WEIGHTED POINCARÉ INEQUALITIES
ON HALF SPACES IN CARNOT GROUPS

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Abstract. We prove some weighted Poincaré inequalities on half spaces for the sublaplacian in Carnot groups. Furthermore, the constants we obtain are sharp.

1. Introduction

The Hardy inequality in $\mathbb{R}^N_+$ reads as follows, for all $u \in C_0^\infty(\mathbb{R}^N_+)$ and $N \geq 3$,

$$\int_{\mathbb{R}^N_+} |\nabla u|^2 \, dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N_+} \frac{u^2}{|x|^2} \, dx,$$

where $\mathbb{R}^N_+ = \{(x_1, \cdots, x_n) | x_1 > 0\}$, and the constant $\frac{N^2}{4}$ in (1) is sharp (see [5] or [4]). This shows that the Hardy constant jumps from $\frac{(N-2)^2}{4}$ to $\frac{N^2}{4}$ when the singularity of the potential reaches the boundary since the Hardy inequality in $\mathbb{R}^N$ is

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx.$$ (2)

Inequality (1) has been generalized by Su et al ([7]) to the cone $\mathbb{R}^N_+ := \mathbb{R}^{N-k} \times (\mathbb{R}_+)^k = \{(x_1, \cdots, x_N) | x_{N-k+1} > 0, \cdots, x_N > 0\}.$

Recently, F. Ferrari and E. Valdinoci ([3]) proved few Poincaré inequalities through the stable solutions of suitable PDEs. By the choice of suitable stable solutions, they obtain some weighted Poincaré inequalities, among other results, for the Kohn’s sublaplace operator in the Heisenberg group $\mathbb{H}^n$ and the sublaplace operator in the Engel group. The sharp constants had been shown by Yang and the first author ([9], see also [8]).

The aim of this note is to prove similar results for the sublaplacian on half space in Carnot groups $G$. Recall that a Carnot group $G$ is a stratified, simply connected nilpotent Lie group with the Lie algebra $g = \bigoplus_{j=1}^r V_j$ satisfying $[V_1, V_j] = V_{j+1}$ for all $1 \leq j \leq r-1$. The integer $r$ is called the step of the group $G$. Set $n_j = \dim V_j$ ($1 \leq j \leq r$). Let $\{X_1, \cdots, X_{n_1}\}$ be a basis of $V_1$ and denote by $\nabla_G = (X_1, \cdots, X_{n_1})$. Set $\xi^{(1)} = \xi_1 X_1 + \cdots + \xi_{n_1} X_{n_1}$ and $|\xi^{(1)}| = \sqrt{\xi_1^2 + \cdots + \xi_{n_1}^2}$. To this end, we have

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THEOREM 1. Let $\alpha \geq 0$. If $G$ is a Carnot group of step $r$, then for $\alpha \geq 0$ and $\phi \in C^0_0(G_+)$, there holds,
\[
\left( \frac{n_1 + 2 + \alpha}{4} - \alpha - 2 \right) \int_{G_+} |\phi|^2 |\xi^{(1)}|^\alpha \leq \int_{G_+} |\nabla_G \phi|^2 |\xi^{(1)}|^\alpha + 2,
\]
where $G_+ = \{ \xi \in G : \xi_1 > 0 \}$ is the half space of $G$. Furthermore, the constant \[
\frac{(n_1 + 2 + \alpha)^2}{4} - \alpha - 2 \text{ in (3)} \text{ is sharp.}
\]

In order to prove Theorem 1.1, we use a new technique which is different from that in [4, 5, 7]. In fact, it seems that the method used in [4, 5, 7] can not be applied to the sublaplacian on Carnot groups. Our result also shows that the sharp constant jumps since the sharp Poincaré inequalities with weights on Carnot groups is (see [8])
\[
\left( \frac{n_1 + \alpha}{2} \right)^2 \int_G |\phi|^2 |\xi^{(1)}|^\alpha \leq \int_G |\nabla_G \phi|^2 |\xi^{(1)}|^\alpha + 2.
\]

2. The proof

We begin by quoting some preliminary facts which will be needed in the sequel and refer to [1] and [2] for more precise information about Carnot group. Let $G$ be a Carnot groups. The Lie algebra $g = \bigoplus_{i=1}^r V_i$ of $G$ satisfies $[V_i, V_j] = V_{j+1}$ for all $1 \leq j \leq r - 1$. As a simply connected nilpotent group, $G$ is differential with $\mathbb{R}^N$, $N = \sum_{i=1}^r \dim V_i = \sum_{i=1}^r n_i$, via the exponential map $\exp : g \rightarrow G$. The Haar measure on $G$ is induced by the exponential mapping from the Lebesgue measure on $g = \mathbb{R}^N$ and coincides with the Lebesgue measure on $\mathbb{R}^N$.

EXAMPLE 1. The Heisenberg group $\mathbb{H}^n$ is the Carnot group of step two whose group structure is given by
\[
(x, t) \circ (x', t') = (x + x', t + t' + 2 \sum_{j=1}^n (x_{2j} x'_{2j-1} - x_{2j-1} x'_{2j})).
\]

The vector fields
\[
X_{2j-1} = \frac{\partial}{\partial x_{2j-1}} + 2x_{2j} \frac{\partial}{\partial t},
\]
\[
X_{2j} = \frac{\partial}{\partial x_{2j}} - 2x_{2j-1} \frac{\partial}{\partial t},
\]
($j = 1, \cdots, n$) are left invariant and generate the Lie algebra of $\mathbb{H}^n$. The horizontal gradient on $\mathbb{H}^n$ is the $(2n)$-dimensional vector given by
\[
\nabla_{\mathbb{H}} = (X_1, \cdots, X_{2n}) = \nabla_x + 2\Lambda x \frac{\partial}{\partial t},
\]
where $\nabla_x = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{2n}})$, $\Lambda$ is a skew symmetric and orthogonal matrix given by
\[
\Lambda = \text{diag}(J_1, \cdots, J_n), \quad J_1 = \cdots = J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Consider \( \xi = (\xi^{(1)}, \ldots, \xi^{(r)}) \in \mathbb{R}^N \) with \( \xi^{(i)} = (\xi^{(i)}_1, \ldots, \xi^{(i)}_{n_i}) \in \mathbb{R}^{n_i} \). For \( j = 1, \ldots, n_1 \), let \( X_j \) be the unique vector field in \( g \) that coincides with \( \partial / \partial \xi^{(1)}_j \) at the origin. The second order differential operator

\[
\Delta_G = -\sum_{j=1}^{n_1} X_j^* X_j = \sum_{j=1}^{n_1} X_j^2
\]

is called a sub-Laplacian on \( G \). We shall denote by the \( \nabla_G = (X_1, \ldots, X_{n_1}) \) the related subelliptic gradient. By the Campbell-Hausdorff formula (see e.g. [6], page 2-4), \( X_j \) can be expressed as the following

\[
X_j = \partial / \partial \xi^{(1)}_j + \sum_{k=2}^{n_1} \sum_{s=1}^{n_k} p^{(j)}_{k,s}(\xi^{(1)}, \ldots, \xi^{(k-1)}) \frac{\partial}{\partial \xi^{(k)}_s}, \tag{5}
\]

where \( p^{(j)}_{k,s}(\xi^{(1)}, \ldots, \xi^{(k-1)}) \) is a polynomial of \( \xi^{(1)}, \ldots, \xi^{(k-1)} \). Therefore, for \( |\xi^{(1)}| \neq 0 \), we have

\[
\Delta_G(\xi_1|\xi^{(1)}|^{\alpha}) = \sum_{j=1}^{n_1} \left( \frac{\partial}{\partial \xi^{(1)}_j} \right)^2 (\xi_1|\xi^{(1)}|^{\alpha})
\]

\[
= \xi_1 \sum_{j=1}^{n_1} \left( \frac{\partial}{\partial \xi^{(1)}_j} \right)^2 |\xi^{(1)}|^{\alpha} + 2 |\xi^{(1)}|^{\alpha} \frac{\partial |\xi^{(1)}|^{\alpha}}{\partial \xi^{(1)}_1}
\]

\[
= \alpha (n_1 + \alpha - 2) \xi_1 |\xi^{(1)}|^{\alpha - 2} + 2 \alpha \xi_1 |\xi^{(1)}|^{\alpha - 2}
\]

\[
= \alpha (n_1 + \alpha) \xi_1 |\xi^{(1)}|^{\alpha - 2}. \tag{6}
\]

Before the proof of main results, we need the following Lemma.

**LEMMA 1.** Let \( f \in C^\infty(\mathbb{R}^N_+) \) and \( \alpha \geq 0 \). There holds

\[
\left( \frac{(N+2+\alpha)^2}{4} - \alpha - 2 \right) \int_{\mathbb{R}^N_+} |x|^\alpha f^2 \, dx \leq \int_{\mathbb{R}^N_+} |\nabla f|^2 |x|^{\alpha + 2} \, dx. \tag{7}
\]

Furthermore, the constant \( \frac{(N+2+\alpha)^2}{4} - \alpha - 2 \) is sharp.

**Proof.** Replacing \( u \) by \( |x|^{\alpha + 2} f \) in (1), we have

\[
\int_{\mathbb{R}^N_+} |\nabla (|x|^{\alpha + 2} f)|^2 \, dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N_+} f^2 |x|^\alpha \, dx. \tag{8}
\]
We compute
\[
\int_{\mathbb{R}^N_+} |\nabla (|x|^\frac{\alpha+2}{2} f)|^2 dx = \int_{\mathbb{R}^N_+} \left( |\nabla f|^2 |x|^\alpha + \alpha + \frac{\alpha+2}{2} \right) u^2 dx
\]
\[
\begin{align*}
\text{Combing (8) and (9) yields (7).} \\
\text{the constant } \frac{(N+2+\alpha)^2}{4} - \alpha - 2 \text{ is sharp since the constant } \frac{N^2}{4} \text{ in (1) is sharp.} \quad \square
\end{align*}
\]

Now we can prove Theorem 1. Using the substitution \( u = |\xi(1)|^\frac{\alpha+2}{2} \phi \), we get
\[
\int_{G_+} |\nabla \psi(1)|^2 |\xi(1)|^{\alpha+2} = \int_{G} |\nabla (|\xi(1)|^\frac{\alpha+2}{2} u)|^2 |\xi(1)|^{\alpha+2}
\]
\[
\begin{align*}
\int_{G_+} |\nabla G(u)|^2 |\xi(1)|^{-\alpha-2} + u^2 |\nabla G(\xi(1))|^{-\alpha+2} + \frac{1}{4} \left( |\nabla G(\xi(1))|^{-\alpha-2}, |\nabla G^2(\xi(1))| \right) |\xi(1)|^{\alpha+2}
\end{align*}
\]
\[
\begin{align*}
\int_{G_+} |\nabla G(u)|^2 + \frac{(\alpha+2)^2}{4} \frac{u^2}{|\xi(1)|^2} - \frac{\alpha+2}{2} \int_{G_+} \langle \nabla G \ln |\xi(1)|, \nabla G^2 \rangle
\end{align*}
\]
\[
\begin{align*}
\int_{G_+} |\nabla G(u)|^2 + \frac{(\alpha+2)^2}{4} \frac{u^2}{|\xi(1)|^2} + \frac{\alpha+2}{2} \int_{G_+} u^2 \Delta G \ln |\xi(1)|
\end{align*}
\]
\[
\begin{align*}
|\nabla G(u)|^2 + \frac{(\alpha+2)^2}{4} \frac{u^2}{|\xi(1)|^2} + \frac{(\alpha+2)(n_1-2) u^2}{2 |\xi(1)|^2}
\end{align*}
\]

(10)

To get the last equation, we use the fact
\[
\Delta G \ln |\xi(1)| = \sum_{j=1}^{n_1} \left( \frac{\partial}{\partial \xi_j} \right)^2 \ln |\xi(1)| = \frac{n_1}{2} \frac{1}{|\xi(1)|^2}.
\]

On the other hand, using the substitution \( u = \xi_1 |\xi(1)|^{-\frac{n_1}{2}} \nu \), we have
\[
\int_{G_+} |\nabla G(u)|^2 = \int_{G_+} \left( |\nabla G\nu|^2 |\xi(1)|^{-n_1} + \nu^2 |\nabla G (\xi_1 |\xi(1)|^{-\frac{n_1}{2}})|^2 \right)
\]
\[
\begin{align*}
\int_{G_+} \left( \nu^2 |\nabla G (\xi_1 |\xi(1)|^{-\frac{n_1}{2}})|^2 - \frac{1}{2} \Delta G (\xi_1^2 |\xi(1)|^{-n_1}) \right)
\end{align*}
\]
\[
\begin{align*}
\int_{G_+} \nu^2 \left( |\nabla G (\xi_1 |\xi(1)|^{-\frac{n_1}{2}})|^2 - \frac{1}{2} \Delta G (\xi_1^2 |\xi(1)|^{-n_1}) \right).
\end{align*}
\]

(11)
Notice that, for \( g \in C^2(G) \),

\[
\Delta_G g^2 = \sum_{j=1}^{n_1} x_j^2 g^2 = 2g \sum_{j=1}^{n_1} x_j^2 g + 2 \sum_{j=1}^{n_1} |x_j g|^2 = 2g \Delta G g + 2 |\nabla G g|^2.
\]

We have, by (6),

\[
\left| \nabla_G \left( \xi_1 |\xi(1)|^{-\frac{n_1}{2}} \right) \right|^2 - \frac{1}{2} \Delta_G \left( \xi_1 |\xi(1)|^{-\frac{n_1}{2}} \right) = -\xi_1 |\xi(1)|^{-\frac{n_1}{2}} \Delta_G \left( |\xi(1)|^{-\frac{n_1}{2}} \right)
\]

\[
= \frac{n_1^2}{4} \xi_1 |\xi(1)|^{-n_1-2}.
\]

Combining (11) and (12) yields

\[
\int_{G_+} |\nabla_G u|^2 \geq \frac{n_1^2}{4} \int_{G_+} \xi_1^2 |\xi(1)|^{-n_1-2} v^2 = \frac{n_1^2}{4} \int_{G_+} \frac{u^2}{|\xi(1)|^2},
\]

(13)

Therefore, we have, by (10) and (13),

\[
\int_{G_+} |\nabla_G \phi|^2 |\xi(1)|^{-\alpha+2} \geq \int_{G_+} \left( \frac{n_1^2}{4} \frac{u^2}{|\xi(1)|^2} + \frac{(\alpha+2)^2}{4} \frac{u^2}{|\xi(1)|^2} + \frac{(\alpha+2)(n_1-2)}{2} \frac{u^2}{|\xi(1)|^2} \right).
\]

\[
= \left( \frac{(n_1+2+\alpha)^2}{4} - \alpha - 2 \right) \int_{G_+} \frac{u^2}{|\xi(1)|^2}
\]

\[
= \left( \frac{(n_1+2+\alpha)^2}{4} - \alpha - 2 \right) \int_{G_+} |\phi|^2 |\xi(1)|^\alpha.
\]

(14)

To finish the proof, we need to show that the constant \( \frac{(n_1+2+\alpha)^2}{4} - \alpha - 2 \) in (3) is sharp. Following [8], we choose the test function \( f(\xi) = u(\xi(1))g(\xi(2), \cdots, \xi(r)) \), where \( u(\cdot) \in C_0^\infty(\mathbb{R}^{n_1}) \) and \( g = \prod_{k=2}^{r} \prod_{s=1}^{n_k} w_{k,s}(\xi^{(k)}) \) with \( w_{k,s}(\cdot) \in C_0^\infty(\mathbb{R}) \) for all \( 2 \leq k \leq r \) and \( 1 \leq s \leq n_k \). For convenience, we set \( w_{k,s} \equiv 1 \) for all \( 1 \leq s \leq n_k \) if \( k = 1 \). By (5),

\[
\int_{G_+} |\nabla_G f|^2 |\xi(1)|^{-\alpha+2} d\xi = \sum_{j=1}^{n_1} \left\{ \int_{G_+} \left( \frac{\partial u}{\partial \xi^{(1)}_j} \right)^2 g^2 |\xi(1)|^{-\alpha+2} d\xi + \int_{G_+} u^2 \left( \sum_{k=2}^{r} \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi^{(k)}} \right)^2 |\xi(1)|^{-\alpha+2} d\xi + (*) \right\},
\]

where

\[
(*) = \int_{G_+} \frac{\partial u^2}{\partial \xi^{(1)}_j} g \sum_{k=2}^{r} \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi^{(k)}} |\xi(1)|^{-\alpha+2} d\xi
\]

\[
= \frac{1}{2} \sum_{k=2}^{r} \sum_{s=1}^{n_k} \int_{G_+} \frac{\partial u^2}{\partial \xi^{(1)}_j} p_{k,s}^j (\xi^{(1)}, \cdots, \xi^{(k-1)}) \frac{\partial g^2}{\partial \xi^{(k)}} |\xi(1)|^{-\alpha+2} d\xi.
\]
Since $w_{k,s}(\cdot) \in C_0^\infty(\mathbb{R})$ for all $2 \leq k \leq r$ and $1 \leq s \leq n_k$, we have

$$
\int_{\mathbb{R}} \frac{\partial w_{k,s}^2(\xi^{(k)}_s)}{\partial \xi^{(k)}_s} d\xi^{(k)}_s = 0.
$$

Therefore

$$
\int_{\mathbb{R}} \frac{\partial g^2}{\partial \xi^{(k)}_s} d\xi^{(k)}_s = 0.
$$

Thus \((*) = 0\) and

$$
\int_{G_+} |\nabla_G f|^2 |\xi^{(1)}_s|^{\alpha+2} d\xi = \sum_{j=1}^{n_1} \left\{ \int_{G_+} \left( \frac{\partial u}{\partial \xi^{(1)}_j} \right)^2 g^2 |\xi^{(1)}_s|^{\alpha+2} d\xi + \int_{G_+} u^2 \left| \sum_{k=2}^{n_k} \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi^{(k)}_s} \right|^2 |\xi^{(1)}_s|^{\alpha+2} d\xi \right\}.
$$

Let $C_G$ be a positive constant such that

$$
\left| \sum_{k=2}^{n_k} \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi^{(k)}_s} \right|^2 \leq C_G \left( \sum_{k=2}^{n_k} \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi^{(k)}_s} \right)^2.
$$

We have, by (15),

$$
\frac{\int_{G_+} |\nabla_G f|^2 |\xi^{(1)}_s|^{\alpha+2} d\xi}{\int_{G_+} f^2 |\xi^{(1)}_s|^{\alpha} d\xi} \leq C_G \sum_{k=2}^{n_k} \int_{G_+} u^2 \left| \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi^{(k)}_s} \right|^2 |\xi^{(1)}_s|^{\alpha+2} d\xi,
$$

Where $\mathbb{R}_{n_1}^+ = \{ (\xi_1, \ldots, \xi_{n_1}) \in \mathbb{R}^{n_1} : \xi_1 > 0 \}$.

Since for all $2 \leq k \leq r$ and $1 \leq s \leq n_k$, we have

$$
\inf_{w_{k,s} \in C_0^\infty(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}}|w_{k,s}'|^2 d\xi^{(k)}_s}{\int_{\mathbb{R}}|w_{k,s}|^2 d\xi^{(k)}_s} = 0,
$$

we have

$$
\inf_{w_{k,s} \in C_0^\infty(\mathbb{R}) \setminus \{0\}} \frac{\int_{G_+} u^2 \left| \sum_{s=1}^{n_k} p_{k,s}^j \frac{\partial g}{\partial \xi^{(k)}_s} \right|^2 |\xi^{(1)}_s|^{\alpha+2} d\xi}{\int_{G_+} f^2 |\xi^{(1)}_s|^{\alpha} d\xi} = \frac{1}{\int_{\mathbb{R}_+^{n_1+\cdots+n_k-1}} u^2 \prod_{l=1}^{k-1} \prod_{i=1}^{n_l} w_{l,i}^2 |\xi^{(1)}_s|^{\alpha+2} d\xi} \inf_{w_{k,s} \in C_0^\infty(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}}|w_{k,s}'|^2 d\xi^{(k)}_s}{\int_{\mathbb{R}}|w_{k,s}|^2 d\xi^{(k)}_s} = 0.
$$
Therefore, by Lemma 1,

\[
\inf_{\phi \in C^\infty_0(G_+)} \frac{\int_{G_+} |\nabla_G \phi|^p \xi^{(1)}(1)^{\alpha+2}}{\int_{G_+} |\phi|^p \xi^{(1)}(1)^{\alpha}} \leq \inf_{u \in C^\infty_0(R_{n+1}^+)} \frac{\int_{R_{n+1}^+} \sum_{j=1}^{n+1} \left( \frac{\partial u}{\partial \xi_j} \right)^2 \xi^{(1)}(1)^{\alpha+2} \, d\xi^{(1)}}{\int_{R_{n+1}^+} u^2 \xi^{(1)}(1)^{\alpha+2} \, d\xi^{(1)}}
\]

\[
= \left( \frac{n_1 + 2 + \alpha}{4} - \alpha - 2 \right).
\]

The proof of Theorem 1 is thereby completed.

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