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# ON GENERALIZATIONS OF A TRIGONOMETRIC INEQUALITY

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*Abstract.* This paper generalizes a trigonometric inequality to sine integral and double trigonometric series satisfying MVBVF($\mathbb{R}_+$) condition and MVBVDS condition.

## 1. Introduction

Let $\{a_n\}_{n=1}^{\infty}$ be a non-negative sequence, and write

$$\sum_{n=1}^{\infty} a_n \sin nx, \quad x \in [-\pi, \pi) \tag{1}$$

as a sine series. In [1], Chaundry and Jolliffe proved the following theorem

THEOREM 1. *If* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ *is decreasing, then series* (1) *converges uniformly in* $x$ *if and only if*

$$na_n \to 0, \quad \text{as} \quad n \to \infty. \tag{2}$$

The monotonicity condition in Theorem 1 was relaxed by a number of authors to QM (quasi-monotone) condition, RBV (rest bounded variation) condition, GBV (group bounded variation) condition and NBV (non-onesided bounded variation) condition. Finally, to MVBV (mean value bounded variation, [12]) condition (see [11] for more details).

DEFINITION 1. A non-negative sequence $\mathrm{A} = \{a_n\}_{n=1}^{\infty}$ is said to be a mean value bounded variation sequence, in symbol: $\mathrm{A} \in$ MVBVS, if there exist constants $K := K(\mathrm{A})$ and $\lambda \geqslant 2$, depending only upon the sequence A, such that

$$\sum_{k=n}^{2n} |\Delta a_k| \leqslant \frac{K}{n} \sum_{k=[\lambda^{-1}n]}^{\lambda n} a_k \tag{3}$$

hold for all $n = 1, 2, \cdots$, where $[\cdot]$ means the integer part and $\Delta a_k := a_k - a_{k+1}$, $k = 1, 2, \cdots$.

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Theorem 1 was generlized under MVBV condition in [12]:

THEOREM 2. *If* $\{a_n\}_{n=1}^{\infty}$ *belongs to the class* MVBVS*, then series* (1) *converges uniformly in* $x \in [-\pi, \pi)$ *if and only if condition* (2) *is satisfied.*

Meantime, they proved this condition is the weakest one to generalize monotonicity and cannot be weakened further in uniform convergence for sine series:

THEOREM 3. *Let* $\{M_n\}_{n=1}^{\infty}$ *be a given non-negative increasing sequence tending to infinity. Then there exists a sine series of the form* (1) *satisfying* (2) *such that for any given* $\lambda \geqslant 2$*,*

$$\lim_{n\to\infty} \frac{\sum\limits_{k=n}^{2n} |\Delta a_k|}{\frac{M_n}{n} \sum\limits_{k=[\lambda^{-1}n]}^{\lambda n} a_k} = 0, \tag{4}$$

*however, the series is not uniformly convergent.*

In [6], F. Móricz simulated and studied the Theorem 1 about sine integral

$$\int_0^{\infty} f(x) \sin tx \mathrm{d}x, \tag{5}$$

where $f : \mathbb{R}_+ := [0,\infty) \to \mathbb{C}$ is a measurable function with the property $xf(x) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ (If for each $a \in \mathbb{R}_+$, $\int_0^a |f(x)|\mathrm{d}x < \infty$, we say that $f(x) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$). F. Móricz gave the definition of MVBVF($\mathbb{R}_+$) as follows:

DEFINITION 2. A function: $f : \mathbb{R}_+ \to \mathbb{C}$ is said to be of mean value bounded variation, in symbols: $f \in$MVBVF($\mathbb{R}_+$), if $f$ is absolutely continuous on every interval $[a,b]$, where $0 < a < b < \infty$ (shortly: $f$ is locally absolutely continuous on $\mathbb{R}_+$); and if there exist constants $K_1$ and $\lambda \geqslant 2$, depending only of $f$, such that for all large enough $a \in \mathbb{R}_+$

$$\int_a^{2a} |f'(x)|\mathrm{d}x \leqslant \frac{K_1}{a} \int_{\lambda^{-1}a}^{\lambda a} |f(x)|\mathrm{d}x. \tag{6}$$

Meantime, he generalized Theorem 2 about sine integral under MVBVF($\mathbb{R}_+$) as follows:

THEOREM 4. *Assume* $f \in$ MVBVF$(\mathbb{R}_+)$ *with property* $xf(x) \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.
*(i) If* $f : \mathbb{R}_+ \to \mathbb{C}$ *and condition*

$$xf(x) \to 0 \quad as \quad x \to \infty \tag{7}$$

*is satisfied, then integral* (5) *converges uniformly in* $t$.

*(ii) Conversely, if* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *and integral* (5) *converges uniformly in* $t$*, then condition* (7) *is satisfied.*

Let $\{c_{kl}\}_{k,l=1}^{\infty}$ be a double sequence of complex numbers (in symbols: $c_{kl}\subset\mathbb{C}$), consider the double sine series

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}c_{kl}\sin kx\sin ly. \tag{8}$$

We use the standard notations for the difference operators:

$$\begin{aligned}\Delta_{10}c_{kl} &:= c_{kl}-c_{k+1,l}, \quad \Delta_{01}c_{kl}:=c_{kl}-c_{k,l+1},\\ \Delta_{11}c_{kl} &:= \Delta_{01}(\Delta_{10}c_{kl})=\Delta_{10}(\Delta_{01}c_{kl})\\ &= c_{kl}-c_{k+1,l}-c_{k,l+1}+c_{k+1,l+1}, \quad k,l=1,2,\cdots.\end{aligned}$$

Recalling that a double sequence $c_{kl}\subset\mathbb{C}$ is said to be monotonically decreasing if

$$c_{kl}\geqslant 0, \quad \Delta_{10}c_{kl}\geqslant 0, \quad \Delta_{01}c_{kl}\geqslant 0, \quad \Delta_{11}c_{kl}\geqslant 0$$

for all $k,l=1,2,\cdots$. It is clear that if

$$c_{kl}\to 0 \quad as \quad k+l\to\infty, \tag{9}$$

then we have

$$\begin{aligned}\Delta_{10}c_{nm} &= \sum_{l=m}^{\infty}\Delta_{11}c_{nl}, \quad \Delta_{01}c_{nm}=\sum_{k=n}^{\infty}\Delta_{11}c_{km},\\ c_{nm} &= \sum_{k=n}^{\infty}\sum_{l=m}^{\infty}\Delta_{11}c_{kl}, \quad n,m=1,2,\cdots.\end{aligned} \tag{10}$$

The two-dimensional extension of the Theorem (1) was proved in [9] as follows:

THEOREM 5. *If $\{c_{kl}\}_{k,l=1}^{\infty}\subset\mathbb{R}_+$ is a monotonically decreasing double sequence, then regular convergence of double sine series* (8) *is uniform in $(x,y)$ if and only if*

$$klc_{kl}\to 0 \quad as \quad k+l\to\infty. \tag{11}$$

In [3], P. Kórus and F. Móricz relaxed the monotonicity condition in Theorem 5 and introduced the class MVBVDS as follows:

DEFINITION 3. A double sequence $\{c_{kl}\}\subset\mathbb{C}$ is said to belong to the class MVB-VDS (mean value bounded variation double sequences) if there exist constants $K_2$ and $\lambda\geqslant 2$ both depending only on $\{c_{kl}\}$ such that

$$\sum_{k=n}^{2n-1}|\Delta_{10}c_{km}|\leqslant\frac{K_2}{n}\sum_{k=[\lambda^{-1}n]}^{[\lambda n]}|c_{km}|, \quad n\geqslant\lambda,\ m=1,2,\cdots, \tag{12}$$

$$\sum_{l=m}^{2m-1}|\Delta_{01}c_{nl}|\leqslant\frac{K_2}{m}\sum_{l=[\lambda^{-1}m]}^{[\lambda m]}|c_{nl}|, \quad m\geqslant\lambda,\ n=1,2,\cdots, \tag{13}$$

$$\sum_{k=n}^{2n-1}\sum_{l=m}^{2m-1}|\Delta_{11}c_{kl}|\leqslant\frac{K_2}{nm}\sum_{k=[\lambda^{-1}n]}^{[\lambda n]}\sum_{l=[\lambda^{-1}m]}^{[\lambda m]}|c_{km}|, \quad n\geqslant\lambda,\ m\geqslant\lambda. \tag{14}$$

They proved

THEOREM 6. *$\{c_{kl}\} \subset \mathbb{C}$ belongs to the class* MVBVDS *and satisfies condition* (11)*, then the regular convergence of the double sine series* (8) *is uniform in* $(x,y)$.

*Conversely, if $\{c_{kl}\} \subset \mathbb{R}_+$ belongs to the class* MVBVDS *and the regular convergence of* (8) *is uniform in* $(x,y)$*, then condition* (11) *is satisfied.*

Throughout the paper, $K_1$ is used to denote a positive constant that may not be necessarily the same at each occurrence. Sometimes, to avoid confusion, we also use $C_1, K_2, K_3, \cdots$ to denote different constants.

In consideration of Theorem 1 was generalized from trigonometric series to sine integral and double trigonometric series, we generalize an important trigonometric inequality to integral inequality and double inequality. In section 2, we introduce the trigonometric inequality and some related work and give our main result. In section 3, we give the proof of our main result.

## 2. The trigonometric inequality

As known, one important tool in Fourier analysis is the following well-known trigonometric inequality (see, e.g., [11])

$$\sup_{n\geqslant 1}\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| \leqslant 3\sqrt{\pi}.$$

we apply the inequality to prove the $L^1$ convergence and Bernstein inequality etc.

In [7], Telyakovskii generalized the inequality and proved the following theorem:

THEOREM 7. *Let $\{n_m\}$ be a subsequence of natural numbers satisfying $1 = n_1 < n_2 < n_3 < \cdots$ and*

$$\sum_{j=m}^{\infty}\frac{1}{n_j} \leqslant \frac{A}{n_m}, \tag{15}$$

*where $m = 1, 2, \cdots$, $A > 1$ is a positive constant, then for any $x$ we have*

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1}\frac{\sin kx}{k}\right| \leqslant K_3 A.$$

In [5], Leindler gave a generalized result and established the following theorem:

THEOREM 8. *Let a positive sequence $\mathbf{C} = \{c_n\}_{n=1}^{\infty} \in RBVS$ satisfy*

$$nc_n \leqslant K_4, n = 1, 2, \cdots, \tag{16}$$

*where $K_4$ is a positive constant, suppose $\{n_m\}$ satisfies* (15)*, then for any $x$, we have*

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx\right| \leqslant K(\mathbf{C})A.$$

In [4], Le and Zhou established the above theorem under GBV condition. Finally, in [8], Wang and Zhao derived the theorem under MVBV condition and proved the condition cannot be weakened further.

THEOREM 9. *Let* $\mathbf{C}=\{c_n\}_{n=1}^{\infty}\in \mathrm{MVBVS}$ *be a positive sequence satisfying* (3) *and* (16)*. If* $\{n_m\}$ *satisfies* (15)*, then for any* $x$ *we have*

$$\sum_{j=1}^{\infty}\left|\sum_{k=n_j}^{n_{j+1}-1} c_k \sin kx\right| \leqslant K(\mathbf{C})A.$$

THEOREM 10. *Let* $M_n$ *be an any given non-negative increasing sequence tending to infinity. Then for any given* $\lambda \geqslant 2$*, there exists a positive sequence* $\mathbf{C}=\{c_n\}_{n=1}^{\infty}$ *satisfying*

$$\lim_{n\to}\frac{\sum\limits_{k=n}^{2n}|\Delta a_k|}{\frac{M_n}{n}\sum\limits_{k=[\lambda^{-1}n]}^{\lambda n} a_k}=0,$$

*however, the trigonometric inequality in Theorem 9 does not hold for some sequence* $n_m$ *satisfying* (15)*.*

Now, we simulate the trigonometric inequality and give the following integral inequality.

LEMMA 1. $\sup\limits_{a>0}\left|\int_0^a \frac{\sin xy}{x}\mathrm{d}x\right| \leqslant 4$*, for all* $y\in[0,\infty)$*.*

*Proof:* We need only prove for $y\in(0,\infty)$.
Case i: $0<y\leqslant \frac{1}{a}$.

$$\left|\int_0^a \frac{\sin xy}{x}\mathrm{d}x\right| = \left|\int_0^{ay}\frac{\sin x}{x}\mathrm{d}x\right| \leqslant \int_0^{ay}\left|\frac{\sin x}{x}\right|\mathrm{d}x \leqslant \int_0^1\left|\frac{\sin x}{x}\right|\mathrm{d}x < 1.$$

Case ii: $y>\frac{1}{a}$.

$$\begin{aligned}\left|\int_0^a \frac{\sin xy}{x}\mathrm{d}x\right| &= \left|\int_0^{ay}\frac{\sin x}{x}\mathrm{d}x\right| \leqslant \left|\int_0^1\frac{\sin x}{x}\mathrm{d}x\right| + \left|\int_1^{ay}\frac{\sin x}{x}\mathrm{d}x\right| \\ &\leqslant 1+\left|\int_1^{ay}\frac{1}{x}\mathrm{d}\cos x\right| \leqslant 1+\frac{|\cos ay|}{ay}+|\cos 1|+\left|\int_1^{ay}\frac{1}{x^2}\mathrm{d}x\right| \leqslant 4. \quad \square\end{aligned}$$

Our first result generalizes Lemma 1. Namely, we obtain that the following theorem is true.

THEOREM 11. *Assume a non-negative function* $f(x)\in \mathrm{MVBVF}(\mathbb{R}_+)$*, if*

$$xf(x)\leqslant C_1, \tag{17}$$

*holds for all* $x \in (0,\infty)$*, where* $C_1$ *is a positive constant depending on* $f$*. Meanwhile,* $\{a_i\}_{i=1}^{\infty}$ *is a sequence satisfying* $a_0 = 0, a_1 = 1 < a_2 < a_3 < \cdots$ *and there exists a positive constant* $A$ *depending on the sequence such that*

$$\sum_{j=i}^{\infty} \frac{1}{a_j} \leqslant \frac{A}{a_i}, \quad i = 1,2,\cdots, \tag{18}$$

*then for any* $y \in [0,\infty)$

$$\sum_{i=0}^{\infty} \left| \int_{a_i}^{a_{i+1}} f(x) \sin xy \mathrm{d}x \right| \leqslant C_2 A, \tag{19}$$

*where* $C_2$ *is a positive constant depending on* $f$ *only.*

Our second result simulates Theorem 9 and proves the MVBVF($\mathbb{R}_+$) condition cannot be weakened further under $f$ being a function of local bounded variation.

THEOREM 12. *Let* $M(x)$ *be a given non-negative increasing function tending to infinity. Then for any given* $\lambda \geqslant 2$*, there exists a positive local bounded variation function* $f(x)$ *satisfying*

$$\lim_{x\to\infty} \frac{\int_x^{2x} |f'(t)| \mathrm{d}t}{\frac{M(x)}{x} \int_{\lambda^{-1}x}^{\lambda x} f(t) \mathrm{d}t} = 0, \tag{20}$$

*however, (19) in Theorem 11 does not hold for some sequence* $\{a_i\}$ *satisfying (18).*

Our third result generalizes Theorem 9 to two-dimensional as follows:

THEOREM 13. *Let* $\mathbf{C} = \{c_{nm}\}_{n,m=1}^{\infty} \in \mathrm{MVBVDS}$ *be a non-negative double positive with*

$$nmc_{nm} \leqslant K, \quad n = 1,2,\cdots; \ m = 1,2\cdots, \tag{21}$$

*where* $K$ *is a positive constant, suppose* $\{n_i\}$*,* $\{m_j\}$ *with the conditions*

$$\begin{aligned} &\sum_{i=k}^{\infty} \frac{1}{n_i} \leqslant \frac{A}{n_k}, \quad k = 1,2,\cdots, \ A > 1, \\ &\sum_{j=l}^{\infty} \frac{1}{m_j} \leqslant \frac{B}{m_l}, \quad l = 1,2,\cdots, \ B > 1, \end{aligned} \tag{22}$$

*where* $\{n_i\}$ *and* $\{m_j\}$ *are subsequences of natural numbers satisfying* $1 = n_1 < n_2 < n_3 < \cdots$*,* $1 = m_1 < m_2 < m_3 < \cdots$*, then for any* $x$ *and* $y$*, we have*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left| \sum_{k=n_i}^{n_{i+1}-1} \sum_{l=m_j}^{m_{j+1}-1} c_{kl} \sin kx \sin ly \right| \leqslant K_1(\mathbf{C}) AB. \tag{23}$$

## 3. The proof of the main result

### 3.1. The proof of the Theorem 11

We need only prove for $y \in (0,\infty)$. Select integer $k$ such that $Y := \frac{1}{y} \in (a_k, a_{k+1}]$.

$$\begin{aligned}\sum_{i=0}^{\infty}\left|\int_{a_i}^{a_{i+1}} f(x)\sin xy\mathrm{d}x\right| &\leqslant \int_0^Y |f(x)\sin xy|\,\mathrm{d}x + \left|\int_Y^{a_{k+1}} f(x)\sin xy\mathrm{d}x\right| \\ &\quad + \sum_{i=k+1}^{\infty}\left|\int_{a_i}^{a_{i+1}} f(x)\sin xy\mathrm{d}x\right| \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

By (17), we have

$$I_1 \leqslant \int_0^Y f(x)xy\mathrm{d}x \leqslant C_1 yY \leqslant C_1.$$

Applying (6) and (17), we obtain

$$\begin{aligned}I_2 &= Y\left|\int_Y^{a_{k+1}} f(x)\mathrm{d}\cos xy\right| \\ &\leqslant Yf(a_{k+1}) + Yf(Y) + Y\left|\int_Y^{a_{k+1}} f'(x)\cos xy\mathrm{d}x\right| \\ &\leqslant a_{k+1}f(a_{k+1}) + Yf(Y) + Y\int_Y^{\infty}\left|f'(x)\right|\mathrm{d}x \\ &\leqslant 2C_1 + Y\sum_{j=0}^{\infty}\int_{2^jY}^{2^{j+1}Y}\left|f'(x)\right|\mathrm{d}x \\ &\leqslant 2C_1 + Y\sum_{j=0}^{\infty}\frac{K_1}{2^jY}\int_{\lambda^{-1}2^jY}^{\lambda 2^jY} f(x)\mathrm{d}x \\ &\leqslant 2C_1 + K_1C_1\sum_{j=0}^{\infty}\int_{\lambda^{-1}2^jY}^{\lambda 2^jY}\frac{1}{x}\mathrm{d}x \\ &\leqslant (2+4K_1\ln\lambda)C_1.\end{aligned}$$

When $i \geqslant k+1$,

$$\begin{aligned}\left|\int_{a_i}^{a_{i+1}} f(x)\sin xy\mathrm{d}x\right| &= Y\left|\int_{a_i}^{a_{i+1}} f(x)\mathrm{d}\cos xy\right| \\ &\leqslant Y\left(f(a_i) + f(a_{i+1}) + \int_{a_i}^{a_{i+1}}|f'(x)|\mathrm{d}x\right) \\ &\leqslant Y\left(\frac{C_1}{a_i} + \frac{C_1}{a_{i+1}} + \int_{a_i}^{\infty}|f'(x)|\mathrm{d}x\right)\end{aligned}$$

$$\begin{aligned}
&\leqslant Y\left(\frac{2C_1}{a_i}+\sum_{j=0}^{\infty}\int_{2^j a_i}^{2^{j+1}a_i}|f'(x)|\mathrm{d}x\right)\\
&\leqslant Y\left(\frac{2C_1}{a_i}+\sum_{j=0}^{\infty}\frac{K_1}{2^j a_i}\int_{\lambda^{-1}2^j a_i}^{\lambda 2^j a_i}f(x)\mathrm{d}x\right)\\
&\leqslant \frac{Y}{a_i}(2+4K_1\ln\lambda)C_1.
\end{aligned}$$

Then, by (18)

$$I_3=\sum_{i=k+1}^{\infty}\frac{Y}{a_i}(2+4K_1\ln\lambda)C_1\leqslant\frac{AY}{a_{k+1}}(2+4K_1\ln\lambda)C_1\leqslant(2+4K_1\ln\lambda)C_1A.$$

Combining the above estimates, we have proved the required inequality. □

### 3.2. The proof of the Theorem 12

Without loss of generality, we can assume that $M(x)\geqslant 10$ when $x\in(0,1)$, therefore, $M(x)\geqslant 10$ when $x\in[1,\infty)$. Set $a_1=1$, $a_2=10$, and $a_{j+1}=2[\sqrt{M(4a_j)}]a_j$, $j=2,3,\cdots$. Let $f(x)=1$, when $x\in(0,40)$. For $j\geqslant 2$ and $k=1,2,\cdots,2[\sqrt{M(4a_j)}]-1$,

$$f(x)=\begin{cases}\dfrac{1}{\sqrt{\log M(\sqrt{4a_j})x}}, & x\in[4ka_j,(4k+2)a_j),\\ \dfrac{1}{8\sqrt{\log M(\sqrt{4a_j})x}}, & x\in[(4k+2)a_j,4(k+1)a_j).\end{cases}$$

Defining accordingly a sine integral $\int_0^\infty f(x)\sin xy\mathrm{d}x$, we will show this integral is exactly what required to prove Theorem 12. This construction implies that inequality (17) is satisfied for such a function $f(x)$ and inequality (18) is also satisfied for such a sequence $\{a_j\}$. For any given $x\geqslant 40$, there exist a $j\geqslant 2$ and a $k$, $k=1,2,\cdots,2[\sqrt{M(4a_j)}]-1$, such that $4ka_j\leqslant x<4(k+1)a_j$, then $8ka_j\leqslant 2x<8(k+1)a_j$. Divide the argument into two cases.

Case 1: $1\leqslant k\leqslant[\sqrt{M(4a_j)}]-1$, then $2x\leqslant 8[\sqrt{M(4a_j)}]a_j=4a_{j+1}$. We check that

$$\begin{aligned}
\int_x^{2x}|f'(t)|\mathrm{d}t\leqslant\int_{4ka_j}^{8(k+1)a_j}|f'(t)|\mathrm{d}t&=\sum_{i=k}^{2k+1}\int_{4ia_j}^{4(i+1)a_j}|f'(t)|\mathrm{d}t\\
&\leqslant\sum_{i=k}^{2k+1}\frac{1}{\sqrt{\log M(4a_j)}}\frac{1}{4ia_j}\leqslant\frac{K}{a_j\sqrt{\log M(4a_j)}}.
\end{aligned}$$

At the same time,

$$\begin{aligned}\int_x^{2x} f(t)\mathrm{d}t &\geqslant \int_{4(k+1)a_j}^{8ka_j} f(t)\mathrm{d}t = \sum_{i=k+1}^{2k-1}\int_{4ia_j}^{4(i+1)a_j} f(t)\mathrm{d}t\\ &\geqslant \frac{1}{8\sqrt{\log M(4a_j)}}\sum_{i=k+1}^{2k-1}\int_{4ia_j}^{4(i+1)a_j}\frac{1}{t}\mathrm{d}t\\ &= \frac{1}{8\sqrt{\log M(4a_j)}}\int_{4(k+1)a_j}^{8ka_j}\frac{1}{t}\mathrm{d}t \geqslant \frac{1}{8\sqrt{\log M(4a_j)}}\frac{(4k-4)a_j}{8ka_j}\\ &\geqslant \frac{1}{32\sqrt{\log M(4a_j)}}.\end{aligned}$$

Thus, by noting that $4a_j \leqslant 4ka_j \leqslant x \leqslant 4(k+1)a_j$, $k \leqslant [\sqrt{M(4a_j)}]-1$, for any $\lambda \geqslant 2$, combining with the two above inequalities, we have

$$\frac{\int_x^{2x}|f'(t)|\mathrm{d}t}{\frac{M(x)}{x}\int_{\lambda^{-1}x}^{\lambda x} f(t)\mathrm{d}t} \leqslant \frac{\int_x^{2x}|f'(t)|\mathrm{d}t}{\frac{M(x)}{x}\int_x^{2x} f(t)\mathrm{d}t} \leqslant \frac{Kx}{a_jM(x)} \leqslant \frac{Ka_j\sqrt{M(4a_j)}}{a_jM(x)} \leqslant \frac{K}{\sqrt{M(x)}}.$$

and the last quantity in the above inequalities obviously tends to zero as $x \to \infty$.

Case (2): $[\sqrt{M(4a_j)}] \leqslant k < 2[\sqrt{M(4a_j)}]-1$. Then $2x \leqslant 16[\sqrt{M(4a_j)}]a_j < 8a_{j+1}$. Similarly, we check for this case that

$$\begin{aligned}\int_x^{2x}|f'(t)|\mathrm{d}t &\leqslant \int_{4ka_j}^{8a_{j+1}}|f'(t)|\mathrm{d}t = \int_{4ka_j}^{4a_{j+1}}|f'(t)|\mathrm{d}t + \int_{4a_{j+1}}^{8a_{j+1}}|f'(t)|\mathrm{d}t\\ &= \sum_{i=k}^{2[\sqrt{M(4a_j)}]-1}\int_{4ia_j}^{4(i+1)a_j}|f'(t)|\mathrm{d}t + \int_{4a_{j+1}}^{8a_{j+1}}|f'(t)|\mathrm{d}t\\ &\leqslant \sum_{i=k}^{2[\sqrt{M(4a_j)}]-1}\frac{1}{\sqrt{\log M(4a_j)}}\frac{1}{4ia_j} + \frac{1}{a_{j+1}\sqrt{\log M(4a_{j+1})}}\\ &\leqslant \frac{K}{a_j\sqrt{\log M(4a_j)}}\sum_{i=[\sqrt{M(4a_j)}]}^{2[\sqrt{M(4a_j)}]-1}\frac{1}{i} + \frac{1}{a_j\sqrt{\log M(4a_j)}}\\ &\leqslant \frac{K}{a_j\sqrt{\log M(4a_j)}}.\end{aligned}$$

On the other hand, by noting that $\frac{x}{2} < 2(k+1)a_j \leqslant 4[\sqrt{M(4a_j)}]a_j$, we achieve that

$$\begin{aligned}\int_{\frac{x}{2}}^{2x} f(t)\mathrm{d}t &\geqslant \int_{4[\sqrt{M(4a_j)}]a_j}^{8ka_j} f(t)\mathrm{d}t = \sum_{i=[\sqrt{M(4a_j)}]}^{2k-1} \int_{4ia_j}^{4(i+1)a_j} f(t)\mathrm{d}t \\ &\geqslant \frac{1}{8\sqrt{\log M(4a_j)}} \sum_{i=[\sqrt{M(4a_j)}]}^{2k-1} \int_{4ia_j}^{4(i+1)a_j} \frac{1}{t}\mathrm{d}t \\ &\geqslant \frac{1}{8\sqrt{\log M(4a_j)}} \sum_{i=[\sqrt{M(4a_j)}]}^{2[\sqrt{M(4a_j)}]} \frac{1}{i+1} \geqslant \frac{K}{\sqrt{\log M(4a_j)}}.\end{aligned}$$

Therefore, for any $\lambda \geqslant 2$, it follows that

$$\frac{\int_x^{2x} |f'(t)|\mathrm{d}t}{\frac{M(x)}{x}\int_{\lambda^{-1}x}^{\lambda x} f(t)\mathrm{d}t} \leqslant \frac{\int_x^{2x} |f'(t)|\mathrm{d}t}{\frac{M(x)}{x}\int_{\frac{x}{2}}^{2x} f(t)\mathrm{d}t} \leqslant \frac{Kx}{a_jM(x)} \leqslant \frac{Ka_j\sqrt{M(4a_j)}}{a_jM(x)} \leqslant \frac{K}{\sqrt{M(x)}}.$$

Combining these two cases, in any circumstance, for any $\lambda \geqslant 2$, we have proved

$$\lim_{x\to\infty} \frac{\int_x^{2x} |f'(t)|\mathrm{d}t}{\frac{M(x)}{x}\int_{\lambda^{-1}x}^{\lambda x} f(t)\mathrm{d}t} = 0.$$

Choose $y_0 = \frac{\pi}{2a_j}$, we have for $k = 1,2,\cdots,2[\sqrt{M(4a_j)}]-1$ that

$$\begin{aligned}\int_{4ka_j}^{(4k+2)a_j} f(x)\sin xy_0\mathrm{d}x &= \int_{4ka_j}^{(4k+2)a_j} f(x)\sin\frac{\pi x}{2a_j}\mathrm{d}x \geqslant \int_{(4k+\frac{1}{2})a_j}^{(4k+\frac{3}{2})a_j} f(x)\sin\frac{\pi x}{2a_j}\mathrm{d}x \\ &\geqslant \frac{\sqrt{2}}{2}\int_{(4k+\frac{1}{2})a_j}^{(4k+\frac{3}{2})a_j} f(x)\mathrm{d}x \geqslant \frac{\sqrt{2}}{2\sqrt{\log M(4a_j)}}\int_{(4k+\frac{1}{2})a_j}^{(4k+\frac{3}{2})a_j} \frac{1}{x}\mathrm{d}x \\ &\geqslant \frac{\sqrt{2}}{\sqrt{\log M(4a_j)}}\frac{1}{8k+3}.\end{aligned}$$

On the other hand, for all $y \in (0,\infty)$,

$$\begin{aligned}\left|\int_{(4k+2)a_j}^{4(k+1)a_j} f(x)\sin xy\mathrm{d}x\right| &\leqslant \int_{(4k+2)a_j}^{4(k+1)a_j} f(x)\mathrm{d}x \leqslant \frac{1}{8\sqrt{\log M(4a_j)}}\int_{(4k+2)a_j}^{4(k+1)a_j} \frac{1}{x}\mathrm{d}x \\ &\leqslant \frac{1}{8\sqrt{\log M(4a_j)}}\frac{2a_j}{(4k+2)a_j} \leqslant \frac{1}{2\sqrt{\log M(4a_j)}}\frac{1}{8k+3}.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\int_{4a_j}^{4a_{j+1}} f(x)\sin xy_0 \mathrm{d}x\right| \\
&\quad = \left|\sum_{k=1}^{2[\sqrt{M(4a_j)}]-1} \left(\int_{4ka_j}^{(4k+2)a_j} f(x)\sin xy_0 \mathrm{d}x + \int_{(4k+2)a_j}^{4(k+1)a_j} f(x)\sin xy \mathrm{d}x\right)\right| \\
&\quad \geqslant \sum_{k=1}^{2[\sqrt{M(4a_j)}]-1} \left(\int_{4ka_j}^{(4k+2)a_j} f(x)\sin xy_0 \mathrm{d}x - \left|\int_{(4k+2)a_j}^{4(k+1)a_j} f(x)\sin xy \mathrm{d}x\right|\right) \\
&\quad \geqslant \frac{1}{2\sqrt{\log M(4a_j)}} \sum_{k=1}^{2[\sqrt{M(4a_j)}]-1} \frac{1}{8k+3} \geqslant \frac{K\log M(4a_j)}{\sqrt{\log M(4a_j)}} \\
&\quad \geqslant K\sqrt{\log M(4a_j)} \to \infty.
\end{aligned}$$

At the same time,

$$\left|\int_{a_{j+1}}^{4a_{j+1}} f(x)\sin xy_0 \mathrm{d}x\right| \leqslant \int_{a_{j+1}}^{4a_{j+1}} f(x)\mathrm{d}x \leqslant \frac{1}{\sqrt{\log M(4a_j)}} \int_{a_{j+1}}^{4a_{j+1}} \frac{1}{x}\mathrm{d}x \leqslant \frac{K}{\sqrt{\log M(4a_j)}} \to 0.$$

Similarly,

$$\left|\int_{a_j}^{4a_j} f(x)\sin xy_0 \mathrm{d}x\right| \to 0.$$

That is to say

$$\left|\int_{a_j}^{a_{j+1}} f(x)\sin xy_0 \mathrm{d}x\right| \to \infty (\text{as } j \to \infty).$$

Thus

$$\sup_{y\in(0,\infty)} \sum_{j=0}^{\infty} \left|\int_{a_j}^{a_{j+1}} f(x)\sin xy \mathrm{d}x\right| \geqslant \left|\int_{a_j}^{a_{j+1}} f(x)\sin xy_0 \mathrm{d}x\right| \to \infty (\text{as } j \to \infty).$$

The conclusion of Theorem 11 cannot hold in this case. □

### 3.3. The proof of the Theorem 13

To prove the Theorem 13, we establish the following Lemma:

LEMMA 2. *If $\{c_{nm}\}$ satisfies the conditions of Theorem 13, then for all $n < N$, $m < M$, $(x,y) \in (0,\pi]^2$, we have*

$$\left|\sum_{k=n}^{N} c_{km}\sin kx\right| \leqslant \frac{6KK_2\pi[1+\ln(2\lambda^2)]}{nmx} := \frac{K_4}{nmx}; \tag{24}$$

$$\left|\sum_{l=m}^{M} c_{nl}\sin ly\right| \leqslant \frac{6KK_2\pi[1+\ln(2\lambda^2)]}{nmy} := \frac{K_4}{nmy}; \tag{25}$$

$$\left|\sum_{k=n}^{N}\sum_{l=m}^{M} c_{kl}\sin kx\sin ly\right| \leqslant \frac{36KK_2\pi^2(1+\ln(2\lambda^2))^2}{xynm} := \frac{K_5}{xynm}. \tag{26}$$

*Proof:* Let $D_n(x) := \sum_{k=1}^{n}\sin kx$, we know $|D_n(x)| \leqslant \frac{\pi}{x}$, $x \in (0,\pi)$. Noticing that $nmc_{nm} \leqslant K$ implies that $c_{nm} \to 0$, $n \to \infty$, we get

$$c_{nm} = \sum_{k=n}^{\infty}\Delta_{10}c_{km} \leqslant \sum_{k=n}^{\infty}|\Delta_{10}c_{km}|. \tag{27}$$

Similarly,

$$c_{nm} \leqslant \sum_{l=m}^{\infty}|\Delta_{10}c_{nl}|. \tag{28}$$

By using Abel's transformation, we see that

$$\begin{aligned}\left|\sum_{k=n}^{N} c_{km}\sin kx\right| &= \left|\sum_{k=n}^{N-1} D_k(x)\Delta_{10}c_{km} + c_{Nm}D_N(x) - c_{nm}D_{n-1}(x)\right|\\ &\leqslant \frac{\pi}{x}\left(\sum_{k=n}^{N-1}|\Delta_{10}c_{km}| + c_{Nm} + c_{nm}\right) \leqslant \frac{3\pi}{x}\sum_{k=n}^{\infty}|\Delta_{10}c_{km}|\\ &\leqslant \frac{3\pi}{x}\sum_{i=0}^{\infty}\sum_{k=2^i n}^{2^{i+1}n-1}|\Delta_{10}c_{km}| \leqslant \frac{3K_2\pi}{x}\sum_{i=0}^{\infty}\frac{1}{2^i n}\sum_{k=[\lambda^{-1}2^i n]}^{[\lambda 2^i n]} c_{km}\\ &\leqslant \frac{3KK_2\pi}{nmx}\sum_{i=0}^{\infty}\frac{1}{2^i}\sum_{k=[\lambda^{-1}2^i n]}^{[\lambda 2^i n]}\frac{1}{k} \leqslant \frac{6KK_2(1+\ln(2\lambda^2))}{nmx}.\end{aligned}$$

Similarly,

$$\left|\sum_{l=m}^{M} c_{nl}\sin ly\right| \leqslant \frac{2KK_2\pi[1+\ln(2\lambda^2)]}{nmy}.$$

Applying double Abel's transformation, (10), (27) and (28), we can obtain that

$$\begin{aligned}\left|\sum_{k=n}^{N}\sum_{l=m}^{M} c_{kl}\sin kx\sin ly\right| &= \left|\sum_{k=n}^{N-1}\sum_{l=m}^{M-1} D_k(x)D_l(y)\Delta_{11}c_{kl} + \sum_{k=n}^{N-1} D_k(x)D_M(y)\Delta_{10}c_{kM}\right.\\ &\quad \left. - \sum_{k=n}^{N-1} D_k(x)D_{m-1}(y)\Delta_{10}c_{km} + \sum_{l=m}^{M-1} D_N(x)D_l(y)\Delta_{01}c_{Nl}\right.\end{aligned}$$

$$
\begin{aligned}
&-\sum_{l=m}^{M-1} D_{n-1}(x)D_l(y)\Delta_{01}c_{nl} + C_{NM}D_N(x)D_M(y) - c_{nM}D_{n-1}(x)D_M(y)\\
&-C_{Nm}D_N(x)D_{m-1}(y) + c_{nm}D_{n-1}(x)D_{m-1}(y)\Bigg|\\
&\leqslant \frac{9\pi^2}{xy}\sum_{k=n}^{\infty}\sum_{l=m}^{\infty}|\Delta_{11}c_{kl}| \leqslant \frac{9\pi^2}{xy}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=2^i n}^{2^{i+1}n-1}\sum_{l=2^j m}^{2^{j+1}m-1}|\Delta_{11}c_{kl}|\\
&\leqslant \frac{9\pi^2}{xy}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2^i n 2^j m}\sum_{k=[\lambda^{-1}2^i n]}^{[\lambda 2^i n]}\sum_{l=[\lambda^{-1}2^j m]}^{[\lambda 2^j m]} c_{kl}\\
&\leqslant \frac{36KK_2\pi^2(1+\ln(2\lambda^2))^2}{xynm}. \quad \square
\end{aligned}
$$

Now, we start to prove Theorem 13. The case $x=0$ or $y=0$ or $x=\pi$ or $y=\pi$ are trivial. Let $(x,y)\in(0,\pi)^2$. Select $n$ and $m$, $p$ and $q$ in turn such that $\frac{\pi}{n+1}\leqslant x<\frac{\pi}{n}$, $n_p\leqslant n<n_{p+1}$, $\frac{\pi}{m+1}\leqslant y<\frac{\pi}{m}$, $m_q\leqslant m<m_{q+1}$. Thus ,we can write

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\sum_{k=n_i}^{n_{i+1}-1}\sum_{l=m_j}^{m_{j+1}-1} c_{kl}\sin kx\sin ly\right|\\
&\leqslant \sum_{k=1}^{n}\sum_{l=1}^{m}|c_{kl}\sin kx\sin ly| + \sum_{k=1}^{n}|\sin kx|\left|\sum_{l=m+1}^{m_{q+1}-1} c_{kl}\sin ly\right| + \sum_{k=1}^{n}|\sin kx|\sum_{j=q+1}^{\infty}\left|\sum_{l=m_j}^{m_{j+1}-1} c_{kl}\sin ly\right|\\
&+\sum_{l=1}^{m}|\sin ly|\left|\sum_{k=n+1}^{n_{p+1}-1} c_{kl}\sin kx\right| + \left|\sum_{k=n+1}^{n_{p+1}-1}\sum_{l=m+1}^{m_{q+1}-1} c_{kl}\sin kx\sin ly\right|\\
&+\sum_{j=q+1}^{\infty}\left|\sum_{k=n+1}^{n_{p+1}-1}\sum_{l=m_j}^{m_{j+1}-1} c_{kl}\sin kx\sin ly\right| + \sum_{l=1}^{m}|\sin ly|\sum_{i=p+1}^{\infty}\left|\sum_{k=n_i}^{n_{i+1}-1} c_{kl}\sin kx\right|\\
&+\sum_{i=p+1}^{\infty}\left|\sum_{k=n_i}^{n_{i+1}-1}\sum_{l=m+1}^{m_{q+1}-1} c_{kl}\sin kx\sin ly\right| + \sum_{i=p+1}^{\infty}\sum_{j=q+1}^{\infty}\left|\sum_{k=n_i}^{n_{i+1}-1}\sum_{l=m_j}^{m_{j+1}-1} c_{kl}\sin kx\sin ly\right|\\
&:= I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.
\end{aligned}
$$

Since $nmc_{nm}\leqslant K$, we easily obtain

$$I_1 \leqslant \sum_{k=1}^{n}\sum_{l=1}^{m} klc_{kl}xy \leqslant Knxmy \leqslant K\pi^2.$$

By (25), we get

$$I_2 \leqslant \sum_{k=1}^{n} kx\frac{K_4}{k(m+1)y} = \frac{nxK_4}{(m+1)y} \leqslant K_4.$$

By (25) and (22), we get

$$I_3 \leqslant \sum_{k=1}^{n} kx \sum_{j=q+1} \frac{K_4}{km_j y} \leqslant \frac{nxK_4B}{ym_{q+1}} \leqslant \frac{nxK_4B}{y(m+1)} \leqslant K_4B.$$

$I_4$ is the symmetric counterpart of $I_2$. Thus,

$$I_4 \leqslant K_4.$$

By (26), we get

$$I_5 \leqslant \frac{K_5}{x(n+1)y(m+1)} \leqslant \frac{K_5}{\pi^2}.$$

By (26) and (22), we get

$$I_6 \leqslant \sum_{j=q+1}^{\infty} \frac{K_5}{xy(n+1)m_j} \leqslant \frac{K_5B}{x(n+1)ym_{q+1}} \leqslant \frac{K_5B}{x(n+1)y(m+1)} \leqslant \frac{K_5B}{\pi^2}.$$

$I_7$ is the symmetric counterpart of $I_3$. Thus

$$I_7 \leqslant K_4A.$$

$I_8$ is the symmetric counterpart of $I_6$. Thus

$$I_8 \leqslant \frac{K_5A}{\pi^2}.$$

By (26) and (22), we get

$$I_q \leqslant \sum_{i=p+1}^{\infty} \sum_{j=q+1}^{\infty} \frac{K_5}{xyn_im_j} \leqslant \frac{K_5AB}{xn_{p+1}ym_{q+1}} \leqslant \frac{K_5AB}{x(n+1)y(m+1)} \leqslant \frac{K_5AB}{\pi^2}.$$

Combining the above estimates $I_1$—$I_9$, we have proved the required inequality. □

REMARK 1. Recently, Feng and Zhou proved the following Theorem in [2]:

THEOREM 14. *Let a real sequence* $\{a_n\} \in$ MVBVS. *Then, for all* $n$ *and* $x \in [0,\pi]$,

$$\sum_{k=1}^{n} a_k \sin kx = O(1)$$

*holds if and only if*

$$na_n = O(1).$$

In [10], Zhao generalized Theorem 14 and established the following theorem:

THEOREM 15. *Assume a non-negative function* $f \in \mathrm{MVBVF}(\mathbb{R}_+)$*, then for any* $a > A,\ t \in [0,\infty)$,

$$\left|\int_a^{\infty} f(x)\sin xt\,\mathrm{d}x\right| = O(1) \tag{29}$$

*holds if and only if*

$$x|f(x)| = O(1). \tag{30}$$

That is to say, Theorem 15 emphasizes (29) and (30) is an equivalence relation under $f(x) \in \mathrm{MVBVF}(\mathbb{R}_+)$. However, our result emphasizes MVBV condition cannot be weakened further to ensure (19) holds. Moreover, our result is different from Theorem 15 on two aspects: First, $xf(x) = O(1)$ is a condition in our paper rather than a condition in Theorem 15. Second, our inequality (19) is different from (29), which means our object is different. Thus, on the basis of two papers' conclusions, we give an open question:

THEOREM 16. *Assume a non-negative* $f(x) \in \mathrm{MVBVF}(\mathbb{R}_+)$ *the sequence* $\{a_i\}_{i=1}^{\infty}$ *satisfies* $a_0 = 0, a_1 = 1 < a_2 < a_3 < \cdots$ *and there exists a positive constant* $A$ *depending on the sequence such that*

$$\sum_{j=i}^{\infty} \frac{1}{a_j} \leqslant \frac{A}{a_i}, \quad i = 1,2,\cdots,$$

*then for all* $y \in [0,\infty)$,

$$\sum_{i=0}^{\infty}\left|\int_{a_i}^{a_{i+1}} f(x)\sin xy\,\mathrm{d}x\right| = O(1)$$

*holds if and only if*

$$xf(x) = O(1).$$

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