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A SHARP DOUBLE INEQUALITY INVOLVING TRIGONOMETRIC FUNCTIONS AND ITS APPLICATIONS

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Abstract. We present the best possible parameters $p,q \in (0,1]$ such that the double inequality $\frac{1}{3p^2}\cos(px)+1-\frac{1}{3p^2} < \frac{\sin(x)}{x} < \frac{1}{3q^2}\cos(qx)+1-\frac{1}{3q^2}$ holds for all $x \in (0,\pi/2)$. As applications, some new inequalities for the sine integral, Catalan constant and Schwab-Borchardt mean are found.

1. Introduction

It is well known that the double inequality

$$\cos^{1/3}(x) < \frac{\sin(x)}{x} < \frac{2+\cos(x)}{3} \tag{1.1}$$

holds for all $x \in (0,\pi/2)$. Recently, the improvements, refinements and generalizations for inequality (1.1) have attracted the attention of many researchers.

Iyengar et al. [1] proved that $p = \frac{1}{\sqrt{3}}$ and $q = \frac{2}{\pi}\arccos\left(\frac{2}{\pi}\right)$ are the best possible constants such that the double inequality

$$\cos(px) \leqslant \frac{\sin(x)}{x} \leqslant \cos(qx)$$

holds for all $x \in (0,\pi/2)$.

Qi et al. [2] established that

$$\frac{\sin(x)}{x} > \cos^2\left(\frac{x}{2}\right)$$

for all $x \in (0,\pi/2)$.

Neuman and Sándor [3] gave an improvement for the first inequality in (1.1) as follows:

$$\frac{\sin(x)}{x} > \left(\frac{1+\cos(x)}{2}\right)^{2/3} = \cos^{4/3}\left(\frac{x}{2}\right), \quad x \in \left(0,\frac{\pi}{2}\right). \tag{1.2}$$

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Inequality (1.2) was also proved by Lv et al. in [4]. Klén et al. [5] proved that

$$\cos^2\left(\frac{x}{2}\right) \leqslant \frac{\sin(x)}{x} \leqslant \cos^3\left(\frac{x}{3}\right) \tag{1.3}$$

for $x \in (-\sqrt{27/5}, \sqrt{27/5})$.

In [6], Yang found that $p = p_0$ and $q = 1/3$ are the best possible parameters in $(0,1]$ such that the double inequality

$$\cos^{1/p}(px) < \frac{\sin(x)}{x} < \cos^{1/q}(qx)$$

holds for all $x \in (0, \pi/2)$, where $p_0 = 0.3473\ldots$ is the unique solution of the equation $p\log(2/\pi) = \log[\cos(p\pi/2)]$ on $(0,1)$. Yang [6] also proved that the double inequality

$$\cos^{\alpha}\left(\frac{x}{3}\right) < \frac{\sin(x)}{x} < \cos^{\beta}\left(\frac{x}{3}\right)$$

holds for all $x \in (0, \pi/2)$ if and only if $\alpha \geqslant 2(\log\pi - \log 2)/(\log 4 - \log 3) = 3.139\ldots$ and $\beta \leqslant 3$.

Zhu [7] and Yang [8] proved that the double inequality

$$\left(\frac{2}{3} + \frac{1}{3}\cos^p(x)\right)^{1/p} < \frac{\sin(x)}{x} < \left(\frac{2}{3} + \frac{1}{3}\cos^q(x)\right)^{1/q}$$

holds for all $x \in (0, \pi/2)$ if and only if $p \leqslant 4/5$ and $q \geqslant (\log 3 - \log 2)/(\log\pi - \log 2) = 0.8978\ldots$.

Lv et al. [4] proved that the double inequality

$$\cos^{\lambda}\left(\frac{x}{2}\right) < \frac{\sin(x)}{x} < \cos^{\mu}\left(\frac{x}{2}\right)$$

holds for all $x \in (0, \pi/2)$ if and only if $\lambda \geqslant 4/3$ and $\mu \leqslant 2(\log\pi - \log 2)/\log 2 = 1.3030\ldots$.

Very recently, Yang et al. [9] proved that the double inequality

$$\left[\frac{2}{3}\cos^{2p}\left(\frac{x}{2}\right) + \frac{1}{3}\right]^{1/p} < \frac{\sin(x)}{x} < \left[\frac{2}{3}\cos^{2q}\left(\frac{x}{2}\right) + \frac{1}{3}\right]^{1/q}$$

holds for all $x \in (0, \pi/2)$ if and only if $p \leqslant \log(\pi - 2)/\log 2$ and $q \geqslant 1/5$.

The main purpose of this paper is to find the best possible parameters $p, q \in (0,1]$ such that the double inequality $\frac{1}{3p^2}\cos(px) + 1 - \frac{1}{3p^2} < \frac{\sin(x)}{x} < \frac{1}{3q^2}\cos(qx) + 1 - \frac{1}{3q^2}$ holds for all $x \in (0, \pi/2)$, and present several new inequalities for the sine integral, Catalan constant and Schwab-Borchardt.

2. Lemmas

In order to prove our main result we need several lemmas, which we present in this section.

LEMMA 2.1. Let

$$U_p(x) = \frac{1}{3p^2}\cos(px) + 1 - \frac{1}{3p^2}. \tag{2.1}$$

Then $U_p(x)$ is strictly increasing with respect to $p \in (0,1]$ for fixed $x \in (0,\pi/2)$.

Proof. It follows from (2.1) that

$$\begin{aligned} \frac{\partial U_p(x)}{\partial p} &= \frac{1}{3p^3}[2 - 2\cos(px) - px\sin(px)] \\ &= \frac{2x}{3p^2}\left[\frac{\sin\left(\frac{px}{2}\right)}{\frac{px}{2}} - \cos\left(\frac{px}{2}\right)\right]\sin\left(\frac{px}{2}\right) > 0 \end{aligned} \tag{2.2}$$

for $p \in (0,1]$ and $x \in (0,\pi/2)$.

Therefore, Lemma 2.1 follows from (2.2). □

LEMMA 2.2. Let $p \in (0,1]$ and the function $F_p(x)$ be defined on $(0,\pi/2)$ by

$$F_p(x) = \frac{\sin(x)}{x} - \left[\frac{1}{3p^2}\cos(px) + 1 - \frac{1}{3p^2}\right]. \tag{2.3}$$

Then the following statements are true:

(1) If $F_p(x) < 0$ for all $x \in (0,\pi/2)$, then $p \geqslant \sqrt{15}/5$;

(2) If $F_p(x) > 0$ for all $x \in (0,\pi/2)$, then $p \leqslant p_0$, where $p_0 = 0.7708\ldots$ is the unique solution of the equation

$$\frac{2}{\pi} - \left[\frac{1}{3p^2}\cos\left(\frac{p\pi}{2}\right) + 1 - \frac{1}{3p^2}\right] = 0 \tag{2.4}$$

on $(0,1/2)$.

Proof. (1) If $F_p(x) < 0$ for all $x \in (0,\pi/2)$, then (2.3) leads to

$$\lim_{x\to 0^+}\frac{F_p(x)}{x^4} = \frac{3-5p^2}{360} \leqslant 0$$

and $p \geqslant \sqrt{15}/5$.

(2) If $F_p(x) > 0$ for all $x \in (0,\pi/2)$, then it follows from (2.3) that

$$F_p\left(\frac{\pi}{2}^-\right) = \frac{2}{\pi} - \left[\frac{1}{3p^2}\cos\left(\frac{p\pi}{2}\right) + 1 - \frac{1}{3p^2}\right] > 0. \tag{2.5}$$

From Lemma 2.1 and (2.5) we clearly see that the function $p \to F_p(\pi/2^-)$ is strictly decreasing on $(0,1]$.

Note that

$$F_{1/2}\left(\frac{\pi}{2}^-\right) = \frac{2}{\pi} - \frac{2\sqrt{2}}{3} + \frac{1}{3} > 0, \quad F_1\left(\frac{\pi}{2}^-\right) = \frac{2}{\pi} - \frac{2}{3} < 0. \tag{2.6}$$

Inequality (2.6) and the monotonicity of the function $p \to F_p(\pi/2^-)$ lead to the conclusion that equation (2.4) has a unique solution $p = p_0 \in (1/2,1)$. Numerical computations using Mathematical software show that $p_0 = 0.7708\ldots$. Moreover, inequalities (2.5) and (2.6) together with the monotonicity of the function $p \to F_p(\pi/2^-)$ imply that $p \leqslant p_0$. $\square$

LEMMA 2.3. Let $c \in (0,3/5]$ and the sequence $\{a_n(c)\}$ $(n = 1,2,\cdots)$ be defined by

$$a_n(c) = 3 - (2n+1)c^{n-1}. \tag{2.7}$$

Then the following statements are true:

(1) $a_n(c) \geqslant 0$ for all $n \geqslant 1$;

(2) $1 < a_{n+1}(c)/a_n(c) \leqslant a_{n+1}(3/5)/a_n(3/5) \leqslant 11/5$ for all $n \geqslant 3$.

Proof. (1) From (2.7) we clearly see that

$$a_1(c) = 0, \tag{2.8}$$

$$\begin{aligned} a_{n+1}(c) - a_n(c) &= [(2n+1) - (2n+3)c]c^{n-1} \\ &\geqslant \left[(2n+1) - \frac{3(2n+3)}{5}\right]c^{n-1} = \frac{4(n-1)}{5}c^{n-1} \geqslant 0 \end{aligned} \tag{2.9}$$

for all $n \geqslant 1$.

Therefore, Lemma 2.3(1) follows easily from (2.8) and (2.9).

(2) We first prove that the sequence $\{a_{n+1}(c)/a_n(c)\}$ is strictly decreasing with respect to $n \geqslant 3$ for any fixed $c \in (0,3/5]$.

From part (1) and (2.9) we clearly see that

$$a_n(c) > 0 \tag{2.10}$$

for all $n \geqslant 3$.

It follows from (2.7) and (2.10) that

$$\begin{aligned} \frac{a_{n+1}(c)}{a_n(c)} - \frac{a_{n+2}(c)}{a_{n+1}(c)} &= \frac{c^{n-1}}{a_n(c)a_{n+1}(c)}\left[4c^{n+1} + 6n(c-1)^2 + 15c^2 - 18c + 3\right] \\ &\geqslant \frac{c^{n-1}}{a_n(c)a_{n+1}(c)}\left[4c^{n+1} + 18(c-1)^2 + 15c^2 - 18c + 3\right] \\ &= \frac{c^{n-1}}{a_n(c)a_{n+1}(c)}\left[4c^{n+1} + 33(1-c)\left(\frac{7}{11} - c\right)\right] > 0 \end{aligned} \tag{2.11}$$

for all $n \geqslant 3$.

Inequality (2.11) implies that the sequence $\{a_{n+1}(c)/a_n(c)\}$ is strictly decreasing with respect to $n \geqslant 3$ for any fixed $c \in (0,3/5]$.

Next we prove that the function $g_n(c) = a_{n+1}(c)/a_n(c)$ is strictly increasing with respect to $c \in (0,3/5]$ for all $n \geqslant 3$.

From (2.7) and (2.10) we have

$$\frac{\partial g_n(c)}{\partial c} = a_n^{-2}(c)c^{n-2}h_n(c), \tag{2.12}$$

where

$$h_n(c) = (4n^2+8n+3)c^n - 3n(2n+3)c + (6n^2-3n-3),$$

$$\frac{\partial h_n(c)}{\partial c} = -n(2n+3)a_n(c) < 0 \tag{2.13}$$

for all $n \geqslant 3$.

Inequality (2.13) implies that

$$h_n(c) \geqslant h_n\left(\frac{3}{5}\right) = \left(\frac{3}{5}\right)^n \left(4n^2+8n+3\right) + \frac{3}{5}\left(4n^2-14n-5\right) \tag{2.14}$$

for all $n \geqslant 3$ and $c \in (0,3/5]$.

Note that

$$h_3\left(\frac{3}{5}\right) = \frac{876}{125} > 0 \tag{2.15}$$

and

$$4n^2 - 14n - 5 = 2n(2n-7) - 5 \geqslant 2n - 5 > 0 \tag{2.16}$$

for $n \geqslant 4$.

It follows from (2.12) and (2.14)–(2.16) that $g_n(c)$ is strictly increasing with respect to $c \in (0,3/5]$ for all $n \geqslant 3$.

From (2.7) and above discussion we get

$$1 = \frac{a_{n+1}(0)}{a_n(0)} < \frac{a_{n+1}(c)}{a_n(c)} \leqslant \frac{a_{n+1}(3/5)}{a_n(3/5)} \leqslant \frac{a_4(3/5)}{a_3(3/5)} = \frac{11}{5}$$

for $n \geqslant 3$ and $c \in (0,3/5]$. $\square$

3. Main result

THEOREM 3.1. Let $p,q \in (0,1]$. Then the double inequality

$$\frac{1}{3p^2}\cos(px) + 1 - \frac{1}{3p^2} < \frac{\sin(x)}{x} < \frac{1}{3q^2}\cos(qx) + 1 - \frac{1}{3q^2} \tag{3.1}$$

holds for all $x \in (0,\pi/2)$ if and only if $p \leqslant p_0$ and $q \geqslant q_0 = \sqrt{15}/5$, where $p_0 = 0.7708\ldots$ is the unique solution of equation (2.4) on $(1/2,1)$.

Proof. Let $\lambda \in (0, \sqrt{3/5}]$ and $f_\lambda(x) = x^{-4}F_\lambda(x)$, where $F_\lambda(x)$ is defined by (2.3). Then making use of power series expansions and (2.3) we get

$$\begin{aligned} f_\lambda(x) &= \left[\frac{\sin(x)}{x} - \left(\frac{1}{3\lambda^2}\cos(\lambda x) + 1 - \frac{1}{3\lambda^2}\right)\right]x^{-4} \qquad (3.2)\\ &= \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} - \left(\frac{1}{3\lambda^2}\sum_{n=0}^{\infty}\frac{(-1)^n(\lambda x)^{2n}}{(2n)!} + 1 - \frac{1}{3\lambda^2}\right)\right]x^{-4}\\ &= \sum_{n=2}^{\infty}\frac{(-1)^n\left[3-(2n+1)\lambda^{2n-2}\right]}{3(2n+1)!}x^{2n-4}\\ &= \frac{3-5\lambda^2}{360} + \sum_{n=3}^{\infty}\frac{(-1)^n a_n(\lambda^2)}{3(2n+1)!}x^{2n-4},\\ \frac{\partial f_\lambda(x)}{\partial x} &= \sum_{n=3}^{\infty}\frac{(-1)^n(2n-4)a_n(\lambda^2)}{3(2n+1)!}x^{2n-5} = \sum_{n=3}^{\infty}(-1)^n u_n(x), \qquad (3.3)\end{aligned}$$

where $a_n(\lambda^2)$ is defined by (2.7) and

$$u_n(x) = \frac{(2n-4)a_n(\lambda^2)}{3(2n+1)!}x^{2n-5}. \qquad (3.4)$$

Equation (3.4) and Lemma 2.3 lead to

$$\begin{aligned}\frac{u_{n+1}(x)}{u_n(x)} &= \frac{2n-2}{(2n-4)(2n+3)(2n+2)}\frac{a_{n+1}(\lambda^2)}{a_n(\lambda^2)}x^2 \qquad (3.5)\\ &< \frac{1}{(2\times 3-4)(2\times 3+3)} \times \frac{11}{5} \times \frac{\pi^2}{4} = \frac{11\pi^2}{360} < 1\end{aligned}$$

for $n \geqslant 3$ and $x \in (0, \pi/2)$.

From (3.3) and (3.5) we clearly see that $\sum_{n=3}^{\infty}(-1)^n u_n(x)$ is a Leibniz's alternating series and $f_\lambda(x)$ is strictly decreasing with respect to x on the interval $(0, \pi/2)$. In particular, it follows from $p_0, q_0 \in (0, \sqrt{3/5}]$ that both $f_{p_0}(x)$ and $f_{q_0}(x)$ are strictly decreasing with respect to x on the interval $(0, \pi/2)$. Then from (3.2) we get

$$\left[\frac{\sin(x)}{x} - \left(\frac{1}{3p_0^2}\cos(p_0x) + 1 - \frac{1}{3p_0^2}\right)\right]x^{-4} = f_{p_0}(x) > f_{p_0}\left(\frac{\pi}{2}^-\right) = 0, \qquad (3.6)$$

$$\left[\frac{\sin(x)}{x} - \left(\frac{1}{3q_0^2}\cos(q_0x) + 1 - \frac{1}{3q_0^2}\right)\right]x^{-4} = f_{q_0}(x) < f_{q_0}\left(0^+\right) = 0 \qquad (3.7)$$

for $x \in (0, \pi/2)$.

Therefore, Theorem 3.1 follows easily from (3.6) and (3.7) together with Lemma 2.2. □

REMARK 3.1. Let $\lambda \in (0, \sqrt{3/5}]$, $c_0(\lambda) = f_\lambda(\pi/2^-) = 16[2/\pi - 1 + 1/(3\lambda^2) - \cos(\lambda\pi/2)/(3\lambda^2)]/\pi^4$ and $c_1 = f_\lambda(0^+) = (3-5\lambda^2)/360$. Then from (3.2) and the monotonicity of $f_\lambda(x)$ with respect to x on $(0,\pi/2)$ we know that the double inequality

$$\left[\frac{1}{3\lambda^2}\cos(\lambda x)+1-\frac{1}{3\lambda^2}\right]+c_0(\lambda)x^4<\frac{\sin(x)}{x}<\left[\frac{1}{3\lambda^2}\cos(\lambda x)+1-\frac{1}{3\lambda^2}\right]+c_1(\lambda)x^4 \tag{3.8}$$

holds for all $x\in(0,\pi/2)$ with the best possible constants $c_0(\lambda)$ and $c_1(\lambda)$. In particular, if $\lambda=p_0=0.7708\ldots$ and $\lambda=q_0=\sqrt{15/5}$, then $c_0(p_0)=0$, $c_1(p_0)=(3-5p_0^2)/360=8.020\ldots\times10^{-5}$, $c_0(q_0)=-7.261\ldots\times10^{-5}$ and $c_1(q_0)=0$, and the double inequalities

$$\left[\frac{1}{3p_0^2}\cos(p_0x)+1-\frac{1}{3p_0^2}\right]+c_0(p_0)x^4<\frac{\sin(x)}{x}<\left[\frac{1}{3p_0^2}\cos(p_0x)+1-\frac{1}{3p_0^2}\right]+c_1(p_0)x^4 \tag{3.9}$$

and

$$\left[\frac{1}{3q_0^2}\cos(q_0x)+1-\frac{1}{3q_0^2}\right]+c_0(q_0)x^4<\frac{\sin(x)}{x}<\left[\frac{1}{3q_0^2}\cos(q_0x)+1-\frac{1}{3q_0^2}\right]+c_1(q_0)x^4 \tag{3.10}$$

hold for all $x\in(0,\pi/2)$ with the best possible parameters $c_0(p_0)$, $c_1(p_0)$, $c_0(q_0)$ and $c_1(q_0)$.

Letting $p=3/4>\sqrt{2}/2>2/3>\sqrt{3}/3\in(0,p_0)$ and $q=\sqrt{3/5}<\sqrt{2/3}<\sqrt{3}/2<1\in[\sqrt{15}/5,1]$, then Lemma 2.1 and Theorem 3.1 lead to Corollary 3.1.

COROLLARY 3.1. *The inequalities*

$$\begin{aligned}\cos\left(\frac{\sqrt{3}x}{3}\right)&<\frac{3}{4}\cos\left(\frac{2x}{3}\right)+\frac{1}{4}<\frac{2}{3}\cos\left(\frac{\sqrt{2}x}{2}\right)+\frac{1}{3}<\frac{16}{27}\cos\left(\frac{3x}{4}\right)+\frac{11}{27}\\&<\frac{\sin(x)}{x}<\frac{5}{9}\cos\left(\frac{\sqrt{15}x}{5}\right)+\frac{4}{9}<\cos^2\left(\frac{\sqrt{6}x}{6}\right)\\&<\frac{4}{9}\cos\left(\frac{\sqrt{3}x}{2}\right)+\frac{5}{9}<\frac{1}{3}\cos(x)+\frac{2}{3}\end{aligned}$$

hold for $x\in(0,\pi/2)$.

4. Applications

In this section, we present several new inequalities for the sine integral, Catalan constant and Schwab-Borchardt by use of Theorem 3.1.

The sine integral is given by

$$Si(t)=\int_0^t\frac{\sin(x)}{x}dx,$$

some estimates for the sine integral can be found in the literature [10–13]. From (3.8) we get a new estimate for the sine integral as follows.

REMARK 4.1. Let $\lambda \in (0, \sqrt{3/5}]$, $c_0(\lambda) = 16[2/\pi - 1 + 1/(3\lambda^2) - \cos(\lambda\pi/2)/(3\lambda^2)]/\pi^4$ and $c_1 = (3-5\lambda^2)/360$. Then the double inequality

$$\frac{\sin(\lambda t)}{3\lambda^3} + \left(1 - \frac{1}{3\lambda^2}\right)t + \frac{c_0(\lambda)}{5}t^5 < Si(t) < \frac{\sin(\lambda t)}{3\lambda^3} + \left(1 - \frac{1}{3\lambda^2}\right)t + \frac{c_1(\lambda)}{5}t^5 \quad (4.1)$$

holds for $t \in (0, \pi/2)$. Let $p = 0^+$ and $p = 2/3$, then (4.1) leads to

$$t - \frac{1}{18}t^3 + \frac{2\pi^3 - 48\pi + 96}{15\pi^5}t^5 < Si(t) < t - \frac{1}{18}t^3 + \frac{1}{600}t^5, \quad (4.2)$$

$$\frac{9}{8}\sin\left(\frac{2t}{3}\right) + \frac{1}{4}t + \frac{2(16-5\pi)}{5\pi^5}t^5 < Si(t) < \frac{9}{8}\sin\left(\frac{2t}{3}\right) + \frac{1}{4}t + \frac{7}{16200}t^5. \quad (4.3)$$

In particular, let $t \to \frac{\pi}{2}^-$, then (4.2) and (4.3) become

$$1.370\ldots = \frac{2\pi+1}{5} - \frac{\pi^3}{360} < Si\left(\frac{\pi}{2}\right) < \frac{\pi}{2} - \frac{\pi^3}{144} + \frac{\pi^5}{19200} = 1.371\ldots,$$

$$1.370\ldots = \frac{\pi + 9\sqrt{3}}{16} + \frac{1}{5} < Si\left(\frac{\pi}{2}\right) < \frac{\pi}{8} + \frac{7\pi^5}{518400} + \frac{9\sqrt{3}}{16} = 1.371\ldots.$$

The Catalan constant

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{1}{2}\int_0^{\pi/2} \frac{x}{\sin(x)}dx = 0.9159655941772190\ldots \quad (4.4)$$

is a mysterious constant in mathematics and physics [14–17].

It follows from Corollary 3.1 and (4.4) that

$$\int_0^{\pi/2} \frac{9}{10\cos\left(\frac{\sqrt{15}x}{5}\right) + 8} < G < \int_0^{\pi/2} \frac{27}{32\cos\left(\frac{3x}{4}\right) + 22}. \quad (4.5)$$

Inequality (4.5) leads to an estimation for the Catalan constant G.

REMARK 4.2. The Catalan constant G satisfies the two-sided inequality

$$0.9158\ldots = \frac{\sqrt{15}}{4}\log\frac{4\cos\left(\frac{\sqrt{15}\pi}{10}\right) + 3\sin\left(\frac{\sqrt{15}\pi}{10}\right) + 5}{4\cos\left(\frac{\sqrt{15}\pi}{10}\right) - 3\sin\left(\frac{\sqrt{15}\pi}{10}\right) + 5} < G$$
$$< \frac{\sqrt{15}}{5}\log\frac{11\sqrt{2-\sqrt{2}} + 3\sqrt{15}\sqrt{2+\sqrt{2}} + 32}{11\sqrt{2-\sqrt{2}} - 3\sqrt{15}\sqrt{2+\sqrt{2}} + 32} = 0.9167\ldots.$$

The Schwab-Borchardt mean $SB(a,b)$ [18–20] of two positive real numbers a and b is given by

$$SB(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\arccos\left(\frac{a}{b}\right)}, & a < b, \\ a, & a = b, \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b. \end{cases} \quad (4.6)$$

Let $b > a > 0$ and $x = \arccos(a/b)$, then from Corollary 3.1 and (4.6) together with the facts

$$\frac{\sqrt{b^2-a^2}}{\arccos\left(\frac{a}{b}\right)} = b\frac{\sin(x)}{x},$$

$$\frac{16}{27}\cos\left(\frac{3x}{4}\right)+\frac{11}{27} = \frac{16}{27}\cos\left[\frac{3\arccos\left(\frac{a}{b}\right)}{4}\right]+\frac{11}{27}$$

and

$$\frac{5}{9}\cos\left(\frac{\sqrt{15}x}{5}\right)+\frac{4}{9} = \frac{5}{9}\cos\left[\frac{\sqrt{15}\arccos\left(\frac{a}{b}\right)}{5}\right]+\frac{4}{9}$$

we get

$$\frac{16b}{27}\cos\left[\frac{3\arccos\left(\frac{a}{b}\right)}{4}\right]+\frac{11b}{27} < SB(a,b) < \frac{5b}{9}\cos\left[\frac{\sqrt{15}\arccos\left(\frac{a}{b}\right)}{5}\right]+\frac{4b}{9}. \quad (4.7)$$

The following Remark 4.3 can be derived immediately from the first inequality in (4.7).

REMARK 4.3. Let $b > a > 0$, then

$$\begin{aligned} SB(a,b) &> \frac{8b^{1/4}}{27}\sqrt{2b^{3/2}+\sqrt{2(a+b)(b-2a)^2}}+\frac{11b}{27} \\ &\geqslant \frac{11+8\sqrt{2}}{27}b = 0.8264\cdots\times b. \end{aligned}$$

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REFERENCES

[1] K. S. K. IYENGAR, B. S. MADHAVA RAO AND T. S. NANJUNDIAH, *Some trigonometrical inequalities*, Half-Yearly J. Mysore Univ. Sect. B., N. S., 1945, **6**: 1–12.

[2] F. QI, L.-H. CUI AND S.-L. XU, *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl., 1999, **2** (4): 517–528.

[3] E. NEUMAN AND J. SÁNDOR, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens*, Wilker, and Huygens inequalities, Math. Inequal. Appl., 2010, **13** (4): 715–723.

[4] Y. LV, G. WANG, Y. CHU, *A note on Jordan type inequalities for hyperbolic functions*, Appl. Math. Lett., 2012, **25** (3): 505–508.

[5] R. KLÉN, M. VISURI AND M. VUORINEN, *On Jordan type inequalities for hyperbolic functions*, J. Inequal. Appl., **2010**, Article ID 362548, 14 pages.

[6] Z.-H. YANG, *New sharp Jordan type inequalities and their applications*, Gulf. J. Math., 2014, **2** (1): 1–10.

[7] L. ZHU, *A source of inequalities for circular functions*, Comput. Math. Appl., 2009, **58** (10): 1998–2004.

[8] Z.-H. YANG, *Sharp bounds for Seiffert mean in terms of weighted power means of arithmetic mean and geometric mean*, Math. Inequal. Appl., 2014, **17** (2): 499–514.
[9] Z.-H. YANG, Y.-M. CHU, Y.-Q. SONG AND Y.-M. LI, *A Sharp double inequality for trigonometric functions and its applications*, Abstr. Appl. Anal., **2014**, Article ID 592085, 9 pages.
[10] S.-H. WU AND L. DEBNATH, *A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality*, Appl. Math. Lett., 2006, **19** (12): 1378–1384.
[11] S.-H. WU, *Sharpness and generalization of Jordan's inequality and its applications*, Taiwanese J. Math., 2008, **12** (2): 325–336.
[12] Z.-H. YANG, *Refinements of a two-sided inequality for trigonometric functions*, J. Math. Inequal., 2013, **7** (4): 601–615.
[13] Z.-H. YANG AND Y.-M. CHU, *A note on Jordan, Adamović-Mitrinović, and Cusa inequalities*, Abstr. Appl. Anal., **2014**, Article ID 364076, 12 pages.
[14] D. M. BRADLEY, *A class of series acceleration formulae for Catalan's constant*, Ramanujan J., 1999, **3** (2): 159–173.
[15] D. BORWEIN, J. M. BORWEIN, M. L. GLASSER AND J. G. WAN, *Moments of Ramanujan's generalized elliptic integrals and extensions of Catalan's constant*, J. Math. Anal. Appl., 2011, **384** (2): 478–496.
[16] D. H. BAILEY, J. M. BORWEIN, A. MATTINGLY AND G. WIGHTWICK, *The computation of previously inaccessible digits of π^2 and Catalan's constant*, Notices Amer. Math. Soc., 2013, **60** (7): 844–854.
[17] Z.-H. YANG, Y.-M. CHU AND X.-J. TAO, *A double inequality for the Trigamma function and it applications*, Abstr. Appl. Anal., **2014**, Article ID 702718, 9 pages.
[18] E. NEUMAN AND J. SÁNDOR, *On the Schwab-Borchardt mean*, Math. Pannon, 2003, **14** (2): 253–266.
[19] E. NEUMAN AND J. SÁNDOR, *On the Schwab-Borchardt mean II*, Math. Pannon, 2006, **17** (1): 49-59.
[20] E. NEUMAN, *Inequalities for the Schwab-Borchardt mean and their applications*, J. Math. Inequal., 2011, **5** (4): 601–609.

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