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ON THE NORMS OF r-CIRCULANT MATRICES WITH THE HYPERHARMONIC NUMBERS

MUSTAFA BAHŞI

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Abstract. In this paper, we study norms of circulant matrices $H = \text{Circ}(H_0^{(k)},\ H_1^{(k)},\ldots,H_{n-1}^{(k)})$, $\widehat{H} = \text{Circ}(H_k^{(0)},\ H_k^{(1)},\ldots,H_k^{(n-1)})$ and $r-$circulant matrices $H_r = \text{Circr}(H_0^{(k)},H_1^{(k)},\ldots,H_{n-1}^{(k)})$, $\widehat{H_r} = \text{Circr}(H_k^{(0)},H_k^{(1)},\ldots,H_k^{(n-1)})$, where $H_n^{(k)}$ denotes the nth hyperharmonic number of order r.

1. Introduction

The circulant matrices and $r-$circulant matrices play important role in signal processing, coding theory, image processing, linear forecast and so on. An $n \times n$ matrix C_r is called an $r-$circulant matrix if it is of the form

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \ldots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{bmatrix}.$$

The matrix C_r is determined by its first row elements and r, thus we denote $C_r = \text{Circr}(c_0, c_1, \ldots, c_{n-1})$. When we take $r = 1$, the matrix $C_1 = C$ is called a circulant matrix. We denote $C_1 = C = \text{Circ}(c_0, c_1, \ldots, c_{n-1})$. Circulant matrices are especially tractable class of matrices since their inverses, conjugate transposes, sums and products are also circulant. Moreover, circulant matrices are normal matrices [9]. Also, by means of [9,13], it is well known that the eigenvalues of C are

$$\underset{0 \leqslant m \leqslant n-1}{\lambda_m} = \sum_{k=0}^{n-1} c_k w^{-mk} \tag{1}$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$, and the corresponding eigenvectors are

$$\underset{0 \leqslant m \leqslant n-1}{x_m} = \left(1, w^m, w^{2m}, \ldots, w^{(n-1)m}\right)^T. \tag{2}$$

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Recently, there have been many papers on the inverse, determinants and norms of special matrices with special elements such as Fibonacci and Lucas numbers [2,4,14-19,22-24]. Shen and Cen [16] have given upper and lower bounds for the spectral norms of r-circulant matrices in the forms $A = C_r(F_0, F_1, ..., F_{n-1})$ and $B = C_r(L_0, L_1, ..., L_{n-1})$. Yazlik and Taskara [22, 23] have obtained upper and lower bounds for the spectral norm of circulant and $r-$circulant matrix with the generalized $k-$Horadam numbers. Bahşi and Solak [4] have computed the spectral norms of circulant and $r-$circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers in the forms $F_r = \text{Circr}(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$ and $L_r = \text{Circr}(L_0^{(k)}. \ L_1^{(k)}, \ldots, L_{n-1}^{(k)})$. As for us, in this paper, we compute the spectral norms of circulant and $r-$circulant matrices with the hyperharmonic numbers.

The main contents of this paper are organized as follows: In Section 2, we give some preliminaries, definitions and lemmas related to our study. In Section 3, we derive some bounds for the spectral norms of $r-$circulant matrices with the hyperharmonic numbers of the forms $H_r = \text{Circr}(H_0^{(k)}, H_1^{(k)}, \ldots, H_{n-1}^{(k)})$, $\widehat{H_r} = \text{Circr}(H_k^{(0)}, H_k^{(1)}, \ldots, H_k^{(n-1)})$ and their Hadamard and Kronecker products. For this, we firstly compute the spectral and Euclidean norms of circulant matrices of the forms $H = \text{Circ}(H_0^{(k)}, \ H_1^{(k)}, \ldots, H_{n-1}^{(k)})$ and $\widehat{H} = \text{Circ}(H_k^{(0)}, \ H_k^{(1)}, \ldots, H_k^{(n-1)})$. Moreover, we give some examples related to special cases of our results.

2. Preliminaries

The harmonic numbers are important in a wide range of diverse fields such as analysis of algorithms in computer science, number theory and combinatorial problems. Also, they are closely related to the some special functions such as Riemann zeta function [6]. The harmonic numbers are defined by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n = 1, 2, \ldots. \tag{3}$$

A generating function for the harmonic numbers is $\frac{-\ln(1-x)}{1-x}$. The first few harmonic numbers are $0, 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{12}, \ldots$. The harmonic numbers have many interesting properties $[6, 7, 20, 21]$. For $n \geqslant 1$, two of them are:

$$\sum_{k=1}^{n-1} H_k = nH_n - n \text{ and } \sum_{k=0}^{n} \binom{n}{k} H_k = 2^n \left(H_n - \sum_{k=1}^{n} \frac{1}{k2^k} \right). \tag{4}$$

The harmonic numbers have been generalized in many ways $[1, 5-8, 10, 21]$. One of them is the hyperharmonic number. The nth hyperharmonic number of order r, $H_n^{(r)}$, defined as $[8]$: for $n, r \geqslant 1$

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \text{ with } H_n^{(0)} = \frac{1}{n} \text{ and } H_0^{(n)} = H_0^{(0)} = 0. \tag{5}$$

From the definition of $H_n^{(r)}$, we have $H_1^{(r)} = 1$, and $H_n^{(1)} = \sum_{k=1}^{n} \frac{1}{k} = H_n$ where H_n is nth ordinary harmonic number. The hyperharmonic numbers have the recurrence relation:

$$H_n^{(r)} = H_n^{(r-1)} + H_{n-1}^{(r)}.$$

Also, Conway and Guy [8] gave an equality as follows:

$$H_n^{(r)} = \binom{n+r-1}{r-1} \left(H_{n+r-1} - H_{r-1}\right), \tag{6}$$

Benjamin and et all. [5] gave

$$H_n^{(r)} = \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} \frac{1}{s},$$

$$\sum_{s=1}^{r} H_n^{(s)} = H_{n+1}^{(r)} - \frac{1}{n+1} \tag{7}$$

and Bahşi and Solak [3] have studied on matrices with the hyperharmonic numbers.

Now we give some definitions and lemmas related to our study.

DEFINITION 1. Let $A = (a_{ij})$ be any $m \times n$ matrix. The *Euclidean norm* of A is

$$\|A\|_E = \sqrt{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)}.$$

DEFINITION 2. Let $A = (a_{ij})$ be any $m \times n$ matrix. The *spectral norm* of A is

$$\|A\|_2 = \sqrt{\max_i \lambda_i \left(A^H A\right)},$$

where $\lambda_i \left(A^H A\right)$ are eigenvalues of $A^H A$ and A^H is conjugate transpose of A.

There are two well known relations between Euclidean norm and spectral norm as the following:

$$\frac{1}{\sqrt{n}} \|A\|_E \leqslant \|A\|_2 \leqslant \|A\|_E \tag{8}$$

$$\|A\|_2 \leqslant \|A\|_E \leqslant \sqrt{n} \|A\|_2. \tag{9}$$

DEFINITION 3. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then their Hadamard product $A \circ B$ is defined

$$A \circ B = [a_{ij} b_{ij}].$$

DEFINITION 4. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $p \times r$ matrices, respectively. Then their Kronecker product $A \otimes B$ is defined

$$A \otimes B = [a_{ij} B].$$

LEMMA 1. *[12] Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \circ B\|_2 \leqslant \|A\|_2 \|B\|_2 .$$

LEMMA 2. *[12] Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \circ B\|_2 \leqslant r_1 (A) c_1 (B)$$

where $r_1 (A) = \max\limits_{1 \leqslant i \leqslant m} \sqrt{\sum\limits_{j=1}^{n} |a_{ij}|^2}$ *and* $c_1 (B) = \max\limits_{1 \leqslant j \leqslant n} \sqrt{\sum\limits_{i=1}^{m} |b_{ij}|^2}$.

LEMMA 3. *[12] Let A and B be two $m \times n$ matrices. Then we have*

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2 .$$

LEMMA 4. *[11] Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of $A^H A$ are $|\lambda_1|^2, |\lambda_2|^2, \dots, |\lambda_n|^2$.*

3. Main results

THEOREM 1. *The spectral norm of the matrix $H = \mathrm{Circ}(H_0^{(k)},\ H_1^{(k)}, \dots, H_{n-1}^{(k)})$ is*

$$\|H\|_2 = H_{n-1}^{(k+1)}.$$

Proof. Since the circulant matrix H is normal, its spectral norm is equal to its spectral radius. Furthermore, by considering H is irreducible and its entries are non-negative, we have that the spectral radius (or spectral norm) of the matrix H is equal to its Perron root. We select an n-dimensional column vector $v = (1, 1, \dots, 1)^T$, then

$$Hv = \left(\sum_{s=0}^{n-1} H_s^{(k)} \right) v.$$

Obviously, $\sum\limits_{s=0}^{n-1} H_s^{(k)}$ is an eigenvalue of H associated with v and it is the Perron root of H. Hence, from (5) we have

$$\|H\|_2 = \sum_{s=0}^{n-1} H_s^{(k)} = H_{n-1}^{(k+1)}. \quad \square$$

EXAMPLE 1. From Theorem 1, we have

$$\|H\|_2 = \begin{cases} H_{n-1}, & \text{if } k = 0, \\ nH_n - n, & \text{if } k = 1. \end{cases}$$

COROLLARY 1. *For the Euclidean norm of the matrix* $H=\operatorname{Circ}(H_0^{(k)},\ H_1^{(k)},\ldots,H_{n-1}^{(k)})$, *we have*

$$H_{n-1}^{(k+1)} \leqslant \|H\|_E \leqslant \sqrt{n}H_{n-1}^{(k+1)}.$$

Proof. This follows from Theorem 1 and the relation (9). □

COROLLARY 2. *The sum of squares of hyperharmonic numbers holds*

$$\frac{1}{\sqrt{n}}H_{n-1}^{(k+1)} \leqslant \sqrt{\sum_{s=0}^{n-1}\left(H_s^{(k)}\right)^2} \leqslant H_{n-1}^{(k+1)}. \tag{10}$$

Proof. The proof is trivial from the definition of Euclidean norm and Corollary 1. □

THEOREM 2. *The spectral norm of the matrix* $\widehat{H}=\operatorname{Circ}(H_k^{(0)},\ H_k^{(1)},\ldots,H_k^{(n-1)})$ *is*

$$\left\|\widehat{H}\right\|_2 = H_{k+1}^{(n-1)} + \frac{1}{k(k+1)} \ \textit{for}\ \ k>0.$$

Proof. This theorem can be proved by using a similar method to method of the proof of Theorem 1. But, we will use another method. Since $\widehat{H}$ is a circulant matrix, from (1) its eigenvalues are of the form

$$\underset{0\leqslant m\leqslant n-1}{\lambda_m} = \sum_{s=0}^{n-1} H_k^{(s)} e^{\frac{-2\pi ims}{n}}.$$

From (7)

$$\lambda_0 = \sum_{s=0}^{n-1} H_k^{(s)} = H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}. \tag{11}$$

Also, we have

$$\underset{1\leqslant m\leqslant n-1}{|\lambda_m|} = \left|\sum_{s=0}^{n-1} H_k^{(s)} e^{\frac{-2\pi ims}{n}}\right| \leqslant \sum_{s=0}^{n-1}\left|H_k^{(s)}\right|\left|e^{\frac{-2\pi ims}{n}}\right| \leqslant \sum_{s=0}^{n-1}\left|H_k^{(s)}\right| = \sum_{s=0}^{n-1} H_k^{(s)}. \tag{12}$$

By using Lemma 4 and the fact that the matrix $\widehat{H}$ is a normal matrix, we have

$$\left\|\widehat{H}\right\|_2 = \max_{0\leqslant m\leqslant n-1}|\lambda_m| = \max\left(|\lambda_0|,\max_{1\leqslant m\leqslant n-1}|\lambda_m|\right). \tag{13}$$

From (11), (12) and (13), we have

$$\left\|\widehat{H}\right\|_2 = H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}.$$

Thus the proof is completed. □

EXAMPLE 2. By using Theorem 2, we have

$$\left\| \widehat{H} \right\|_2 = \begin{cases} n, & \text{if } k=1, \\ \frac{n^2}{2}, & \text{if } k=2. \end{cases}$$

COROLLARY 3. *The Euclidean norm of the matrix* $\widehat{H} = \mathrm{Circ}(H_k^{(0)},\ H_k^{(1)},\ldots,H_k^{(n-1)})$ *holds*

$$H_{k+1}^{(n-1)} + \frac{1}{k(k+1)} \leqslant \left\| \widehat{H} \right\|_E \leqslant \sqrt{n}(H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}).$$

Proof. This follows from Theorem 2 and the relation (9). □

COROLLARY 4. *The sum of squares of* $H_k^{(s)}$ *holds*

$$\frac{1}{\sqrt{n}}(H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}) \leqslant \sqrt{\sum_{s=0}^{n-1} \left(H_k^{(s)} \right)^2} \leqslant (H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}). \tag{14}$$

Proof. This follows from the definition of Euclidean norm and Corollary 3. □

COROLLARY 5. *The spectral norm of the Hadamard product of* $H = \mathrm{Circ}(H_0^{(m)},$ $H_1^{(m)},\ldots,H_{n-1}^{(m)})$ *and* $\widehat{H} = \mathrm{Circ}(H_k^{(0)},\ H_k^{(1)},\ldots,H_k^{(n-1)})$ *holds*

$$\left\| H \circ \widehat{H} \right\|_2 \leqslant H_{n-1}^{(m+1)}(H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}).$$

Proof. The proof is trivial since $\left\| H \circ \widehat{H} \right\|_2 \leqslant \|H\|_2 \left\| \widehat{H} \right\|_2$. □

COROLLARY 6. *The spectral norm of the Kronecker product of* $H = \mathrm{Circ}(H_0^{(m)},$ $H_1^{(m)},\ldots,H_{n-1}^{(m)})$ *and* $\widehat{H} = \mathrm{Circ}(H_k^{(0)},\ H_k^{(1)},\ldots,H_k^{(n-1)})$ *holds*

$$\left\| H \otimes \widehat{H} \right\|_2 = H_{n-1}^{(m+1)}(H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}).$$

Proof. The proof is trivial since $\left\| H \otimes \widehat{H} \right\|_2 = \|H\|_2 \left\| \widehat{H} \right\|_2$. □

THEOREM 3. *Let* $H_r = \mathrm{Circr}(H_0^{(k)}, H_1^{(k)},\ldots,H_{n-1}^{(k)})$ *be an* $r-$*circulant matrix.*
$i)$ *If* $|r| \geqslant 1$, *then*

$$\frac{1}{\sqrt{n}} H_{n-1}^{(k+1)} \leqslant \|H_r\|_2 \leqslant |r| \left(H_{n-1}^{(k+1)} \right)^2$$

ii) *If* $|r|<1$, *then*

$$\frac{|r|}{\sqrt{n}}H_{n-1}^{(k+1)} \leqslant \|H_r\|_2 \leqslant \sqrt{n-1}H_{n-1}^{(k+1)}.$$

Proof. Since the matrix H_r is of the form

$$H_r=\begin{bmatrix} H_0^{(k)} & H_1^{(k)} & H_2^{(k)} & \cdots & H_{n-2}^{(k)} & H_{n-1}^{(k)} \\ rH_{n-1}^{(k)} & H_0^{(k)} & H_1^{(k)} & \cdots & H_{n-3}^{(k)} & H_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rH_2^{(k)} & rH_3^{(k)} & rH_4^{(k)} & \cdots & H_0^{(k)} & H_1^{(k)} \\ rH_1^{(k)} & rH_2^{(k)} & rH_3^{(k)} & \cdots & rH_{n-1}^{(k)} & H_0^{(k)} \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|H_r\|_E=\sqrt{\sum_{s=0}^{n-1}(n-s)\left(H_s^{(k)}\right)^2+\sum_{s=0}^{n-1}s|r|^2\left(H_s^{(k)}\right)^2}.$$

i) Since $|r|\geqslant 1$, by (10) we have

$$\|H_r\|_E \geqslant \sqrt{\sum_{s=0}^{n-1}(n-s)\left(H_s^{(k)}\right)^2+\sum_{s=0}^{n-1}s\left(H_s^{(k)}\right)^2}=\sqrt{n\sum_{s=0}^{n-1}\left(H_s^{(k)}\right)^2}\geqslant H_{n-1}^{(k+1)}.$$

Hence, from (8)

$$\|H_r\|_2 \geqslant \frac{1}{\sqrt{n}}H_{n-1}^{(k+1)}.$$

Now, let we consider the matrices B and C be as

$$B=\begin{bmatrix} rH_0^{(k)} & 1 & 1 & \cdots & 1 & 1 \\ rH_{n-1}^{(k)} & rH_0^{(k)} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rH_2^{(k)} & rH_3^{(k)} & rH_4^{(k)} & \cdots & rH_0^{(k)} & 1 \\ rH_1^{(k)} & rH_2^{(k)} & rH_3^{(k)} & \cdots & rH_{n-1}^{(k)} & rH_0^{(k)} \end{bmatrix}$$

and

$$C=\begin{bmatrix} H_0^{(k)} & H_1^{(k)} & H_2^{(k)} & \cdots & H_{n-2}^{(k)} & H_{n-1}^{(k)} \\ 1 & H_0^{(k)} & H_1^{(k)} & \ldots & H_{n-3}^{(k)} & H_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & H_0^{(k)} & H_1^{(k)} \\ 1 & 1 & 1 & \cdots & 1 & H_0^{(k)} \end{bmatrix}.$$

That is, $H_r=B\circ C$. Then we compute

$$r_1(B)=\max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{j=1}^{n}\left|b_{nj}\right|^2}=\sqrt{|r|^2\sum_{s=0}^{n-1}\left(H_s^{(k)}\right)^2}$$

and

$$c_1(B) = \max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2} = \sqrt{\sum_{i=1}^{n}\left|c_{in}\right|^2} = \sqrt{\sum_{s=0}^{n-1}\left(H_s^{(k)}\right)^2}.$$

Hence, from (10) and Lemma 2, we have

$$\|H_r\|_2 \leqslant r_1(B)\,c_1(B) \leqslant |r|\left(H_{n-1}^{(k+1)}\right)^2.$$

Thus,

$$\frac{1}{\sqrt{n}}H_{n-1}^{(k+1)} \leqslant \|H_r\|_2 \leqslant |r|\left(H_{n-1}^{(k+1)}\right)^2.$$

ii) Since $|r| < 1,$ by (10) we have

$$\begin{aligned}
\|H_r\|_E &= \sqrt{\sum_{s=0}^{n-1}(n-s)\left(H_s^{(k)}\right)^2 + \sum_{s=0}^{n-1}s\,|r|^2\left(H_s^{(k)}\right)^2} \\
&\geqslant \sqrt{\sum_{s=0}^{n-1}(n-s)\,|r|^2\left(H_s^{(k)}\right)^2 + \sum_{s=0}^{n-1}s\,|r|^2\left(H_s^{(k)}\right)^2} \\
&= |r|\sqrt{n\sum_{s=0}^{n-1}\left(H_s^{(k)}\right)^2} \geqslant |r|\,H_{n-1}^{(k+1)}.
\end{aligned}$$

From (8)

$$\|H_r\|_2 \geqslant \frac{|r|}{\sqrt{n}}H_{n-1}^{(k+1)}.$$

Now, let we consider the matrices D and E be as

$$D = \begin{bmatrix} H_0^{(k)} & 1 & 1 \cdots & 1 & 1 \\ r & H_0^{(k)} & 1 \cdots & 1 & 1 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ r & r & r \cdots & H_0^{(k)} & 1 \\ r & r & r \cdots & r & H_0^{(k)} \end{bmatrix}$$

and

$$E = \begin{bmatrix} H_0^{(k)} & H_1^{(k)} & H_2^{(k)} & \cdots & H_{n-2}^{(k)} & H_{n-1}^{(k)} \\ H_{n-1}^{(k)} & H_0^{(k)} & H_1^{(k)} & \cdots & H_{n-3}^{(k)} & H_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ H_2^{(k)} & H_3^{(k)} & H_4^{(k)} & \cdots & H_0^{(k)} & H_1^{(k)} \\ H_1^{(k)} & H_2^{(k)} & H_3^{(k)} & \cdots & H_{n-1}^{(k)} & H_0^{(k)} \end{bmatrix}.$$

That is, $H_r = D \circ E.$ Then we compute

$$r_1(D) = \max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}\left|d_{ij}\right|^2} = \sqrt{H_0^{(k)} + n - 1} = \sqrt{n-1}$$

and

$$c_1(D) = \max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}\left|d_{ij}\right|^2} = \sqrt{\sum_{s=0}^{n-1}\left(H_s^{(k)}\right)^2}.$$

Hence, from (10) and Lemma 2, we have

$$\|H_r\|_2 \leqslant r_1(D)\,c_1(D) \leqslant \sqrt{n-1}H_{n-1}^{(k+1)}.$$

Thus,

$$\frac{|r|}{\sqrt{n}}H_{n-1}^{(k+1)} \leqslant \|H_r\|_2 \leqslant \sqrt{n-1}H_{n-1}^{(k+1)}.$$

Thus, the proof is completed. □

EXAMPLE 3. By using Theorem 3, if $|r| \geqslant 1$, we have

$$\frac{1}{\sqrt{n}}H_{n-1} \leqslant \|H_r\|_2 \leqslant |r|\,H_{n-1}^2, \ \ if \ \ k=0,$$

$$\sqrt{n}\,(H_n-1) \leqslant \|H_r\|_2 \leqslant n^2\,|r|\,(H_n-1)^2, \ \ if \ \ k=1$$

and if $|r|<1$*, we have*

$$\frac{|r|}{\sqrt{n}}H_{n-1} \leqslant \|H_r\|_2 \leqslant \sqrt{n-1}H_{n-1}, \ \ if \ \ k=0,$$

$$\sqrt{n}\,|r|\,(H_n-1) \leqslant \|H_r\|_2 \leqslant n\sqrt{n-1}\,(H_n-1), \ \ if \ \ k=1.$$

THEOREM 4. *Let* $\widehat{H_r} = \mathrm{Circr}(H_k^{(0)}, H_k^{(1)}, \ldots, H_k^{(n-1)})$ *be an* $r-$*circulant matrix and* $k>0$.

i) If $|r| \geqslant 1$*, then*

$$\frac{1}{\sqrt{n}}\left(H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}\right) \leqslant \left\|\widehat{H_r}\right\|_2 \leqslant |r|\left(H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}\right)^2.$$

ii) If $|r|<1$*, then*

$$\frac{|r|}{\sqrt{n}}\left(H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}\right) \leqslant \left\|\widehat{H_r}\right\|_2 \leqslant \sqrt{n}\left(H_{k+1}^{(n-1)} + \frac{1}{k(k+1)}\right).$$

Proof. Since the matrix $\widehat{H_r}$ is of the form

$$\widehat{H_r} = \begin{bmatrix} H_k^{(0)} & H_k^{(1)} & H_k^{(2)} & \cdots & H_k^{(n-2)} & H_k^{(n-1)} \\ rH_k^{(n-1)} & H_k^{(0)} & H_k^{(1)} & \cdots & H_k^{(n-3)} & H_k^{(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rH_k^{(2)} & rH_k^{(3)} & rH_k^{(4)} & \cdots & H_k^{(0)} & H_k^{(1)} \\ rH_k^{(1)} & rH_k^{(2)} & rH_k^{(3)} & \cdots & rH_k^{(n-1)} & H_k^{(0)} \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\left\|\widehat{H_r}\right\|_E=\sqrt{\sum_{s=0}^{n-1}(n-s)\left(H_k^{(s)}\right)^2+\sum_{s=0}^{n-1}s\,|r|^2\left(H_k^{(s)}\right)^2}.$$

i) Since $|r|\geqslant 1,$ by (14) we obtain

$$\left\|\widehat{H_r}\right\|_E\geqslant\sqrt{\sum_{s=0}^{n-1}(n-s)\left(H_k^{(s)}\right)^2+\sum_{s=0}^{n-1}s\left(H_k^{(s)}\right)^2}=\sqrt{n\sum_{s=0}^{n-1}\left(H_k^{(s)}\right)^2}\geqslant H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}.$$

From (8)

$$\left\|\widehat{H_r}\right\|_2\geqslant\frac{1}{\sqrt{n}}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right).$$

Now, let we consider the matrices B and C be as

$$B=\begin{bmatrix}1 & 1 & 1 & \cdots & 1 & 1\\ rH_k^{(n-1)} & 1 & 1 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ rH_k^{(2)} & rH_k^{(3)} & rH_k^{(4)} & \cdots & 1 & 1\\ rH_k^{(1)} & rH_k^{(2)} & rH_k^{(3)} & \cdots & rH_k^{(n-1)} & 1\end{bmatrix}$$

and

$$C=\begin{bmatrix}H_k^{(0)} & H_k^{(1)} & H_k^{(2)} & \cdots & H_k^{(n-2)} & H_k^{(n-1)}\\ 1 & H_k^{(0)} & H_k^{(1)} & \cdots & H_k^{(n-3)} & H_k^{(n-2)}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 1 & \cdots & H_k^{(0)} & H_k^{(1)}\\ 1 & 1 & 1 & \cdots & 1 & H_k^{(0)}\end{bmatrix}.$$

That is, $\widehat{H_r}=B\circ C.$ Then we have

$$\begin{aligned}r_1(B)&=\max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{j=1}^{n}\left|b_{nj}\right|^2}=\sqrt{1+\sum_{s=1}^{n-1}|r|^2\left(H_k^{(s)}\right)^2}\\ &\leqslant|r|\sqrt{\sum_{s=0}^{n-1}\left(H_k^{(s)}\right)^2}\end{aligned}$$

and

$$c_1(B)=\max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2}=\sqrt{\sum_{i=1}^{n}|c_{in}|^2}=\sqrt{\sum_{s=0}^{n-1}\left(H_k^{(s)}\right)^2}.$$

Hence, from (14) and Lemma 2, we have

$$\left\|\widehat{H_r}\right\|_2\leqslant r_1(B)\,c_1(B)\leqslant|r|\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)^2.$$

Thus,

$$\frac{1}{\sqrt{n}}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)\leqslant\left\|\widehat{H_r}\right\|_2\leqslant|r|\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)^2.$$

ii) Since $|r|<1,$ by (14) we have

$$\begin{aligned}\left\|\widehat{H_r}\right\|_E &= \sqrt{\sum_{s=0}^{n-1}(n-s)\left(H_k^{(s)}\right)^2+\sum_{s=0}^{n-1}s|r|^2\left(H_k^{(s)}\right)^2}\\ &\geqslant \sqrt{\sum_{s=0}^{n-1}(n-s)|r|^2\left(H_k^{(s)}\right)^2+\sum_{s=0}^{n-1}s|r|^2\left(H_k^{(s)}\right)^2}\\ &= \sqrt{n|r|^2\sum_{s=0}^{n-1}\left(H_k^{(s)}\right)^2}\geqslant|r|\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right).\end{aligned}$$

From (8)

$$\left\|\widehat{H_r}\right\|_2\geqslant\frac{|r|}{\sqrt{n}}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right).$$

Now, let the matrices D and E be as

$$D=\begin{bmatrix}1&1&1&\cdots&1&1\\ r&1&1&\cdots&1&1\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ r&r&r&\cdots&1&1\\ r&r&r&\cdots&r&1\end{bmatrix}$$

and

$$E=\begin{bmatrix}H_k^{(0)}&H_k^{(1)}&H_k^{(2)}&\cdots&H_k^{(n-2)}&H_k^{(n-1)}\\ H_k^{(n-1)}&H_k^{(0)}&H_k^{(1)}&\cdots&H_k^{(n-3)}&H_k^{(n-2)}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ H_k^{(2)}&H_k^{(3)}&H_k^{(4)}&\cdots&H_k^{(0)}&H_k^{(1)}\\ H_k^{(1)}&H_k^{(2)}&H_k^{(3)}&\cdots&H_k^{(n-1)}&H_k^{(0)}\end{bmatrix}.$$

That is, $\widehat{H_r}=D\circ E.$ Then we compute

$$r_1(D)=\max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}\left|d_{ij}\right|^2}=\sqrt{n}$$

and

$$c_1(D)=\max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}\left|d_{ij}\right|^2}=\sqrt{\sum_{s=0}^{n-1}\left(H_k^{(s)}\right)^2}.$$

Hence, from (14) and Lemma 2, we have

$$\left\|\widehat{H_r}\right\|_2\leqslant r_1(D)c_1(D)\leqslant\sqrt{n}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right).$$

Thus,

$$\frac{|r|}{\sqrt{n}}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)\leqslant\left\|\widehat{H_r}\right\|_2\leqslant\sqrt{n}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right).$$

This completes the proof. □

EXAMPLE 4. By using Theorem 4, if $|r|\geqslant 1$, we have

$$\sqrt{n}\leqslant\left\|\widehat{H_r}\right\|_2\leqslant|r|\,n^2,\ \text{ if } k=1,$$

$$\frac{n\sqrt{n}}{2}\leqslant\left\|\widehat{H_r}\right\|_2\leqslant\frac{n^4\,|r|}{4},\quad \textit{if}\ \ k=2$$

and if $|r|<1$, *we have*

$$|r|\sqrt{n}\leqslant\left\|\widehat{H_r}\right\|_2\leqslant n\sqrt{n},\quad \textit{if}\ \ k=1,$$

$$\frac{n\sqrt{n}\,|r|}{2}\leqslant\left\|\widehat{H_r}\right\|_2\leqslant\frac{n^2\sqrt{n}}{2},\quad \textit{if}\ \ k=2.$$

COROLLARY 7. *The spectral norm of the Hadamard product of* $H_r=\mathrm{Circr}(H_0^{(m)}, H_1^{(m)},\ldots,H_{n-1}^{(m)})$ *and* $\widehat{H_r}=\mathrm{Circr}(H_k^{(0)},H_k^{(1)},\ldots,H_k^{(n-1)})$ *holds*

i) If $|r|\geqslant 1$, *then*

$$\left\|H_r\circ\widehat{H_r}\right\|_2\leqslant|r|^2\left(H_{n-1}^{(m+1)}\right)^2\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)^2.$$

ii) If $|r|<1$, *then*

$$\left\|H_r\circ\widehat{H_r}\right\|_2\leqslant\sqrt{n\,(n-1)}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)H_{n-1}^{(m+1)}.$$

Proof. The proof is trivial since $\left\|H_r\circ\widehat{H_r}\right\|_2\leqslant\|H_r\|_2\left\|\widehat{H_r}\right\|_2$. □

COROLLARY 8. *The spectral norm of the Kronecker product of* $H_r=\mathrm{Circr}(H_0^{(k)}, H_1^{(k)},\ldots,H_{n-1}^{(k)})$ *and* $\widehat{H_r}=\mathrm{Circr}(H_k^{(0)},H_k^{(1)},\ldots,H_k^{(n-1)})$ *holds*

i) If $|r|\geqslant 1$, *then*

$$\frac{1}{n}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)H_{n-1}^{(m+1)}\leqslant\left\|H_r\otimes\widehat{H_r}\right\|_2\leqslant|r|^2\left(H_{n-1}^{(m+1)}\right)^2\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)^2.$$

ii) If $|r|<1$*, then*

$$\frac{|r|^2}{n}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)H_{n-1}^{(m+1)} \leqslant \left\|H_r \otimes \widehat{H_r}\right\|_2 \leqslant \sqrt{n(n-1)}\left(H_{k+1}^{(n-1)}+\frac{1}{k(k+1)}\right)H_{n-1}^{(m+1)}.$$

Proof. The proof is trivial since $\left\|H_r \otimes \widehat{H_r}\right\|_2 = \|H_r\|_2 \left\|\widehat{H_r}\right\|_2$. $\square$

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Mustafa Bahşi
Aksaray University, Education Faculty
Aksaray, Turkey
e-mail: mhvbahsi@yahoo.com