

NEW WEIGHTED ČEBYŠEV–OSTROWSKI TYPE INTEGRAL INEQUALITIES ON TIME SCALES

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(Communicated by A. Peterson)

Abstract. In this paper we obtain some weighted Čebyšev–Ostrowski type integral inequalities on time scales involving functions whose first derivatives belong to $L_p(a, b)$ ($1 \leq p \leq \infty$). We also give some other interesting inequalities as special cases.

1. Introduction

In [21], Rafiq et al. obtained some weighted Čebyšev–Ostrowski type integral inequalities, involving functions whose first derivatives belong to $L_\infty(a, b)$, as follows.

THEOREM 1.1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose first derivatives f', g' belong to $L_\infty(a, b)$. Then for all $x \in [a, b]$, we have*

$$\left| \frac{1}{m(a, b)} \int_a^b w(t) f(t) g(t) dt - \left(\frac{1}{m(a, b)} \int_a^b w(t) f(t) dt \right) \left(\frac{1}{m(a, b)} \int_a^b w(t) g(t) dt \right) \right| \leq \frac{\|f'\|_\infty \|g'\|_\infty}{m^3(a, b)} \int_a^b w(x) \left(\int_a^b \operatorname{sgn}(t-x)(t-x) w(t) dt \right)^2 dx,$$

and

$$\left| \frac{1}{m(a, b)} \int_a^b w(t) f(t) g(t) dt - \left(\frac{1}{m(a, b)} \int_a^b w(t) f(t) dt \right) \left(\frac{1}{m(a, b)} \int_a^b w(t) g(t) dt \right) \right| \leq \frac{1}{2m^2(a, b)} \int_a^b w(x) [|g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty] \left(\int_a^b \operatorname{sgn}(t-x)(t-x) w(t) dt \right) dx.$$

Mathematics subject classification (2010): 26D15, 26E70.

Keywords and phrases: Weighted Čebyšev–Ostrowski type inequality, weighted Čebyšev type inequality, time scales.

THEOREM 1.2. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose first derivatives f', g' belong to $L_\infty(a, b)$. Then for all $x \in [a, b]$, we have the inequalities*

$$\begin{aligned} & \left| FG - \frac{G}{m(a, b)} \int_a^b w(t) f(t) dt - \frac{F}{m(a, b)} \int_a^b w(t) g(t) dt \right. \\ & \quad \left. + \left(\frac{1}{m(a, b)} \int_a^b w(t) f(t) dt \right) \left(\frac{1}{m(a, b)} \int_a^b w(t) g(t) dt \right) \right| \\ & \leq \frac{\|f'\|_\infty \|g'\|_\infty}{m^2(a, b)} \left(\int_a^{\frac{a+b}{2}} \operatorname{sgn} \left(x - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, x \right) dx \right. \\ & \quad \left. + \int_{\frac{a+b}{2}}^b \operatorname{sgn} \left(x - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, x \right) dx \right)^2, \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a, b)} \int_a^b w(t) [Fg(t) + Gf(t)] dt - 2 \left(\frac{1}{m(a, b)} \int_a^b w(t) f(t) dt \right) \right. \\ & \quad \left. \times \left(\frac{1}{m(a, b)} \int_a^b w(t) g(t) dt \right) \right| \\ & \leq \frac{1}{m^2(a, b)} \int_a^b w(x) (|g(x)| \|f'\|_\infty + |f(x)| \|g'\|_\infty) \\ & \quad \times \left(\int_a^{\frac{a+b}{2}} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) dt + \int_{\frac{a+b}{2}}^b \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) dt \right) dx \end{aligned}$$

where

$$F = m \left(a, \frac{5a+b}{6} \right) f(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) f \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) f(b),$$

$$G = m \left(a, \frac{5a+b}{6} \right) g(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) g \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) g(b),$$

and

$$m(x) = \begin{cases} \int_a^x w(\tau) d\tau, & a \leq x < \frac{a+b}{2}, \\ \frac{5a+b}{6} \\ \int_{\frac{a+5b}{6}}^x w(\tau) d\tau, & \frac{a+b}{2} \leq x \leq b. \end{cases}$$

Ahmad, Mir and Rafiq [2] established weighted Čebyšev-Ostrowski type integral inequalities involving functions whose first derivatives belong to $L_p(a, b)$, ($1 \leq p < \infty$) as follows.

THEOREM 1.3. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose first derivatives f', g' belong to $L_p(a, b)$ space where $1 \leq p < \infty$. Then we have*

$$\begin{aligned} & \left| \frac{1}{m(a, b)} \int_a^b w(t)f(t)g(t)dt - \frac{1}{m^2(a, b)} \left(\int_a^b w(t)f(t)dt \right) \left(\int_a^b w(t)g(t)dt \right) \right| \\ & \leq \frac{\|f'\|_p \|g'\|_p}{m^3(a, b)} \int_a^b w(x) \left(\int_a^x m^q(a, t)dt + \int_x^b m^q(t, b)dt \right)^{\frac{2}{q}} dx \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a, b)} \int_a^b w(t)f(t)g(t)dt - \frac{1}{m^2(a, b)} \left(\int_a^b w(t)f(t)dt \right) \left(\int_a^b w(t)g(t)dt \right) \right| \\ & \leq \frac{1}{2m^2(a, b)} \left[\int_a^b w(x) \left[|g(x)| \|f'\|_p + |f(x)| \|g'\|_p \right] \left(\int_a^x m^q(a, t)dt + \int_x^b m^q(t, b)dt \right)^{\frac{1}{q}} dx \right] \end{aligned}$$

for all $x \in [a, b]$.

THEOREM 1.4. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose first derivatives f', g' belong to $L_p(a, b)$ space where $1 \leq p < \infty$. Then we have*

$$\begin{aligned} & \left| FG - \frac{G}{m(a, b)} \int_a^b w(t)f(t)dt - \frac{F}{m(a, b)} \int_a^b w(t)g(t)dt \right. \\ & \quad \left. + \left(\frac{1}{m(a, b)} \int_a^b w(t)f(t)dt \right) \left(\frac{1}{m(a, b)} \int_a^b w(t)g(t)dt \right) \right| \\ & \leq \frac{\|f'\|_p \|g'\|_p}{m^2(a, b)} \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) dt + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) dt \right. \\ & \quad \left. + \int_{\frac{a+5b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) dt + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) dt \right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t) [Fg(t) + Gf(t)] dt - 2 \frac{1}{m^2(a,b)} \left(\int_a^b w(t)g(t)dt \right) \left(\int_a^b w(t)f(t)dt \right) \right| \\ & \leq \frac{1}{m^2(a,b)} \int_a^b \left[w(x) \left(\|g(x)\| \|f'\|_p + |f(x)| \|g'\|_p \right) \right. \\ & \quad \times \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) dt + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) dt \right. \\ & \quad \left. \left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) dt + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) dt \right)^{\frac{1}{q}} \right] dx \end{aligned}$$

where

$$F = m \left(a, \frac{5a+b}{6} \right) f(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) f \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) f(b),$$

$$G = m \left(a, \frac{5a+b}{6} \right) g(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) g \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) g(b),$$

and

$$m(x) = \begin{cases} \int_a^x w(\tau) d\tau, & a \leq x < \frac{a+b}{2} \\ \int_{\frac{a+b}{2}}^x w(\tau) d\tau, & \frac{a+b}{2} \leq x \leq b. \end{cases}$$

In 1988, Hilger [14] initiated the development of the theory of time scales. Recently, many authors studied the weighted Čebyšev type inequalities, the Grüss inequality, Ostrowski inequalities, Diamond-alpha Grüss type inequalities, the weighted Ostrowski–Grüss inequalities, Ostrowski type inequalities, a generalized Ostrowski’s inequality, an Ostrowski–Grüss type inequality, a perturbed Ostrowski-type inequality, the weighted Ostrowski, trapezoid and Grüss type inequalities, Grüss type inequalities, the weighted Ostrowski and Čebyšev type inequalities, the Čebyšev’s inequality on time scales (see [1, 3, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 19, 22, 23, 21, 24, 25, 26, 27, 28]).

The purpose of this paper is to obtain some weighted Čebyšev–Ostrowski type integral inequalities on time scales involving functions whose first derivatives belong to $L_p(a, b)$ ($1 \leq p \leq \infty$). We also give some other interesting inequalities as special cases.

2. General definitions

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger’s Ph. D. thesis [14], the books [5, 6, 13], and the survey [1].

DEFINITION 1. A time scale \mathbb{T} is an arbitrary nonempty closed subset of \mathbb{R} .

We assume throughout that \mathbb{T} has the topology that is inherited from the standard topology on \mathbb{R} . It also assumed throughout that in \mathbb{T} the interval $[a, b]$ means the set $\{t \in \mathbb{T} : a \leq t \leq b\}$ for the points $a < b$ in \mathbb{T} . Since a time scale may not be connected, we need the following concept of jump operators.

DEFINITION 2. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$, while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) = \inf \{s \in \mathbb{T} : s < t\}$.

The jump operators σ and ρ allow the classification of points in \mathbb{T} as follows.

DEFINITION 3. If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$ then t is called left-dense. Points that are both right-dense and left-dense are called dense.

DEFINITION 4. The mapping $\mu : \mathbb{T} \rightarrow [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$ is called the graininess function. The set \mathbb{T}^k is defined as follows: if \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

DEFINITION 5. Let $f : \mathbb{T} \rightarrow \mathbb{R}$. f is called differentiable at $t \in \mathbb{T}^k$, with (delta) derivative $f^\Delta(t) \in \mathbb{R}$, if for any given $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t) [\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = f(t + 1) - f(t)$.

THEOREM 2.1. Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t).$$

DEFINITION 6. The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous (denote $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.

It follows from [5, Theorem 1.74] that every rd-continuous function has an anti-derivative.

DEFINITION 7. Let $f \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then $F : \mathbb{T} \rightarrow \mathbb{R}$ is called the antiderivative of f on \mathbb{T} if it satisfies $F^\Delta(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we define the Δ -integral of f as

$$\int_a^b f(s)\Delta s = F(t) - F(a), \quad t \in \mathbb{T}.$$

THEOREM 2.2. Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then

$$(1) \int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t,$$

$$(2) \int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$$

$$(3) \int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$$

$$(4) \int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t,$$

THEOREM 2.3. If f is Δ -integrable on $[a, b]$, then so is $|f|$, and

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b |f(t)|\Delta t.$$

DEFINITION 8. Let $h_k : \mathbb{T}^2 \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by $h_0(t, s) = 1$, for all $s, t \in \mathbb{T}$ and then recursively by $h_{k+1}(t, s) = \int_s^t h_{k+1}(\tau, s)\Delta \tau$, for all $s, t \in \mathbb{T}$.

THEOREM 2.4. (Hölder's Inequality) Let $a, b \in \mathbb{T}$. For rd-continuous functions $f : [a, b] \rightarrow \mathbb{R}$ we have

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^p \Delta t \right\}^{\frac{1}{p}} \left\{ \int_a^b |g(t)|^q \Delta t \right\}^{\frac{1}{q}}$$

where $p > 1$ and $q = \frac{p}{p-1}$.

3. Main results

THEOREM 3.1. *Let \mathbb{T} be a time scale and $a, b \in \mathbb{T}$ $a < b$ and $w : [a, b] \rightarrow [0, \infty)$ be non-negative and integrable. If $f, g \in C_{rd}^1([a, b], \mathbb{R})$ such that $f^\Delta, g^\Delta \in L^\infty((a, b))$, then for all $t \in [a, b]$, we have*

$$\begin{aligned} & \left| \frac{1}{m(a, b)} \int_a^b w(t) f(t) g(t) \Delta t - \frac{1}{m^2(a, b)} \left(\int_a^b w(t) f(t) \Delta t \right) \left(\int_a^b w(t) g(\sigma(t)) \Delta t \right) \right. \\ & \quad \left. - \frac{1}{m^2(a, b)} \left(\int_a^b w(t) g(t) \Delta t \right) \left(\int_a^b w(t) f(\sigma(t)) \Delta t \right) \right. \\ & \quad \left. + \frac{1}{m^2(a, b)} \left(\int_a^b w(t) f(\sigma(t)) \Delta t \right) \left(\int_a^b w(t) g(\sigma(t)) \Delta t \right) \right| \\ & \leq \frac{MN}{m^3(a, b)} \int_a^b w(x) \left(\int_a^b \operatorname{sgn}(t-x) (\sigma(t)-x) w(t) \Delta t \right)^2 \Delta x \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a, b)} \int_a^b w(t) f(t) g(t) \Delta t - \frac{1}{2m^2(a, b)} \left(\int_a^b w(t) g(t) \Delta t \right) \left(\int_a^b w(t) f(\sigma(t)) \Delta t \right) \right. \\ & \quad \left. - \frac{1}{2m^2(a, b)} \left(\int_a^b w(t) f(t) \Delta t \right) \left(\int_a^b w(t) g(\sigma(t)) \Delta t \right) \right| \\ & \leq \frac{1}{2m^2(a, b)} \int_a^b w(x) [|g(x)|M + |f(x)|N] \left(\int_a^b \operatorname{sgn}(t-x) (\sigma(t)-x) w(s) \Delta t \right) \Delta x \end{aligned} \tag{3.2}$$

where $M = \sup_{a < t < b} |f^\Delta(t)| < \infty$ and $N = \sup_{a < t < b} |g^\Delta(t)| < \infty$.

Proof. We have (see also [9])

$$f(x) - \frac{1}{m(a, b)} \int_a^b w(t) f(\sigma(t)) \Delta t = \frac{1}{m(a, b)} \int_a^b P_w(x, t) f^\Delta(t) \Delta t \tag{3.3}$$

Replacing f by g in (3.3), we have

$$g(x) - \frac{1}{m(a, b)} \int_a^b w(t) g(\sigma(t)) \Delta t = \frac{1}{m(a, b)} \int_a^b P_w(x, t) g^\Delta(t) \Delta t \tag{3.4}$$

where

$$P_w(x, t) = \begin{cases} \int_a^t w(\tau)\Delta\tau, & a \leq t < x, \\ \int_b^t w(\tau)\Delta\tau, & x \leq t \leq b. \end{cases} \tag{3.5}$$

We also have (see also [9])

$$\int_a^b |P_w(x, t)| \Delta s = \int_a^b (\sigma(t) - x) w(t) \operatorname{sgn}(t - x) \Delta t. \tag{3.6}$$

Now, multiplying both sides of (3.3) and (3.4), we obtain

$$\begin{aligned} & f(x)g(x) - \frac{f(x)}{m(a, b)} \int_a^b w(t)g(\sigma(t)) \Delta t - \frac{g(x)}{m(a, b)} \int_a^b w(t)f(\sigma(t)) \Delta t \\ & + \frac{1}{m^2(a, b)} \left(\int_a^b w(t)f(\sigma(t)) \Delta t \right) \left(\int_a^b w(t)g(\sigma(t)) \Delta t \right) \\ & = \frac{1}{m^2(a, b)} \left(\int_a^b P_w(x, t) f^\Delta(t) \Delta t \right) \left(\int_a^b P_w(x, t) g^\Delta(t) \Delta t \right) \end{aligned} \tag{3.7}$$

Multiplying with $\frac{w(x)}{m(a, b)}$ and integrating over $[a, b]$ and taking absolute values, we get

$$\begin{aligned} & \left| \frac{1}{m(a, b)} \int_a^b w(t)f(t)g(t)\Delta t - \frac{1}{m^2(a, b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t)) \Delta t \right) \right. \\ & \left. - \frac{1}{m^2(a, b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t)) \Delta t \right) \right. \\ & \left. + \frac{1}{m^2(a, b)} \left(\int_a^b w(t)f(\sigma(t)) \Delta t \right) \left(\int_a^b w(t)g(\sigma(t)) \Delta t \right) \right| \\ & \leq \frac{1}{m^3(a, b)} \int_a^b w(x) \left(\int_a^b |P_w(x, t)| |f^\Delta(t)| \Delta t \right) \left(\int_a^b |P_w(x, t)| |g^\Delta(t)| \Delta t \right) \Delta x \\ & \leq \frac{MN}{m^3(a, b)} \int_a^b w(x) \left(\int_a^b \operatorname{sgn}(t - x) (\sigma(t) - x) w(t) \Delta t \right)^2 \Delta x \end{aligned} \tag{3.8}$$

which completes the proof of inequality (3.1).

Next, multiplying (3.3) by $g(x)$ and (3.4) by $f(x)$, adding the resulting identities and multiplying the final result with $\frac{w(x)}{2m(a,b)}$ and integrating over $[a, b]$, we have

$$\begin{aligned} & \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t)\Delta t - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \\ & - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \\ & = \frac{1}{2m^2(a,b)} \int_a^b w(x) \left[g(x) \left(\int_a^b P_w(x,t) f^\Delta(t) \Delta t \right) + f(x) \left(\int_a^b P_w(x,t) g^\Delta(t) \Delta t \right) \right] \Delta x \end{aligned} \tag{3.9}$$

taking absolute values, we get

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t)\Delta t - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \right. \\ & \left. - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) \left[|g(x)| \left(\int_a^b |P_w(x,t)| |f^\Delta(t)| \Delta t \right) \right. \\ & \left. + |f(x)| \left(\int_a^b |P_w(x,t)| |g^\Delta(t)| \Delta t \right) \right] \Delta x \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) [|g(x)|M + |f(x)|N] \left(\int_a^b |P_w(x,t)| \Delta t \right) \Delta x \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) [|g(x)|M + |f(x)|N] \left(\int_a^b \operatorname{sgn}(t-x)(\sigma(t)-x)w(t)\Delta t \right) \Delta x \end{aligned} \tag{3.10}$$

Thus, we get the inequality (3.2). \square

COROLLARY 3.1. *In the case $\mathbb{T} = \mathbb{R}$ in Theorem 3.1, we have*

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t)dt - \left(\frac{1}{m(a,b)} \int_a^b w(t)f(t)dt \right) \left(\frac{1}{m(a,b)} \int_a^b w(t)g(t)dt \right) \right| \\ & \leq \frac{MN}{m^3(a,b)} \int_a^b w(x) \left(\int_a^b \operatorname{sgn}(t-x)(t-x)w(t)dt \right)^2 dx \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t)dt - \left(\frac{1}{m(a,b)} \int_a^b w(t)g(t)dt \right) \left(\frac{1}{m(a,b)} \int_a^b w(t)f(t)dt \right) \right| \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) [|g(x)|M + |f(x)|N] \left(\int_a^b \operatorname{sgn}(t-x)(t-x)w(t)dt \right) dx \end{aligned} \tag{3.12}$$

where $M = \sup_{a < t < b} |f'(t)| < \infty$ and $N = \sup_{a < t < b} |g'(t)| < \infty$. This inequality can be found in [21] as Theorem 1.

COROLLARY 3.2. In the case $\mathbb{T} = \mathbb{Z}$ in Theorem 3.1, we have

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t)g(t) - \frac{1}{m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)f(t) \right) \left(\sum_{t=a}^{b-1} w(t)g(t+1) \right) \right. \\ & \quad \left. - \frac{1}{m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)g(t) \right) \left(\sum_{t=a}^{b-1} w(t)f(t+1) \right) \right. \\ & \quad \left. + \frac{1}{m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)f(t+1) \right) \left(\sum_{t=a}^{b-1} w(t)g(t+1) \right) \right| \\ & \leq \frac{MN}{m^3(a,b)} \sum_{x=a}^{b-1} w(x) \left(\sum_{t=a}^{b-1} \operatorname{sgn}(t-x)(t+1-x)w(t) \right)^2 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t)g(t) - \frac{1}{2m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)g(t) \right) \left(\sum_{t=a}^{b-1} w(t)f(t+1) \right) \right. \\ & \quad \left. - \frac{1}{2m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)f(t) \right) \left(\sum_{t=a}^{b-1} w(t)g(t+1) \right) \right| \\ & \leq \frac{1}{2m^2(a,b)} \sum_{x=a}^{b-1} w(x) [|g(x)|M + |f(x)|N] \left(\sum_{t=a}^{b-1} \operatorname{sgn}(t-x)(t+1-x)w(t) \right) \end{aligned} \tag{3.14}$$

where $M = \sup_{a < t < b-1} |\Delta f(t)| < \infty$ and $N = \sup_{a < t < b-1} |\Delta g(t)| < \infty$.

COROLLARY 3.3. In the case $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Theorem 3.1, we have

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_{q^m}^{q^n} w(t)f(t)g(t)\Delta t - \frac{1}{m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)f(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)g(\sigma(t))\Delta t \right) \right. \\ & \quad \left. - \frac{1}{m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)g(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)f(\sigma(t))\Delta t \right) \right| \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)f(\sigma(t))\Delta t \right) \left(\int_{q^m}^{q^n} w(t)g(\sigma(t))\Delta t \right) \Big| \\
 & \leq \frac{MN}{m^3(a,b)} \int_{q^m}^{q^n} w(x) \left(\int_{q^m}^{q^n} \text{sgn}(t-x)(\sigma(t)-x)w(s)\Delta t \right)^2 \Delta x \tag{3.15}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \frac{1}{m(a,b)} \int_{q^m}^{q^n} w(t)f(t)g(t)\Delta t - \frac{1}{2m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)g(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)f(\sigma(t))\Delta t \right) \right. \\
 & \quad \left. - \frac{1}{2m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)f(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)g(\sigma(t))\Delta t \right) \right| \\
 & \leq \frac{1}{2m^2(a,b)} \int_{q^m}^{q^n} w(x) [|g(x)|M + |f(x)|N] \left(\int_{q^m}^{q^n} \text{sgn}(t-x)(\sigma(t)-x)w(s)\Delta t \right) \Delta t \tag{3.16}
 \end{aligned}$$

where $M = \sup_{q^m < t < q^n} \left| \frac{f(q^{k+1}) - f(q^k)}{(q-1)q^k} \right| < \infty$ and $N = \sup_{q^m < t < q^n} \left| \frac{g(q^{k+1}) - g(q^k)}{(q-1)q^k} \right| < \infty$.

THEOREM 3.2. *Let \mathbb{T} be a time scale and $a, b \in \mathbb{T}$ $a < b$ and $w : [a, b] \rightarrow [0, \infty)$ be non-negative and integrable. If $f, g \in C_{rd}^1([a, b], \mathbb{R})$ such that $f^\Delta, g^\Delta \in L^\infty((a, b))$, then for all $t \in [a, b]$, we have*

$$\begin{aligned}
 & \left| FG - \frac{G}{m(a,b)} \int_a^b w(t)f(\sigma(t))\Delta t - \frac{F}{m(a,b)} \int_a^b w(t)g(\sigma(t))\Delta t \right. \\
 & \quad \left. + \left(\frac{1}{m(a,b)} \int_a^b w(t)f(\sigma(t))\Delta t \right) \left(\frac{1}{m(a,b)} \int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \\
 & \leq \frac{MN}{m^2(a,b)} \left[\int_a^{\frac{a+b}{2}} \text{sgn}\left(t - \frac{5a+b}{6}\right) m\left(\frac{5a+b}{6}, t\right) \Delta t \right. \\
 & \quad \left. + \int_{\frac{a+b}{2}}^b \text{sgn}\left(t - \frac{a+5b}{6}\right) m\left(\frac{a+5b}{6}, t\right) \Delta t \right]^2, \tag{3.17}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \frac{F}{m(a,b)} \int_a^b w(t)g(t)\Delta t + \frac{G}{m(a,b)} \int_a^b w(t)f(t)\Delta t \right. \\
 & \quad - \frac{1}{m(a,b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \\
 & \quad \left. - \frac{1}{m(a,b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \\
 & \leq \frac{1}{m^2(a,b)} \int_a^b w(x) (|g(x)|M + |f(x)|N) \left(\int_a^{\frac{a+b}{2}} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) \Delta t \right. \\
 & \quad \left. + \int_{\frac{a+b}{2}}^b \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) \Delta t \right) \Delta x \tag{3.18}
 \end{aligned}$$

where $M = \sup_{a < t < b} |f^\Delta(t)| < \infty$, $N = \sup_{a < t < b} |g^\Delta(t)| < \infty$,

$$F = m \left(a, \frac{5a+b}{6} \right) f(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) f \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) f(b),$$

$$G = m \left(a, \frac{5a+b}{6} \right) g(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) g \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) g(b)$$

and

$$m(t) = \begin{cases} \int_a^t w(\tau)\Delta\tau, & a \leq t < \frac{a+b}{2} \\ \frac{5a+b}{6} \\ \int_t^b w(\tau)\Delta\tau, & \frac{a+b}{2} \leq t \leq b. \\ \frac{a+5b}{6} \end{cases}$$

Proof. Using item 4 of Theorem 2.2, we have

$$F - \frac{1}{m(a,b)} \int_a^b w(t)f(\sigma(t))\Delta t = \frac{1}{m(a,b)} \int_a^b m(t)f^\Delta(t)\Delta t \tag{3.19}$$

and

$$G - \frac{1}{m(a,b)} \int_a^b w(t)g(\sigma(t))\Delta t = \frac{1}{m(a,b)} \int_a^b m(t)g^\Delta(t)\Delta t \tag{3.20}$$

where

$$F = m\left(a, \frac{5a+b}{6}\right) f(a) + m\left(\frac{5a+b}{6}, \frac{a+5b}{6}\right) f\left(\frac{a+b}{2}\right) + m\left(\frac{a+5b}{6}, b\right) f(b),$$

$$G = m\left(a, \frac{5a+b}{6}\right) g(a) + m\left(\frac{5a+b}{6}, \frac{a+5b}{6}\right) g\left(\frac{a+b}{2}\right) + m\left(\frac{a+5b}{6}, b\right) g(b)$$

and

$$m(t) = \begin{cases} \int_a^t w(\tau)\Delta\tau, & a \leq t \leq \frac{a+b}{2} \\ \int_{\frac{5a+b}{6}}^t w(\tau)\Delta\tau, & \frac{a+b}{2} < t \leq b. \end{cases} \tag{3.21}$$

We also have

$$\begin{aligned} \int_a^b |m(t)| \Delta t &= \int_a^{\frac{a+b}{2}} \left| \int_a^t w(\tau)\Delta\tau \right| \Delta t + \int_{\frac{a+b}{2}}^b \left| \int_{\frac{5a+b}{6}}^t w(\tau)\Delta\tau \right| \Delta t \\ &= \int_a^{\frac{5a+b}{6}} \left(\int_t^{\frac{5a+b}{6}} w(\tau)\Delta\tau \right) \Delta t + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left(\int_{\frac{5a+b}{6}}^t w(\tau)\Delta\tau \right) \Delta t \\ &\quad + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left(\int_t^{\frac{a+5b}{6}} w(\tau)\Delta\tau \right) \Delta t + \int_{\frac{a+5b}{6}}^b \left(\int_{\frac{a+5b}{6}}^t w(\tau)\Delta\tau \right) \Delta t \\ &= \int_a^{\frac{a+b}{2}} \operatorname{sgn}\left(t - \frac{5a+b}{6}\right) m\left(\frac{5a+b}{6}, t\right) \Delta t + \int_{\frac{a+b}{2}}^b \operatorname{sgn}\left(t - \frac{a+5b}{6}\right) m\left(\frac{a+5b}{6}, t\right) \Delta t \end{aligned} \tag{3.22}$$

Now, multiplying both sides of (3.19) and (3.20), we obtain

$$\begin{aligned} FG - \frac{G}{m(a,b)} \int_a^b w(t)f(\sigma(t))\Delta t - \frac{F}{m(a,b)} \int_a^b w(t)g(\sigma(t))\Delta t \\ + \left(\frac{1}{m(a,b)} \int_a^b w(t)f(\sigma(t))\Delta t \right) \left(\frac{1}{m(a,b)} \int_a^b w(t)g(\sigma(t))\Delta t \right) \\ = \frac{1}{m^2(a,b)} \left(\int_a^b m(t)f^\Delta(t)\Delta t \right) \left(\int_a^b m(t)g^\Delta(t)\Delta t \right) \end{aligned} \tag{3.23}$$

taking absolute values and using (3.22), we get the inequality (3.17).

Next, multiplying (3.19) by $g(x)$ and (3.20) by $f(x)$, adding the resulting identities, we have

$$\begin{aligned}
 & Fg(x) + Gf(x) - \frac{g(x)}{m(a,b)} \int_a^b w(t)f(\sigma(t))\Delta t - \frac{f(x)}{m(a,b)} \int_a^b w(t)g(\sigma(t))\Delta t \\
 &= \frac{g(x)}{m(a,b)} \int_a^b m(t)f^\Delta(t)\Delta t + \frac{f(x)}{m(a,b)} \int_a^b m(t)g^\Delta(t)\Delta t.
 \end{aligned}
 \tag{3.24}$$

Multiplying the final result with $\frac{w(x)}{m(a,b)}$, integrating over $[a, b]$ and taking absolute values, we get

$$\begin{aligned}
 & \left| \frac{1}{m(a,b)} \int_a^b w(t) [Fg(t) + Gf(t)] \Delta t - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \right. \\
 & \quad \left. - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \\
 & \leq \frac{1}{m^2(a,b)} \int_a^b w(x) \left(|g(x)| \int_a^b |m(t)| |f^\Delta(t)| \Delta t + |f(x)| \int_a^b |m(t)| |g^\Delta(t)| \Delta t \right) \Delta x \\
 & \leq \frac{1}{m^2(a,b)} \int_a^b w(x) (|g(x)|M + |f(x)|N) \left(\int_a^b |m(t)| \Delta t \right) \Delta x \\
 & \leq \frac{1}{m^2(a,b)} \int_a^b w(x) (|g(x)|M + |f(x)|N) \left(\int_a^{\frac{a+b}{2}} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) \right. \\
 & \quad \left. + \int_{\frac{a+b}{2}}^b \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) \Delta t \right) \Delta x
 \end{aligned}
 \tag{3.25}$$

Thus, we obtain the inequality (3.18). \square

COROLLARY 3.4. *In the case $\mathbb{T} = \mathbb{R}$ in Theorem 3.2, we have*

$$\begin{aligned}
 & \left| FG - \frac{G}{m(a,b)} \int_a^b w(t)f(t) dt - \frac{F}{m(a,b)} \int_a^b w(t)g(t) dt \right. \\
 & \quad \left. + \left(\frac{1}{m(a,b)} \int_a^b w(t)f(t) dt \right) \left(\frac{1}{m(a,b)} \int_a^b w(t)g(t) dt \right) \right|
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{MN}{m^3(a,b)} \left(\int_a^{\frac{a+b}{2}} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) dt + \right. \\ &\quad \left. \int_{\frac{a+b}{2}}^b \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) dt \right)^2 \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &\left| \frac{F}{m(a,b)} \int_a^b w(t)g(t)dt + \frac{G}{m(a,b)} \int_a^b w(t)f(t)dt \right. \\ &\quad \left. - \frac{2}{m(a,b)} \left(\int_a^b w(t)g(t)dt \right) \left(\int_a^b w(t)f(t)dt \right) \right| \\ &\leq \frac{1}{m^2(a,b)} \int_a^b w(x) (|g(x)|M + |f(x)|N) \left(\int_a^{\frac{a+b}{2}} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) dt \right. \\ &\quad \left. + \int_{\frac{a+b}{2}}^b \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) dt \right) dx \end{aligned} \tag{3.27}$$

where $M = \sup_{a < t < b} |f'(t)| < \infty$ and $N = \sup_{a < t < b} |g'(t)| < \infty$.

$$F = m \left(a, \frac{5a+b}{6} \right) f(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) f \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) f(b),$$

$$G = m \left(a, \frac{5a+b}{6} \right) g(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) g \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) g(b).$$

This inequality can be found in [21] as Theorem 2.

COROLLARY 3.5. In the case $\mathbb{T} = \mathbb{Z}$ in Theorem 3.2, we have

$$\begin{aligned} &\left| FG - \frac{G}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t+1) - \frac{F}{m(a,b)} \sum_{t=a}^{b-1} w(t)g(t+1) \right. \\ &\quad \left. + \left(\frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t+1) \right) \left(\frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t)g(t+1) \right) \right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{MN}{m^3(a,b)} \left(\sum_{t=a}^{\frac{a+b}{2}-1} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) \right. \\ &\quad \left. + \sum_{t=\frac{a+b}{2}-1}^{b-1} \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) \right)^2 \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} &\left| \frac{F}{m(a,b)} \sum_{t=a}^{b-1} w(t)g(t) + \frac{G}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t) \right. \\ &\quad - \frac{1}{m(a,b)} \left(\sum_{t=a}^{b-1} w(t)g(t) \right) \left(\sum_{t=a}^{b-1} w(t)f(t+1) \right) \\ &\quad \left. - \frac{1}{m(a,b)} \left(\sum_{t=a}^{b-1} w(t)f(t) \right) \left(\sum_{t=a}^{b-1} w(t)g(t+1) \right) \right| \\ &\leq \frac{1}{m^2(a,b)} \sum_{x=a}^{b-1} w(x) (|g(x)|M + |f(x)|N) \left(\sum_{t=a}^{\frac{a+b}{2}-1} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) \right. \\ &\quad \left. + \sum_{t=\frac{a+b}{2}}^{b-1} \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) \right) \end{aligned} \tag{3.29}$$

where $M = \sup_{a < t < b-1} |\Delta f(t)| < \infty$ and $N = \sup_{a < t < b-1} |\Delta g(t)| < \infty$.

$$F = m \left(a, \frac{5a+b}{6} \right) f(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) f \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) f(b),$$

$$G = m \left(a, \frac{5a+b}{6} \right) g(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) g \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) g(b).$$

COROLLARY 3.6. In the case $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Theorem 3.2, we have

$$\begin{aligned} &\left| FG - \frac{G}{m(a,b)} \int_{q^m}^{q^n} w(t)f(\sigma(t)) \Delta t - \frac{F}{m(a,b)} \int_{q^m}^{q^n} w(t)g(\sigma(t)) \Delta t \right. \\ &\quad \left. + \left(\frac{1}{m(a,b)} \int_{q^m}^{q^n} w(t)f(\sigma(t)) \Delta t \right) \left(\frac{1}{m(a,b)} \int_{q^m}^{q^n} w(t)g(\sigma(t)) \Delta t \right) \right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{MN}{2m^3(a,b)} \left(\int_{q^m}^{\frac{q^m+q^n}{2}} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) \Delta t \right. \\ &\quad \left. + \int_{\frac{q^m+q^n}{2}}^{q^n} \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) \Delta t \right)^2 \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} &\left| \frac{F}{m(a,b)} \int_{q^m}^{q^n} w(t)g(t)\Delta t + \frac{G}{m(a,b)} \int_{q^m}^{q^n} w(t)f(t)\Delta t \right. \\ &\quad - \frac{1}{m(a,b)} \left(\int_{q^m}^{q^n} w(t)g(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)f(\sigma(t))\Delta t \right) \\ &\quad \left. - \frac{1}{m(a,b)} \left(\int_{q^m}^{q^n} w(t)f(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)g(\sigma(t))\Delta t \right) \right| \\ &\leq \frac{1}{m^2(a,b)} \int_{q^m}^{q^n} w(x) (|g(x)|M + |f(x)|N) \left(\int_{q^m}^{\frac{q^m+q^n}{2}} \operatorname{sgn} \left(t - \frac{5a+b}{6} \right) m \left(\frac{5a+b}{6}, t \right) \Delta t \right. \\ &\quad \left. + \int_{\frac{q^m+q^n}{2}}^{q^n} \operatorname{sgn} \left(t - \frac{a+5b}{6} \right) m \left(\frac{a+5b}{6}, t \right) \Delta t \right) \Delta x \end{aligned} \tag{3.31}$$

where $M = \sup_{q^m < t < q^n} \left| \frac{f(q^{k+1}) - f(q^k)}{(q-1)q^k} \right| < \infty$ and $N = \sup_{q^m < t < q^n} \left| \frac{g(q^{k+1}) - g(q^k)}{(q-1)q^k} \right| < \infty$.

$$\begin{aligned} F &= m \left(q^m, \frac{5q^m+q^n}{6} \right) f(q^m) + m \left(\frac{5q^m+q^n}{6}, \frac{q^m+5q^n}{6} \right) f \left(\frac{q^m+q^n}{2} \right) \\ &\quad + m \left(\frac{q^m+5q^n}{6}, q^n \right) f(q^n), \\ G &= m \left(q^m, \frac{5q^m+q^n}{6} \right) g(q^m) + m \left(\frac{5q^m+q^n}{6}, \frac{q^m+5q^n}{6} \right) g \left(\frac{q^m+q^n}{2} \right) \\ &\quad + m \left(\frac{q^m+5q^n}{6}, q^n \right) g(q^n). \end{aligned}$$

THEOREM 3.3. Let \mathbb{T} be a time scale and $a, b \in \mathbb{T}$ $a < b$ and $w : [a, b] \rightarrow [0, \infty)$ be non-negative and integrable. If $f, g \in C_{rd}^1([a, b], \mathbb{R})$ such that $f^\Delta, g^\Delta \in L^p([a, b])$,

then for all $t \in [a, b]$, we have

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t)\Delta t - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \right. \\ & \left. - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \right. \\ & \left. + \frac{1}{m^2(a,b)} \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \\ & \leq \frac{\|f^\Delta\|_p \|g^\Delta\|_p}{m^3(a,b)} \int_a^b w(x) \left(\int_a^x m^q(a,t) \Delta t + \int_x^b m^q(t,b) \Delta t \right)^{\frac{2}{q}} \Delta x \end{aligned} \quad (3.32)$$

and

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t)\Delta t - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \right. \\ & \left. - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) \left[|g(x)| \|f^\Delta\|_p + |f(x)| \|g^\Delta\|_p \right] \\ & \quad \times \left(\int_a^x m^q(a,t) \Delta t + \int_x^b m^q(t,b) \Delta t \right)^{\frac{1}{q}} \Delta x \end{aligned} \quad (3.33)$$

where $\|f^\Delta\|_p = \left(\int_a^b |f^\Delta(t)|^p \Delta t \right)^{\frac{1}{p}} < \infty$ and $\|g^\Delta\|_p = \left(\int_a^b |g^\Delta(t)|^p \Delta t \right)^{\frac{1}{p}} < \infty$.

Proof. Using (3.5), we obtain

$$\begin{aligned} \left(\int_a^b |p_w(x,t)|^q \Delta t \right)^{\frac{1}{q}} &= \left(\int_a^x \left| \int_a^t w(u) \Delta u \right|^q \Delta t + \int_x^b \left| \int_b^t w(u) \Delta u \right|^q \Delta t \right)^{\frac{1}{q}} \\ &= \left(\int_a^x \left(\int_a^t w(u) \Delta u \right)^q \Delta t + \int_x^b \left(\int_t^b w(u) \Delta u \right)^q \Delta t \right)^{\frac{1}{q}} \end{aligned}$$

$$= \left(\int_a^x m^q(a,t) \Delta t + \int_x^b m^q(t,b) \Delta t \right)^{\frac{1}{q}}. \tag{3.34}$$

From (3.7), we have

$$\begin{aligned} & f(x)g(x) - \frac{f(x)}{m(a,b)} \int_a^b w(t)g(\sigma(t)) \Delta t - \frac{g(x)}{m(a,b)} \int_a^b w(t)f(\sigma(t)) \Delta t \\ & + \frac{1}{m^2(a,b)} \left(\int_a^b w(t)f(\sigma(t)) \Delta t \right) \left(\int_a^b w(t)g(\sigma(t)) \Delta t \right) \\ & = \frac{1}{m^2(a,b)} \left(\int_a^b P_w(x,t) f^\Delta(t) \Delta t \right) \left(\int_a^b P_w(x,t) g^\Delta(t) \Delta t \right) \end{aligned} \tag{3.35}$$

Multiplying with $\frac{w(x)}{m(a,b)}$, integrating over $[a, b]$ and taking absolute values and using Hölder's inequality on time scales, we have

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t) \Delta t - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)f(t) \Delta t \right) \left(\int_a^b w(t)g(\sigma(t)) \Delta t \right) \right. \\ & \left. - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)g(t) \Delta t \right) \left(\int_a^b w(t)f(\sigma(t)) \Delta t \right) \right. \\ & \left. + \frac{1}{m^2(a,b)} \left(\int_a^b w(t)f(\sigma(t)) \Delta t \right) \left(\int_a^b w(t)g(\sigma(t)) \Delta t \right) \right| \\ & \leq \frac{1}{m^3(a,b)} \int_a^b w(x) \left(\int_a^b |P_w(x,t)| |f^\Delta(t)| \Delta t \right) \left(\int_a^b |P_w(x,t)| |g^\Delta(t)| \Delta t \right) \Delta x \\ & \leq \frac{\|f^\Delta\|_p \|g^\Delta\|_p}{m^3(a,b)} \int_a^b w(x) \left(\int_a^b |P_w(x,t)|^q \Delta t \right)^{\frac{2}{q}} \Delta x \end{aligned} \tag{3.36}$$

From (3.36) and (3.34), we get the desired inequality (3.32).

From (3.9), we have

$$\begin{aligned} & \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t) \Delta t - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t)g(t) \Delta t \right) \left(\int_a^b w(t)f(\sigma(t)) \Delta t \right) \\ & - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t)f(t) \Delta t \right) \left(\int_a^b w(t)g(\sigma(t)) \Delta t \right) \end{aligned}$$

$$= \frac{1}{2m^2(a,b)} \int_a^b w(x) \left[g(x) \left(\int_a^b P_w(x,t) f^\Delta(t) \Delta t \right) + f(x) \left(\int_a^b P_w(x,t) g^\Delta(t) \Delta t \right) \right] \Delta x \tag{3.37}$$

taking absolute values and using Hölder’s inequality on time scales, we get

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t) f(t) g(t) \Delta t - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t) g(t) \Delta t \right) \left(\int_a^b w(t) f(\sigma(t)) \Delta t \right) \right. \\ & \quad \left. - \frac{1}{2m^2(a,b)} \left(\int_a^b w(t) f(t) \Delta t \right) \left(\int_a^b w(t) g(\sigma(t)) \Delta t \right) \right| \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) \left[|g(x)| \left(\int_a^b P_w(x,t) |f^\Delta(t)| \Delta t \right) \right. \\ & \quad \left. + |f(x)| \left(\int_a^b P_w(x,t) |g^\Delta(t)| \Delta t \right) \right] \Delta x \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b \left[w(x) |g(x)| \|f^\Delta\|_p \left(\int_a^b |p_w(x,t)|^q \Delta t \right)^{\frac{1}{q}} \right. \\ & \quad \left. + w(x) |f(x)| \|g^\Delta\|_p \left(\int_a^b |p_w(x,t)|^q \Delta t \right)^{\frac{1}{q}} \right] \Delta x \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) \left[\|g(x)\|_p \|f^\Delta\|_p + \|f(x)\|_p \|g^\Delta\|_p |f(x)| \right] \\ & \quad \times \left(\int_a^x m^q(a,t) \Delta t + \int_x^b m^q(t,b) \Delta t \right)^{\frac{1}{q}} \Delta x \end{aligned}$$

Consequently, we get the inequality (3.33). \square

COROLLARY 3.7. *In the case $\mathbb{T} = \mathbb{R}$ in Theorem 3.3, we have*

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t) f(t) g(t) dt - \frac{1}{m^2(a,b)} \left(\int_a^b w(t) f(t) dt \right) \left(\int_a^b w(t) g(t) dt \right) \right| \\ & \leq \frac{\|f'\|_p \|g'\|_p}{m^3(a,b)} \int_a^b w(x) \left(\int_a^x m^q(a,t) dt + \int_x^b m^q(t,b) dt \right)^{\frac{2}{q}} dx \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_a^b w(t)f(t)g(t)dt - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)g(t)dt \right) \left(\int_a^b w(t)f(t)dt \right) \right| \\ & \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) \left[|g(x)| \|f'\|_p + |f(x)| \|g'\|_p \right] \left(\int_a^x m^q(a,t)dt + \int_x^b m^q(t,b)dt \right)^{\frac{1}{q}} dx \end{aligned} \tag{3.39}$$

where $\|f'\|_p = \left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} < \infty$ and $\|g'\|_p = \left(\int_a^b |g'(t)|^p dt \right)^{\frac{1}{p}} < \infty$. This inequality can be found in [2] as Theorem 3.

COROLLARY 3.8. In the case $\mathbb{T} = \mathbb{Z}$ in Theorem 3.3, we have

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t)g(t) - \frac{1}{m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)f(t) \right) \left(\sum_{t=a}^{b-1} w(t)g(t+1) \right) \right. \\ & \quad - \frac{1}{m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)g(t) \right) \left(\sum_{t=a}^{b-1} w(t)f(t+1) \right) \\ & \quad \left. + \frac{1}{m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)f(t+1) \right) \left(\sum_{t=a}^{b-1} w(t)g(t+1) \right) \right| \\ & \leq \frac{\|\Delta f\|_p \|\Delta g\|_p}{m^3(a,b)} \sum_{x=a}^{b-1} w(x) \left(\sum_{t=a}^{x-1} m^q(a,t) + \sum_{t=x}^{b-1} m^q(t,b) \right)^{\frac{2}{q}} \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t)g(t) - \frac{1}{2m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)g(t) \right) \left(\sum_{t=a}^{b-1} w(t)f(t+1) \right) \right. \\ & \quad \left. - \frac{1}{2m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)f(t) \right) \left(\sum_{t=a}^{b-1} w(t)g(t+1) \right) \right| \\ & \leq \frac{1}{2m^2(a,b)} \sum_{x=a}^{b-1} w(x) \left[|g(x)| \|\Delta f\|_p + |f(x)| \|\Delta g\|_p \right] \left(\sum_{t=a}^{x-1} m^q(a,t) + \sum_{t=x}^{b-1} m^q(t,b) \right)^{\frac{1}{q}} \end{aligned} \tag{3.41}$$

where $\|\Delta f\|_p = \left(\sum_{t=a}^{b-1} |\Delta f(t)|^p \right)^{\frac{1}{p}} < \infty$ and $\|\Delta g\|_p = \left(\sum_{t=a}^{b-1} |\Delta g(t)|^p \right)^{\frac{1}{p}} < \infty$.

COROLLARY 3.9. *In the case $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Theorem 3.3, we have*

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_{q^m}^{q^n} w(t)f(t)g(t)\Delta t - \frac{1}{m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)f(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)g(\sigma(t))\Delta t \right) \right. \\ & \quad \left. - \frac{1}{m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)g(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)f(\sigma(t))\Delta t \right) \right. \\ & \quad \left. + \frac{1}{m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)f(\sigma(t))\Delta t \right) \left(\int_{q^m}^{q^n} w(t)g(\sigma(t))\Delta t \right) \right| \\ & \leq \frac{\|f^\Delta\|_p \|g^\Delta\|_p}{m^3(a,b)} \int_{q^m}^{q^n} w(x) \left(\int_{q^m}^x m^q(a,t)\Delta t + \int_x^{q^n} m^q(t,b)\Delta t \right)^{\frac{2}{q}} \Delta x \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} & \left| \frac{1}{m(a,b)} \int_{q^m}^{q^n} w(t)f(t)g(t)\Delta t - \frac{1}{2m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)g(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)f(\sigma(t))\Delta t \right) \right. \\ & \quad \left. - \frac{1}{2m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)f(t)\Delta t \right) \left(\int_{q^m}^{q^n} w(t)g(\sigma(t))\Delta t \right) \right| \\ & \leq \frac{1}{2m^2(a,b)} \int_{q^m}^{q^n} w(x) \left[|g(x)| \|f^\Delta\|_p + |f(x)| \|g^\Delta\|_p \right] \\ & \quad \times \left(\int_{q^m}^x m^q(a,t)\Delta t + \int_x^{q^n} m^q(t,b)\Delta t \right)^{\frac{1}{q}} \Delta x. \end{aligned} \tag{3.43}$$

THEOREM 3.4. *Let \mathbb{T} be a time scale and $a, b \in \mathbb{T}$ $a < b$ and $w : [a, b] \rightarrow [0, \infty)$ be non-negative and integrable. If $f, g \in C_{rd}^1([a, b], \mathbb{R})$ such that $f^\Delta, g^\Delta \in L^p([a, b])$, then for all $t \in [a, b]$, we have*

$$\begin{aligned} & \left| FG - \frac{G}{m(a,b)} \int_a^b w(t)f(\sigma(t))\Delta t - \frac{F}{m(a,b)} \int_a^b w(t)g(\sigma(t))\Delta t \right. \\ & \quad \left. + \left(\frac{1}{m(a,b)} \int_a^b w(t)f(\sigma(t))\Delta t \right) \left(\frac{1}{m(a,b)} \int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{\|f^\Delta\|_p \|g^\Delta\|_p}{m^2(a,b)} \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) \Delta t + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) \Delta t \right. \\ &\quad \left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) \Delta t + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) \Delta t \right)^{\frac{2}{q}} \end{aligned} \tag{3.44}$$

and

$$\begin{aligned} &\left| \frac{1}{m(a,b)} \int_a^b w(t) [Fg(t) + Gf(t)] \Delta t - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \right. \\ &\quad \left. - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \\ &\leq \frac{1}{m^2(a,b)} \int_a^b w(x) \left(|g(x)| \|f^\Delta\|_p + |f(x)| \|g^\Delta\|_p \right) \\ &\quad \times \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) \Delta t + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) \Delta t \right. \\ &\quad \left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) \Delta t + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) \Delta t \right)^{\frac{1}{q}} \Delta x \end{aligned} \tag{3.45}$$

where $\|f^\Delta\|_p = \left(\int_a^b |f^\Delta(t)|^p \Delta t \right)^{\frac{1}{p}} < \infty$ and $\|g^\Delta\|_p = \left(\int_a^b |g^\Delta(t)|^p \Delta t \right)^{\frac{1}{p}} < \infty$.

Proof. Using (3.21), we get

$$\begin{aligned} \left(\int_a^b |m(t)|^q \Delta t \right)^{\frac{1}{q}} &= \left(\int_a^{\frac{a+b}{2}} \left| \int_{\frac{5a+b}{6}}^t w(\tau)\Delta\tau \right|^q \Delta t + \int_{\frac{a+b}{2}}^b \left| \int_{\frac{a+5b}{6}}^t w(\tau)\Delta\tau \right|^q \Delta t \right)^{\frac{1}{q}} \\ &= \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) \Delta t + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) \Delta t \right. \\ &\quad \left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) \Delta t + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) \Delta t \right)^{\frac{1}{q}} \end{aligned} \tag{3.46}$$

Using (3.23), we have

$$\begin{aligned}
 & FG - \frac{G}{m(a,b)} \int_a^b w(t) f(\sigma(t)) \Delta t - \frac{F}{m(a,b)} \int_a^b w(t) g(\sigma(t)) \Delta t \\
 & + \left(\frac{1}{m(a,b)} \int_a^b w(t) f(\sigma(t)) \Delta t \right) \left(\frac{1}{m(a,b)} \int_a^b w(t) g(\sigma(t)) \Delta t \right) \\
 & = \frac{1}{m^2(a,b)} \left(\int_a^b m(t) f^\Delta(t) \Delta t \right) \left(\int_a^b m(t) g^\Delta(t) \Delta t \right)
 \end{aligned} \tag{3.47}$$

taking absolute values and using Hölder's inequality on time scales, we get

$$\begin{aligned}
 & \left| FG - \frac{G}{m(a,b)} \int_a^b w(t) f(\sigma(t)) \Delta t - \frac{F}{m(a,b)} \int_a^b w(t) g(\sigma(t)) \Delta t \right. \\
 & \left. + \left(\frac{1}{m(a,b)} \int_a^b w(t) f(\sigma(t)) \Delta t \right) \left(\frac{1}{m(a,b)} \int_a^b w(t) g(\sigma(t)) \Delta t \right) \right| \\
 & \leq \frac{1}{m^2(a,b)} \left(\int_a^b |m(t)| |f^\Delta(t)| \Delta t \right) \left(\int_a^b |m(t)| |g^\Delta(t)| \Delta t \right) \\
 & \leq \frac{\|f^\Delta\|_p \|g^\Delta\|_p}{m^2(a,b)} \left(\int_a^b |m(t)|^q \Delta t \right)^{\frac{2}{q}} \\
 & \leq \frac{\|f^\Delta\|_p \|g^\Delta\|_p}{m^2(a,b)} \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) \Delta t + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) \Delta t \right. \\
 & \quad \left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) \Delta t + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) \Delta t \right)^{\frac{2}{q}}
 \end{aligned}$$

which completes the proof of inequality (3.44).

Using (3.24), we have

$$\begin{aligned}
 & Fg(x) + Gf(x) - \frac{g(x)}{m(a,b)} \int_a^b w(t) f(\sigma(t)) \Delta t - \frac{f(x)}{m(a,b)} \int_a^b w(t) g(\sigma(t)) \Delta t \\
 & = \frac{g(x)}{m(a,b)} \int_a^b m(t) f^\Delta(t) \Delta t + \frac{f(x)}{m(a,b)} \int_a^b m(t) g^\Delta(t) \Delta t.
 \end{aligned} \tag{3.48}$$

Multiplying with $\frac{w(x)}{m(a,b)}$ and integrating over $[a, b]$ and taking absolute values and using Hölder's inequality on time scales, we obtain

$$\begin{aligned}
 & \left| \frac{1}{m(a,b)} \int_a^b w(t) [Fg(t) + Gf(t)] \Delta t \right. \\
 & - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)g(t)\Delta t \right) \left(\int_a^b w(t)f(\sigma(t))\Delta t \right) \\
 & \left. - \frac{1}{m^2(a,b)} \left(\int_a^b w(t)f(t)\Delta t \right) \left(\int_a^b w(t)g(\sigma(t))\Delta t \right) \right| \\
 & \leq \frac{1}{m^2(a,b)} \int_a^b w(x) \left(|g(x)| \int_a^b |m(t)| |f^\Delta(t)| \Delta t + |f(x)| \int_a^b |m(t)| |g^\Delta(t)| \Delta t \right) \Delta x \\
 & \leq \frac{1}{m^2(a,b)} \int_a^b w(x) \left(|g(x)| \|f^\Delta\|_p + |f(x)| \|g^\Delta\|_p \right) \left(\int_a^b |m(t)|^q \Delta t \right)^{\frac{1}{q}} \Delta x \\
 & \leq \frac{1}{m^2(a,b)} \int_a^b w(x) \left(|g(x)| \|f^\Delta\|_p + |f(x)| \|g^\Delta\|_p \right) \\
 & \quad \times \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) \Delta t + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) \Delta t \right. \\
 & \quad \left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) \Delta t + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) \Delta t \right)^{\frac{1}{q}} \Delta x \tag{3.49}
 \end{aligned}$$

□

COROLLARY 3.10. *In the case $\mathbb{T} = \mathbb{R}$ in Theorem 3.4, we have*

$$\begin{aligned}
 & \left| FG - \frac{G}{m(a,b)} \int_a^b w(t)f(t) dt - \frac{F}{m(a,b)} \int_a^b w(t)g(t) dt \right. \\
 & \left. + \left(\frac{1}{m(a,b)} \int_a^b w(t)f(t) dt \right) \left(\frac{1}{m(a,b)} \int_a^b w(t)g(t) dt \right) \right|
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{\|f'\|_p \|g'\|_p}{m^2(a,b)} \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) dt + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) dt \right. \\ &\quad \left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) dt + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) dt \right)^{\frac{2}{q}} \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} &\left| \frac{1}{m(a,b)} \int_a^b w(t) [Fg(t) + Gf(t)] dt - \frac{2}{m^2(a,b)} \left(\int_a^b w(t)g(t) dt \right) \left(\int_a^b w(t)f(t) dt \right) \right| \\ &\leq \frac{1}{m^2(a,b)} \int_a^b w(x) \left(|g(x)| \|f'\|_p + |f(x)| \|g'\|_p \right) \\ &\quad \times \left(\int_a^{\frac{5a+b}{6}} m^q \left(t, \frac{5a+b}{6} \right) dt + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, t \right) dt \right. \\ &\quad \left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(t, \frac{a+5b}{6} \right) dt + \int_{\frac{a+5b}{6}}^b m^q \left(\frac{a+5b}{6}, t \right) dt \right)^{\frac{1}{q}} dx \end{aligned} \tag{3.51}$$

where $\|f'\|_p = \left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} < \infty$ and $\|g'\|_p = \left(\int_a^b |g'(t)|^p dt \right)^{\frac{1}{p}} < \infty$. This inequality can be found in [2] as Theorem 4.

COROLLARY 3.11. In the case $\mathbb{T} = \mathbb{Z}$ in Theorem 3.4, we have

$$\begin{aligned} &\left| FG - \frac{G}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t+1) - \frac{F}{m(a,b)} \sum_{t=a}^{b-1} w(t)g(t+1) \right. \\ &\quad \left. + \left(\frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t)f(t+1) \right) \left(\frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t)g(t+1) \right) \right| \\ &\leq \frac{\|\Delta f\|_p \|\Delta g\|_p}{m^2(a,b)} \left(\sum_{t=a}^{\frac{5a+b}{6}-1} m^q \left(t, \frac{5a+b}{6} \right) + \sum_{t=\frac{5a+b}{6}}^{\frac{a+b}{2}-1} m^q \left(\frac{5a+b}{6}, t \right) \right. \\ &\quad \left. + \sum_{t=\frac{a+b}{2}}^{\frac{a+5b}{6}-1} m^q \left(t, \frac{a+5b}{6} \right) + \sum_{t=\frac{a+5b}{6}}^{b-1} m^q \left(\frac{a+5b}{6}, t \right) \right)^{\frac{2}{q}} \end{aligned} \tag{3.52}$$

and

$$\begin{aligned}
 & \left| \frac{1}{m(a,b)} \sum_{t=a}^{b-1} w(t) [Fg(t) + Gf(t)] \Delta t - \frac{1}{m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)g(t) \right) \left(\sum_{t=a}^{b-1} w(t)f(t+1) \right) \right. \\
 & \quad \left. - \frac{1}{m^2(a,b)} \left(\sum_{t=a}^{b-1} w(t)f(t) \right) \left(\sum_{t=a}^{b-1} w(t)g(t+1) \right) \right| \\
 & \leq \frac{1}{m^2(a,b)} \sum_{x=a}^{b-1} w(x) \left(|g(x)| \|\Delta f\|_p + |f(x)| \|\Delta g\|_p \right) \\
 & \quad \times \left(\sum_{t=a}^{\frac{5a+b}{6}-1} m^q \left(t, \frac{5a+b}{6} \right) + \sum_{t=\frac{5a+b}{6}}^{\frac{a+b}{2}-1} m^q \left(\frac{5a+b}{6}, t \right) \right. \\
 & \quad \left. + \sum_{t=\frac{a+b}{2}}^{\frac{a+5b}{6}-1} m^q \left(t, \frac{a+5b}{6} \right) + \sum_{t=\frac{a+5b}{6}}^{b-1} m^q \left(\frac{a+5b}{6}, t \right) \right)^{\frac{1}{q}} \tag{3.53}
 \end{aligned}$$

where $\|\Delta f\|_p = \left(\sum_{t=a}^{b-1} |\Delta f(t)|^p \right)^{\frac{1}{p}} < \infty$ and $\|\Delta g\|_p = \left(\sum_{t=a}^{b-1} |\Delta g(t)|^p \right)^{\frac{1}{p}} < \infty$,

$$F = m \left(a, \frac{5a+b}{6} \right) f(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) f \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) f(b),$$

$$G = m \left(a, \frac{5a+b}{6} \right) g(a) + m \left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) g \left(\frac{a+b}{2} \right) + m \left(\frac{a+5b}{6}, b \right) g(b).$$

COROLLARY 3.12. In the case $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ ($q > 1$) in Theorem 3.4, we have

$$\begin{aligned}
 & \left| FG - \frac{G}{m(a,b)} \int_{q^m}^{q^n} w(t)f(\sigma(t)) \Delta t - \frac{F}{m(a,b)} \int_{q^m}^{q^n} w(t)g(\sigma(t)) \Delta t \right. \\
 & \quad \left. + \left(\frac{1}{m(a,b)} \int_{q^m}^{q^n} w(t)f(\sigma(t)) \Delta t \right) \left(\frac{1}{m(a,b)} \int_{q^m}^{q^n} w(t)g(\sigma(t)) \Delta t \right) \right| \\
 & \leq \frac{\|f^\Delta\|_p \|g^\Delta\|_p}{m^2(a,b)} \left(\int_{q^m}^{\frac{5q^m+q^n}{6}} m^q \left(t, \frac{5q^m+q^n}{6} \right) \Delta t + \int_{\frac{5q^m+q^n}{6}}^{\frac{q^m+q^n}{2}} m^q \left(\frac{5q^m+q^n}{6}, t \right) \Delta t \right. \\
 & \quad \left. + \int_{\frac{q^m+5q^n}{2}}^{\frac{q^m+5q^n}{6}} m^q \left(t, \frac{q^m+5q^n}{6} \right) \Delta t + \int_{\frac{q^m+5q^n}{6}}^{q^n} m^q \left(\frac{q^m+5q^n}{6}, t \right) \Delta t \right)^{\frac{2}{q}} \tag{3.54}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \frac{1}{m^2(a,b)} \int_{q^m}^{q^n} w(t) [Fg(t) + Gf(t)] \Delta t \right. \\
 & - \frac{1}{m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)g(t) \Delta t \right) \left(\int_{q^m}^{q^n} w(t)f(\sigma(t)) \Delta t \right) \\
 & \left. - \frac{1}{m^2(a,b)} \left(\int_{q^m}^{q^n} w(t)f(t) \Delta t \right) \left(\int_{q^m}^{q^n} w(t)g(\sigma(t)) \Delta t \right) \right| \\
 & \leq \frac{1}{m^2(a,b)} \int_{q^m}^{q^n} w(x) \left(\|g(x)\| \|f^\Delta\|_p + \|f(x)\| \|g^\Delta\|_p \right) \\
 & \times \left(\int_{q^m}^{\frac{5q^m+q^n}{6}} m^q \left(t, \frac{5q^m+q^n}{6} \right) \Delta t + \int_{\frac{5q^m+q^n}{6}}^{\frac{q^m+q^n}{2}} m^q \left(\frac{5q^m+q^n}{6}, t \right) \Delta t \right. \\
 & \left. + \int_{\frac{q^m+5q^n}{2}}^{\frac{q^m+5q^n}{6}} m^q \left(t, \frac{q^m+5q^n}{6} \right) \Delta t + \int_{\frac{q^m+5q^n}{6}}^{q^n} m^q \left(\frac{q^m+5q^n}{6}, t \right) \Delta t \right)^{\frac{1}{q}} \Delta x \quad (3.55)
 \end{aligned}$$

where $\|f^\Delta\|_p = \left(\int_{q^m}^{q^n} |f^\Delta(t)|^p \Delta t \right)^{\frac{1}{p}} < \infty$ and $\|g^\Delta\|_p = \left(\int_{q^m}^{q^n} |g^\Delta(t)|^p \Delta t \right)^{\frac{1}{p}} < \infty$,

$$\begin{aligned}
 F &= m \left(q^m, \frac{5q^m+q^n}{6} \right) f(q^m) + m \left(\frac{5q^m+q^n}{6}, \frac{q^m+5q^n}{6} \right) f \left(\frac{q^m+q^n}{2} \right) \\
 & \quad + m \left(\frac{q^m+5q^n}{6}, q^n \right) f(q^n), \\
 G &= m \left(q^m, \frac{5q^m+q^n}{6} \right) g(q^m) + m \left(\frac{5q^m+q^n}{6}, \frac{q^m+5q^n}{6} \right) g \left(\frac{q^m+q^n}{2} \right) \\
 & \quad + m \left(\frac{q^m+5q^n}{6}, q^n \right) g(q^n).
 \end{aligned}$$

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(Received August 14, 2014)

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