln\mu}\int_\mu^1 \frac{f(\nu)}{\nu}d\nu \leqslant \frac{f(\mu)+f(1)}{2},$$

and

$$f\left(\sqrt{\mu/2}\right) \leqslant \frac{1}{\ln 2 + \ln\mu}\int_{1/2}^\mu \frac{f(\nu)}{\nu}d\nu \leqslant \frac{f(\mu)+f(1/2)}{2}.$$

As a consequence, we get the following estimate.

COROLLARY 2.13. *Let f be a Heinz function. Then for each $\frac{1}{2} \leqslant \mu \leqslant 1$, we have*

$$\begin{aligned}(\ln 2) f\left(\sqrt{\mu/2}\right) &\leqslant \int_{1/2}^1 \frac{f(\nu)}{\nu}d\nu \\ &\leqslant \frac{f(1)-f(1/2)}{2}\ln\frac{1}{\mu} + \frac{f(\mu)+f(1/2)}{2}\ln 2.\end{aligned}$$

Proof. Observe that the inequalities of Corollary 2.12 can be written as

$$\left(\ln\frac{1}{\mu}\right) f(\sqrt{\mu}) \leqslant \int_\mu^1 \frac{f(\nu)}{\nu}d\nu \leqslant \frac{f(\mu)+f(1)}{2}\ln\frac{1}{\mu}$$

$$\left(\ln 2 - \ln\frac{1}{\mu}\right) f(\sqrt{\mu/2}) \leqslant \int_{1/2}^\mu \frac{f(\nu)}{\nu}d\nu \leqslant \frac{f(\mu)+f(1/2)}{2}\left(\ln 2 - \ln\frac{1}{\mu}\right).$$

By adding these inequalities, and recalling that $f(\sqrt{\mu/2}) \leqslant f(\sqrt{\mu})$, f being increasing, we get the result. $\square$

Note that $f(\sqrt{\mu/2})$ increases with μ, hence we infer that

$$f(1/\sqrt{2})\ln 2 \leqslant \int_{1/2}^{1} \frac{f(v)}{v}dv.$$

Moreover, an easy estimate would be obtained for the integral in the following way

$$\int_{1/2}^{1} \frac{f(v)}{v}dv \geqslant f(1/2)\int_{1/2}^{1}\frac{1}{v}dv = f(1/2)\ln 2.$$

However, the estimate we obtain from Corollary 2.13 is better because $f(1/\sqrt{2})\ln 2 \geqslant f(1/2)\ln 2$.

COROLLARY 2.14. *Let f be a Heinz function. Then, for each $\frac{1}{2} \leqslant \mu \leqslant 1$, we have*

$$\int_0^1 f(v)dv \leqslant (f(1)-f(1/2))\ln\frac{1}{\mu} + (f(\mu)+f(1/2))\ln 2.$$

Proof. Since f is symmetric about $\frac{1}{2}$, we have

$$\begin{aligned}\int_0^1 f(v)dv &= 2\int_{1/2}^{1} f(v)dv\\ &\leqslant 2\int_{1/2}^{1}\frac{f(v)}{v}dv\\ &\leqslant 2\left\{\frac{f(1)-f(1/2)}{2}\ln\frac{1}{\mu} + \frac{f(\mu)+f(1/2)}{2}\ln 2\right\},\end{aligned}$$

which completes the proof. □

These ideas enable us to present the following Hermite-Hadamard type inequality.

THEOREM 2.15. *Let f be continuous, convex and increasing on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leqslant (f(b)-f(a))\ln\frac{1}{\mu} + (f(2(b-a)\mu+2a-b)+f(a))\ln 2,$$

for each $\mu \in [\frac{1}{2},1]$.

Proof. Let f be as stated, and define

$$g(x) = f(2(b-a)x+2a-b), \quad \frac{1}{2} \leqslant x \leqslant 1.$$

Then, g is convex and increasing on $[\frac{1}{2},1]$. Hence, using Corollary 2.14 we have

$$\int_{1/2}^{1} f(2(b-a)x+2a-b)dx \leqslant \frac{1}{2}\left\{(f(1)-f(1/2))\ln\frac{1}{\mu} + (f(\mu)+f(1/2))\ln 2\right\}.$$

The change of variable $v = 2(b-a)x+2a-b$ implies the result. □

The function $\varphi(t) = \tan t$, $\tan^{-1}\frac{1}{2} \leqslant t \leqslant \frac{\pi}{4}$ gives the inequality

COROLLARY 2.16. *Let f be a Heinz function. Then for each $\frac{1}{2} \leqslant \mu \leqslant 1$, we have*

$$\frac{1}{\frac{\pi}{4} - \tan^{-1}\mu} \int_{\mu}^{1} \frac{f(\nu)}{\nu^2+1} d\nu \leqslant \frac{f(\mu)+f(1)}{2}.$$

We conclude this idea by presenting the general weighted form of these inequalities.

THEOREM 2.17. *Let f be a Heinz function and let g be continuous, decreasing and positive on $[\frac{1}{2},1]$. Then for each $\frac{1}{2} \leqslant \mu \leqslant 1$, we have*

$$\int_{\mu}^{1} f(\nu)g(\nu)d\nu \leqslant \frac{f(\mu)+f(1)}{2} \int_{\mu}^{1} g(\nu)d\nu.$$

Proof. Let f and g be as stated. Define the functions

$$h(\nu) = \int_{1/2}^{\nu} g(\tau)d\tau, \quad \frac{1}{2} \leqslant \nu \leqslant 1,$$

and $\varphi(t) = h^{-1}(t)$, $0 \leqslant t \leqslant h(1) := b$. Since $g > 0$, h is increasing, hence so is φ. Moreover, since g is decreasing, h^{-1} is increasing and

$$\varphi'(t) = \frac{1}{g(h^{-1}(t))},$$

it follows that φ' is increasing, hence φ is convex. Observe also that $g(\nu) = \frac{1}{\varphi'(\varphi^{-1}(\nu))}$, consequently, by virtue of Theorem 2.8,

$$\begin{aligned}\int_{\mu}^{1} f(\nu)g(\nu)d\nu &= \int_{\mu}^{1} \frac{f(\nu)}{\varphi'(\varphi^{-1}(\nu))} d\nu \\ &\leqslant (b - \varphi^{-1}(\mu)) \frac{f(\mu)+f(\varphi(b))}{2} \\ &= (h(1)-h(\mu)) \frac{f(\mu)+f(1)}{2}.\end{aligned}$$

This completes the proof. □

Now we present a counterpart of the Fejer inequality, where the symmetry condition is released, provided some monotonicity is available.

THEOREM 2.18. *Let f be convex and increasing on $[a,b]$ and let g be continuous, positive and decreasing on $[a,b]$. Then*

$$\int_{a}^{b} f(x)g(x)dx \leqslant \frac{f(a)+f(b)}{2} \int_{a}^{b} g(x)dx.$$

Proof. Let f and g be as stated, and define

$$f_1(t) = f(2(b-a)(t-1)+b) \text{ and } g_1(t) = g(2(b-a)(t-1)+b).$$

Then, using the substitution $t = \frac{x-b}{2(b-a)} + 1$, we have

$$\begin{aligned}\int_a^b f(x)g(x)dx &= 2(b-a)\int_{1/2}^1 f_1(t)g_1(t)dt \\ &\leqslant 2(b-a)\frac{f_1(1/2)+f_1(1)}{2}\int_{1/2}^1 g_1(t)dt \text{ (using Theorem 2.17)} \\ &= 2(b-a)\frac{f(a)+f(b)}{2}\int_a^b g(x)\frac{dx}{2(b-a)} \\ &= \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.\end{aligned}$$

This completes the proof. □

REMARK.

1. It should be noted that the above results can be stated for $0 \leqslant \mu \leqslant 1$ using the symmetry of f.
2. By an appropriate transformation, the above results can be generalized to any interval $[a,b]$. However, we stick to the interval $[0,1]$ being the Heinz-means interval.

We conclude this article by shedding some light on recent refinements of the Heinz inequality. In particular, utilizing the convexity of the function $f(\nu) = |||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|||$, it was proved that [4]

$$f(\nu) \leqslant 2r_0 f\left(\frac{1}{2}\right) + (1-2r_0)f(0), \quad \nu \in (0,1), \quad r_0 = \min\{\nu, 1-\nu\}, \tag{2.7}$$

as a refinement of the Heinz inequality. In the following result, we present the general refinement that governs these refinements. It should be noted that our choice of the dyadic partition is just for convenience, but any partition will follow a similar behavior.

THEOREM 2.19. *Let $n \in \mathbb{N}$, let $\{0, \frac{1}{2^n}, \cdots, \frac{2^{n-1}}{2^n}\}$ be a partition of $[0, \frac{1}{2}]$ and let $\nu \in [0,1]$. Then, if $\nu \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ or $1-\nu \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, we have*

$$f(\nu) \leqslant (k - 2^n\nu)f\left(\frac{k-1}{2^n}\right) + (2^n\nu - k + 1)f\left(\frac{k}{2^n}\right), \tag{2.8}$$

for any convex function f on $(0,1)$, that is symmetric about $\frac{1}{2}$.

Proof. For each $k = 1,\cdots,2^{n-1}$, let

$$\varphi_k(x) = \frac{f(x) - f\left(\frac{k-1}{2^n}\right)}{x - \frac{k-1}{2^n}}.$$

Since f is convex, it follows that φ_k is increasing on $\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$. Consequently, if $v \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, we get $\varphi_k(\nu) \leqslant \varphi_k\left(\frac{k}{2^n}\right)$, which implies

$$\frac{f(\nu) - f\left(\frac{k-1}{2^n}\right)}{\nu - \frac{k-1}{2^n}} \leqslant \frac{f\left(\frac{k}{2^n}\right) - f\left(\frac{k-1}{2^n}\right)}{\frac{k}{2^n} - \frac{k-1}{2^n}}.$$

The inequality of the theorem follows by simplifying this inequality. This completes the proof for $v \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$. Now, if $\nu \in [\frac{1}{2}, 1]$, then $1 - v \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ for some k, and since f is symmetric about $\frac{1}{2}$, the result follows. □

It can be seen now that (2.7) follows from (2.8) when $n = 1$.

It should be remarked that the above proof is inspired from [3], where the result was proved for $n = 2$, as a refinement of (2.7).

In the following result we prove how these refinements become better when n gets bigger.

THEOREM 2.20. *Let f be convex on $(0,1)$ and symmetric about $\frac{1}{2}$. For $n \in \mathbb{N}$ let*

$$I_{k,n} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right], \quad k = 1,\cdots,2^{n-1}.$$

For each such n,k define

$$g_n(\nu) = (k - 2^n\nu) f\left(\frac{k-1}{2^n}\right) + (2^n\nu - k + 1) f\left(\frac{k}{2^n}\right), \quad \nu \in I_{n,k},$$

and

$$g(\nu) = g(1-\nu) \text{ for } \nu \in [\frac{1}{2}, 1].$$

Then $g_{n+1}(\nu) \leqslant g_n(\nu)$ for all $\nu \in [0,1]$.

Proof. We prove this for $\nu \in [0, \frac{1}{2}]$, then by symmetry the result follows. So, let $\nu \in [0, \frac{1}{2}]$ and let $n \in \mathbb{N}$. Then, for some k, we have $\nu \in I_{k,n}$. Since $I_{k,n} = I_{2k-1,n+1} \cup I_{2k,n+1}$, we have $\nu \in I_{2k-1,n+1}$ or $\nu \in I_{2k,n+1}$.

If $\nu \in I_{2k-1,n+1}$, then

$$\begin{aligned} g_{n+1}(\nu) - g_n(\nu) &= (2k - 1 - 2^{n+1}\nu) f\left(\frac{2k-2}{2^{n+1}}\right) + (2^{n+1}\nu - 2k + 2) f\left(\frac{2k-1}{2^{n+1}}\right) \\ &\quad - (k - 2^n\nu) f\left(\frac{k-1}{2^n}\right) - (2^n\nu - k + 1) f\left(\frac{k}{2^n}\right) \\ &= (k - 1 - 2^n\nu)\left\{ f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right) - 2f\left(\frac{2k-1}{2^{n+1}}\right) \right\}. \end{aligned}$$

Now using the increasing function φ_k, of Theorem 2.19, we infer that $\varphi_k\left(\frac{2k-1}{2^{n+1}}\right) \leqslant \varphi_k\left(\frac{k}{2^n}\right)$. Upon simplification we get

$$f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right) - 2f\left(\frac{2k-1}{2^{n+1}}\right) \geqslant 0.$$

Moreover, since $\nu \in I_{k,n}$, it follows that $k-1-2^n\nu \leqslant 0$. Consequently, $g_{n+1}(\nu) - g_n(\nu) \leqslant 0$. This completes the proof when $\nu \in I_{2k-1,n+1}$. Now, if $\nu \in I_{2k,n+1}$, similar computations lead to the same conclusion. This completes the proof. □

The following is a matrix version of these convex inequalities.

COROLLARY 2.21. *Let $A,B \in \mathbb{M}_n^+$, $X \in \mathbb{M}_n$, and let $I_{k,m}$ be the above partition for some $m \in \mathbb{N}$. If $\nu \in I_{k,m}$ or $1-\nu \in I_{k,m}$, let $\alpha = \frac{k-1}{2^m}$. Then*

$$\begin{aligned}\left|\left|\left|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\right|\right|\right|^r \leqslant (k-2^m\nu)\left|\left|\left|A^\alpha XB^{1-\alpha} + A^{1-\alpha}XB^\alpha\right|\right|\right|^r \\ +(2^m\nu - k+1)\left|\left|\left|A^{k/2^m}XB^{1-k/2^m} + A^{1-k/2^m}XB^{k/2^m}\right|\right|\right|^r,\end{aligned}$$

for all $r \geqslant 1$, and any unitarily invariant norm $|||\ |||$.

Proof. Let $f(\nu) = |||A^\nu XB^{1-\nu} + A^\nu XB^{1-\nu}|||$, and $\varphi(t) = t^r$, $r \geqslant 1$. Then, f is convex on $(0,1)$ and φ is increasing and convex on $(0,\infty)$. Consequently, $\varphi \circ f$ is convex on $(0,1)$. By applying Theorem 2.19 on the function $\varphi \circ f$, we get the result. □

In particular, if $m = 1$ we get

$$\left|\left|\left|A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu\right|\right|\right|^r \leqslant (1-2r_0)|||AX+XB|||^r + 2r_0|||2A^{1/2}XB^{1/2}|||^r,$$

or equivalently,

$$\left|\left|\left|\frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2}\right|\right|\right|^r \leqslant (1-2r_0)\left|\left|\left|\frac{AX+XB}{2}\right|\right|\right|^r + 2r_0|||A^{1/2}XB^{1/2}|||^r, \quad (2.9)$$

for $0 \leqslant \nu \leqslant 1$.

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