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SOME INEQUALITIES ON THE PERRON EIGENVALUE AND EIGENVECTORS FOR POSITIVE TENSORS

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Abstract. In this paper, we obtain new bounds for the maximal eigenvalue and eigenvectors of positive tensors and compare these bounds with the known bounds. Numerical experiments are given to validate the efficiency of our new bounds.

1. Introduction

In 2005, Qi [1] and Lim [2] introduced eigenvalues for higher order tensors independently. Since then, eigenvalue problems of tensors have become an important topic of study in numerical multilinear algebra, and they have wide applications in magnetic resonance imaging [3], multilinear pagerank [4], spectral hypergraph theory [5], higher order Markov chains [6, 10], algebraic geometry [7], and so on.

Recently, spectral theory of nonnegative tensors developed rapidly. In particular, the Perron-Frobenius theorem for nonnegative tensor is related to measuring higher order connectivity in hypergraphs [8]. Some nonnegative tensor versions of the Perron-Frobenius theorem were given by Chang et al [9], Yang and Yang [12], and Friedland et al [20], respectively. Subsequently, various algorithms for finding the largest eigenvalue of nonnegative tensors were proposed by [17–18, 23], and some results on the convergence of these algorithms were given, see [19, 21–22], etc.

The remainder of this paper is organized as follows. In section 2, we first recall some definitions and theorems. In section 3, we give new bounds on the maximal eigenvalue and eigenvectors of positive tensors and a numerical example to show the efficiency of our new bounds.

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2. Preliminaries

Let $\mathbb{R}$ be the real field. We consider a tensor $\mathscr{A}$ of order m dimensional n consisting of n^m entries in $\mathbb{R}$:

$$\mathscr{A} = (a_{i_1 i_2 \cdots i_m}), \quad a_{i_1 i_2 \cdots i_m} \in \mathbb{R}, \quad 1 \leqslant i_1, \cdots, i_m \leqslant n. \tag{1}$$

The tensor $\mathscr{A}$ is called nonnegative (positive) if $a_{i_1 i_2 \cdots i_m} \geqslant 0$ $(a_{i_1 i_2 \cdots i_m} \geqslant 0)$. For an n dimension vector $x = (x_1, x_2, \cdots, x_n)^T$, real or complex, $\mathscr{A}x^{m-1}$ is an n dimension vector whose ith component is

$$(\mathscr{A}x^{m-1})_i = \sum_{i_2 \cdots i_m} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}. \tag{2}$$

The unit tensor of order m dimension n is the tensor $\mathscr{I} = (\delta_{i_1 i_2 \cdots i_m})$ with entries as follows:

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots i_m, \\ 0, & \text{otherwise.} \end{cases}$$

In 2005, Qi [1] and Lim [2] independently introduced the notion of eigenvalue problems for symmetric tensors, and the notion has been generalized in Chang et al [9]. Here we use the definition in [1].

We denote the space of all tensors of order m dimension n by $\mathscr{R}^{[m,n]}$. We denote $\mathscr{R}_+^{[m,n]}$ $(\mathscr{R}_{++}^{[m,n]})$ to be the space of all nonnegative (positive) tensors of order m dimension n. $\mathbb{R}^n$ $(\mathbb{C}^n)$ denotes the n dimensional vector space over the real(complex) field.

DEFINITION 1. ([1]) A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue and eigenvector of $\mathscr{A} \in \mathscr{R}^{[m,n]}$ if they satisfy $\mathscr{A}x^{m-1} = \lambda x^{[m-1]}$, where $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \cdots, x_n^{m-1})^T$. Furthermore, we say λ is an H-eigenvalue with the corresponding H-eigenvector x (or (λ, x) is an H-eigenpair) of $\mathscr{A}$ if they are both real.

DEFINITION 2. ([9]) A tensor $\mathscr{A} \in \mathscr{R}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $I \subset 1, 2, \cdots, n$ such that

$$a_{i_1 i_2 \cdots i_m} = 0, \text{ for all } i_1 \in I \text{ and for all } i_2, \cdots, i_m \notin I.$$

If $\mathscr{A}$ is not reducible, then we call $\mathscr{A}$ irreducible.

The following Perron-Frobenius theorem for nonnegative tensors are listed for reference.

THEOREM 1. ([9]) *If $\mathscr{A} \in \mathscr{R}_+^{[m,n]}$, then there exist $\lambda_0 \geqslant 0$ and a nonnegative vector $x_0 \neq 0$ such that*

$$\mathscr{A}x_0^{m-1} = \lambda x_0^{[m-1]}. \tag{3}$$

THEOREM 2. ([9]) *If* $\mathscr{A} \in \mathscr{R}_{+}^{[m,n]}$ *is irreducible, then the pair* (λ_0, x) *in equation (3) satisfies the following:*

(1) The eigenvalue λ_0 *is positive.*

(2) The eigenvector x_0 *is positive, i.e., all components of* x_0 *are positive.*

(3) If λ *is an eigenvalue with a nonnegative eigenvector, then* $\lambda = \lambda_0$. *Moreover, the nonnegative eigenvector is unique up to a multiplicative constant.*

(4) If λ *is an eigenvalue of* $\mathscr{A}$, *then* $|\lambda| \leqslant \lambda_0$.

In [12], Yang and Yang gave the definition and a simple estimate of the spectral radius of a tensor $\mathscr{A}$ and proved that for any nonnegative tensor, the spectral radius is the largest eigenvalue of it, the related definitions and theorem are as follows.

DEFINITION 3. ([12]) Let $\mathscr{A} \in \mathscr{R}^{[m,n]}$. We call $\rho(\mathscr{A})$ the spectral radius of $\mathscr{A}$ if it equals the largest absolute eigenvalue of $\mathscr{A}$, i.e.,

$$\rho(\mathscr{A}) = max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathscr{A}\}.$$

DEFINITION 4. ([12]) $\mathscr{A} \in \mathscr{R}_{+}^{[m,n]}$ is irreducible, a positive eigenvector corresponding to $\rho(\mathscr{A})$ is called a maximal eigenvector of $\mathscr{A}$.

THEOREM 3. ([12]) *If* $\mathscr{A} \in \mathscr{R}_{+}^{[m,n]}$, *then* $\rho(\mathscr{A})$ *is an eigenvalue with a nonnegative eigenvector* y *corresponding to it.*

THEOREM 4. ([12]) *Let* $\mathscr{A} \in \mathscr{R}_{+}^{[m,n]}$. *Then*

$$r = \min_{1 \leqslant i \leqslant n} \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} \leqslant \rho(\mathscr{A}) \leqslant \max_{1 \leqslant i \leqslant n} \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} = R. \tag{4}$$

Recently, Wang and Wu [15] gave new bounds for the spectral radius and maximal eigenvectors of a positive tensor, that is

THEOREM 5. ([15]) *Let* $\mathscr{A} \in \mathscr{R}_{++}^{[m,n]}$ *with maximal eigenvalue* $\rho(\mathscr{A})$. *If* $R > r$, *then*

$$r + m(\frac{1}{\sqrt{\delta}} - 1) \leqslant \rho(\mathscr{A}) \leqslant R - m(1 - \sqrt{\delta}), \tag{5}$$

where $\delta = \max\limits_{r_i < r_j} \frac{r_i}{r_j}$.

REMARK 1. From inequality (5), it is easy to see that $\delta = \max\limits_{r_i < r_j} \frac{r_i}{r_j} < 1$ and $m > 0$, then $\sqrt{\delta} < 1$. Thus, $r + m(\frac{1}{\sqrt{\delta}} - 1) > r$ and $R - m(1 - \sqrt{\delta}) < R$. Namely, the bounds in (5) are sharper than those in Theorem 4.

THEOREM 6. ([15]) *Let* $\mathscr{A} \in \mathscr{R}_{++}^{[m,n]}$ *with maximal eigenvalue* $\rho(\mathscr{A})$*. Let* $\sigma = \sqrt{\frac{r-m}{R-m}}$*. Then*

$$r + m\left(\frac{1}{\sigma} - 1\right) \leqslant \rho(\mathscr{A}) \leqslant R - m(1-\sigma). \tag{6}$$

REMARK 2. Note that the bounds in (6) are shaper than those in Theorem 5,

$$\sigma^2 = \frac{r-m}{R-m} \leqslant \frac{r}{R} \leqslant \max_{r_i < r_j} \frac{r_i}{r_j} = \delta.$$

THEOREM 7. ([15]) *Let* $\mathscr{A} \in \mathscr{R}_{++}^{[m,n]}$ *with maximal eigenvector* $x = (x_1, x_2, \cdots, x_n)$ *and* $\gamma = \max_{i,j} \frac{x_i}{x_j}$*. Then*

$$\sqrt{\frac{R}{r}} \leqslant \gamma^{m-1} \leqslant \max_{s,t,i_2,\cdots,i_m} \frac{a_{si_2\cdots i_m}}{a_{ti_2\cdots i_m}}. \tag{7}$$

The left inequality in (7) is an equality if and only if $r = R$*. Equality holds on the right-hand side of (7) if and only if* $r_p = kr_q$*, for some pair of indices* p *and* q *satisfying* $\frac{a_{pi_2\cdots i_m}}{a_{qi_2\cdots i_m}} = \max_{s,t,i_2,\cdots,i_m} \frac{a_{si_2\cdots i_m}}{a_{ti_2\cdots i_m}} = \frac{M}{m}$*.*

3. Main results

Let $\mathscr{A} \in \mathscr{R}_{+}^{[m,n]}$. If we put

$$r_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}, \quad R = \max_i r_i, \quad r = \min_i r_i,$$

and denote by $\kappa_1, \kappa_2, m, \tau_1, \tau_2, M$ respectively the smallest $a_{i,\cdots,i}$, the smallest $a_{i,i_2,\cdots,i_m}$ $((i_2,\cdots,i_m) \neq (i,\cdots,i))$, the smallest $a_{i,i_2,\cdots,i_m}$ $(i,i_2\cdots,i_m = 1,\cdots,n)$, the greatest $a_{i,\cdots,i}$, the greatest $a_{i,i_2,\cdots,i_m}$ $((i_2,\cdots,i_m) \neq (i,\cdots,i))$ and the greatest $a_{i,i_2,\cdots,i_m}$ $(i,i_2\cdots,i_m = 1,\cdots,n)$, i.e.

$$\kappa_1 = \min_i a_{i,\cdots,i}, \quad \kappa_2 = \min_{(i_2,\cdots,i_m)\neq(i,\cdots,i)} a_{i,i_2,\cdots,i_m}, \quad m = \min_{i,i_2,\cdots,i_m} a_{i,i_2,\cdots,i_m}$$

$$\tau_1 = \max_i a_{i,\cdots,i}, \quad \tau_2 = \max_{(i_2,\cdots,i_m)\neq(i,\cdots,i)} a_{i,i_2,\cdots,i_m}, \quad M = \max_{i,i_2,\cdots,i_m} a_{i,i_2,\cdots,i_m}.$$

It is easy to know that $m = min(\kappa_1, \kappa_2)$.

We first introduce a lemma and give new bounds for maximal eigenvectors of positive tensors. These bounds is derived using a technique due to Ostrowski [13] and Schneider [14].

LEMMA 1. ([11]) *If* $q_1, q_2, \cdots, q_n$ *are positive numbers, then*

$$\min_i \frac{p_i}{q_i} \leqslant \frac{p_1 + p_2 + \cdots + p_n}{q_1 + q_2 + \cdots + q_n} \leqslant \max_i \frac{p_i}{q_i}, \tag{8}$$

for any real numbers $p_1, p_2, \cdots, p_n$*. Equality holds on either side of (8) if and only if all the ratios* $\frac{p_i}{q_i}$ *are equal.*

THEOREM 8. *Let* $\mathscr{A} \in \mathscr{R}_{++}^{[m,n]}$ *with maximal eigenvector* $x = (x_1, x_2, \cdots, x_n)^T$ *and* $\gamma = \max_{i,j} \frac{x_i}{x_j}$. *Then*

$$\sqrt{\frac{R-\kappa_1}{r-\kappa_1}} \leqslant \gamma^{m-1} \leqslant \frac{max(\tau_1+\kappa_2-\kappa_1, \tau_2)}{\kappa_2}. \tag{9}$$

Proof. Let ρ be the spectral radius of $\mathscr{A}$ and $x = (x_1, x_2, \cdots, x_n)^T$ be a positive eigenvector corresponding to ρ. Without loss of generality, we can assume that $0 < x_n \leqslant \cdots \leqslant x_2 \leqslant x_1 = 1$. We have obviously $\gamma = \frac{1}{x_n}$. From Theorem 1, we have

$$\rho x_i^{m-1} = \sum_{i_2,\cdots,i_m}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}, \tag{10}$$

for $i = 1, 2, \cdots, n$.

We now choose i in (10) such that $R = r_i$. Then we have

$$\begin{aligned}
(\rho - a_{i\cdots i})x_i^{m-1} &= \sum_{(i_2,\cdots,i_m)\neq(i,\cdots,i)}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \\
&= a_{i1\cdots 1}x_1^{m-1} + \sum_{\substack{(i_2,\cdots,i_m)\neq(i,\cdots,i) \\ (i_2,\cdots,i_m)\neq(1,\cdots,1)}}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \\
&\geqslant a_{i1\cdots 1} + (R - a_{ii\cdots i} - a_{i1\cdots 1})x_n^{m-1} \\
&= (R - a_{ii\cdots i})x_n^{m-1} + (1 - x_n^{m-1})a_{i1\cdots 1},
\end{aligned}$$

where for $i = 1$ we write 0 for $a_{11\cdots 1}$. As $x_n \leqslant 1$ and $a_{i1\cdots 1} > 0$, it follows that

$$(\rho - a_{i\cdots i}) \geqslant (\rho - a_{i\cdots i})x_i^{m-1} \geqslant (R - a_{ii\cdots i})x_n^{m-1},$$

and therefore

$$x_n^{m-1} \leqslant \frac{\rho - a_{i\cdots i}}{R - a_{ii\cdots i}}.$$

From Theorem 4, we know that $\frac{\rho}{R} \leqslant 1$, and do not decrease the right-hand bound by replacing $a_{ii\cdots i}$ by κ_1 and so

$$x_n^{m-1} \leqslant \frac{\rho - \kappa_1}{R - \kappa_1},$$

that is

$$\gamma^{m-1} = \frac{1}{x_n^{m-1}} \geqslant \frac{R - \kappa_1}{\rho - \kappa_1}. \tag{11}$$

Similarly, taking i in (10) such that $r = r_i$, we have

$$\begin{aligned}(\rho - a_{i\cdots i})x_i^{m-1} &= \sum_{(i_2,\cdots,i_m)\neq(i,\cdots,i)}^{n} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m} \\ &= a_{in\cdots n}x_n^{m-1} + \sum_{\substack{(i_2,\cdots,i_m)\neq(i,\cdots,i)\\(i_2,\cdots,i_m)\neq(n,\cdots,n)}}^{n} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m} \\ &\leqslant a_{in\cdots n}x_n^{m-1} + (r - a_{ii\cdots i} - a_{in\cdots n}) \\ &= (r - a_{ii\cdots i}) - (1 - x_n^{m-1})a_{in\cdots n},\end{aligned}$$

where for $i = n$ we write 0 for $a_{nn\cdots n}$. In any case it follows

$$(\rho - a_{i\cdots i})x_n^{m-1} \leqslant (\rho - a_{i\cdots i})x_i^{m-1} \leqslant r - a_{i\cdots i}, \quad x_n^{m-1} \leqslant \frac{r - a_{i\cdots i}}{\rho - a_{ii\cdots i}}.$$

As $\frac{r}{\rho} \leqslant 1$, we do not decrease the right-hand bound replacing $a_{i\cdots i}$ by κ_1 and therefore

$$x_n^{m-1} \leqslant \frac{r - \kappa_1}{\rho - \kappa_1}, \quad \gamma^{m-1} = \frac{1}{x_n^{m-1}} \geqslant \frac{\rho - \kappa_1}{r - \kappa_1}. \tag{12}$$

From (11) and (12), it follows that

$$\gamma^{2(m-1)} \geqslant \frac{R - \kappa_1}{r - \kappa_1}, \quad \gamma^{m-1} \geqslant \sqrt{\frac{R - \kappa_1}{r - \kappa_1}}. \tag{13}$$

To prove that the upper bound of equation (9), we define a new positive tensor $\mathscr{B} = (\kappa_2 - \kappa_1)\mathscr{I} + \mathscr{A}$, for which ρ is replaced by $\rho + \kappa_2 - \kappa_1$, but for which all x_i in (10) remain unchanged, hence γ is unchanged too. On the other hand, for the positive tensor B, κ_1 and τ_1 become κ_2 and $\tau_1 + \kappa_2 - \kappa_1$, respectively, while κ_2 and τ_2 remain unchanged. Further, m becomes κ_2, and M becomes $max(\tau_1 + \kappa_2 - \kappa_1, \tau_2)$. Applying the upper bound of Theorem 7 for the tensor $\mathscr{B}$, we obtain inequality (9). $\square$

REMARK 3. We next give a simple comparison between the bounds in (8) and the bounds in (7). If $\kappa_1 \leqslant \kappa_2$, then $m = \min(\kappa_1, \kappa_2) = \kappa_1$. Thus, we have $\tau_1 + \kappa_2 - \kappa_1 \leqslant M + \kappa_2 - \kappa_1$ and $\tau_2 \leqslant M \leqslant M + \kappa_2 - \kappa_1$. Further, we have

$$max(\tau_1 + \kappa_2 - \kappa_1, \tau_2) \leqslant M + \kappa_2 - \kappa_1.$$

Hence, we have

$$\frac{max(\tau_1 + \kappa_2 - \kappa_1, \tau_2)}{\kappa_2} \leqslant \frac{M + \kappa_2 - \kappa_1}{\kappa_2}.$$

While

$$\frac{M + \kappa_2 - \kappa_1}{\kappa_2} - \frac{M}{m} = \frac{(\kappa_1 - \kappa_2)(M - \kappa_1)}{\kappa_1 \kappa_2} \leqslant 0.$$

Thus, we have

$$\frac{max(\tau_1 + \kappa_2 - \kappa_1, \tau_2)}{\kappa_2} \leqslant \frac{M + \kappa_2 - \kappa_1}{\kappa_2} \leqslant \frac{M}{m}.$$

If $\kappa_1 > \kappa_2$, then $m = \min(\kappa_1, \kappa_2) = \kappa_2$, and $\frac{M}{m} = \frac{M}{\kappa_2}$. But $\kappa_2 < \kappa_1$ implies that

$$\tau_1 + \kappa_2 - \kappa_1 \leqslant M + \kappa_2 - \kappa_1 \leqslant M.$$

Since $\tau_2 \leqslant M$. Hence, we have

$$\frac{max(\tau_1 + \kappa_2 - \kappa_1, \tau_2)}{\kappa_2} \leqslant \frac{M}{m}.$$

Furthermore, as $\frac{r-\kappa_1}{R-\kappa_1} \leqslant \frac{r}{R}$, then $\sqrt{\frac{R-\kappa_1}{r-\kappa_1}} \geqslant \sqrt{\frac{R}{r}}$. Therefore, the bounds of (9) is shaper than those in Theorem 7.

Based on Theorem 8, we establish new bounds for the spectral radius of positive tensors.

THEOREM 9. *Let* $\mathscr{A} \in \mathscr{R}_{++}^{[m,n]}$ *with maximal eigenvalue* ρ. *Let* $\theta = \sqrt{\frac{r-\kappa_1}{R-\kappa_1}}$. *Then*

$$r + m\Big(\frac{1}{\theta} - 1\Big) \leqslant \rho \leqslant R - m(1-\theta). \tag{14}$$

Proof. Let $x = (x_1, x_2, \cdots, x_n)^T$ be a positive maximal eigenvector of $\mathscr{A}$ corresponding to ρ. Without loss of generality, we can assume that $0 < x_n \leqslant \cdots \leqslant x_2 \leqslant x_1 = 1$. We have obviously $\gamma = \frac{1}{x_n}$. Similar to the proof of Theorem 5, we have

$$\begin{aligned}
\rho x_n^{m-1} &= \sum_{i_2,\cdots,i_m}^{n} a_{ni_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\
&= a_{n1\cdots 1} x_1^{m-1} + \sum_{(i_2,\cdots,i_m)\neq(1,\cdots,1)}^{n} a_{ni_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\
&\geqslant a_{n1\cdots 1} + x_n^{m-1} \sum_{(i_2,\cdots,i_m)\neq(1,\cdots,1)}^{n} a_{ni_2\cdots i_m} \\
&= a_{n1\cdots 1} + x_n^{m-1}(r_n - a_{n1\cdots 1}) \\
&= (1 - x_n^{m-1}) a_{n1\cdots 1} + r_n x_n^{m-1}.
\end{aligned}$$

Therefore,

$$\rho \geqslant r_n + a_{n1\cdots 1}\Big(\frac{1}{x_n^{m-1}} - 1\Big) \geqslant r + m\Big(\frac{1}{x_n^{m-1}} - 1\Big). \tag{15}$$

From Theorem 8, we know that

$$\frac{1}{x_n^{m-1}} = \gamma^{m-1} \geqslant \sqrt{\frac{R-\kappa_1}{r-\kappa_1}} = \frac{1}{\theta}.$$

Hence, we have

$$\rho \geqslant r + m\Big(\frac{1}{\theta} - 1\Big). \tag{16}$$

Similarly,

$$\begin{aligned}\rho=\rho x_1^{m-1} &= \sum_{i_2,\cdots,i_m}^{n} a_{1i_2\cdots i_m}x_{i_2}\cdots x_{i_m}\\ &= a_{1n\cdots n}x_n^{m-1}+\sum_{(i_2,\cdots,i_m)\neq(n,\cdots,n)}^{n} a_{1i_2\cdots i_m}x_{i_2}\cdots x_{i_m}\\ &\leqslant a_{1n\cdots n}x_n^{m-1}+\sum_{(i_2,\cdots,i_m)\neq(n,\cdots,n)}^{n} a_{ni_2\cdots i_m}\\ &= a_{1n\cdots n}x_n^{m-1}+(r_1-a_{1n\cdots n})\\ &= r_1-a_{1n\cdots n}(1-x_n^{m-1})\leqslant r_1-m(1-x_n^{m-1}).\end{aligned}$$

As $x_n^{m-1}\leqslant\theta$, we have

$$\rho\leqslant r_1-m(1-\theta)\leqslant R-m(1-\theta). \tag{17}$$

This completes the proof. □

REMARK 4. It is well known that the function $\frac{r-x}{R-x}$ is decreasing. As $m\leqslant\kappa_1$, then it follows that $\sigma=\sqrt{\frac{r-m}{R-m}}\geqslant\sqrt{\frac{r-\kappa_1}{R-\kappa_1}}=\theta$. Further, we have that $R-m(1-\theta)\leqslant R-m(1-\sigma)$ and $r+m(\frac{1}{\theta}-1)\geqslant r+m(\frac{1}{\sigma}-1)$. That is, the bound in (14) are sharper than that in Theorem 6.

EXAMPLE 1. We now show the efficiency of the new bounds in Theorem 8 and Theorem 9 by the following example which is considered in [16]. Let $\mathscr{A}\in\mathscr{R}_{++}^{[4,2]}$ with entries defined as follows:

$$a_{1111}=\frac{1}{2},\quad a_{2222}=3,\quad \text{and}\quad a_{ijkl}=\frac{1}{3}\quad \text{elsewhere.}$$

We compute the bounds of the maximal eigenvalue ρ of $\mathscr{A}$ given by Theorem 4-Theorem 6, Theorem 9.

Theorem 4: $2.8333\leqslant\rho\leqslant 5.3333$.
Theorem 5: $2.9573\leqslant\rho\leqslant 5.2429$.
Theorem 6: $2.9713\leqslant\rho\leqslant 5.2357$.
Theorem 9: $2.9797\leqslant\rho\leqslant 5.2316$.

And the bounds of the maximal eigenvector of $\mathscr{A}$ given by Theorem 7 and Theorem 8.

Theorem 7: $1.3720\leqslant\gamma^3\leqslant 9$.
Theorem 8: $1.4392\leqslant\gamma^3\leqslant 8.4999$.

4. Conclusions

In this paper, we obtain new bounds for the maximal eigenvalue and eigenvector of positive tensors using a technique due to Ostrowski [13] and Schneider [14], and prove that new bounds are shaper than that in [15]. From the proof of Theorem 9, we see that the bound for the spectral radius of positive tensors $\mathscr{A}$ is closely related to the lower bound for the maximal eigenvector. Hence, Future work should be aimed at improving the lower bound of the maximal eigenvector.

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