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SOME MEASURE THEORETIC ASPECTS OF STEFFENSEN'S AND REVERSED STEFFENSEN'S INEQUALITY

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Abstract. We find necessary and sufficient conditions for Steffensen's and inverse Steffensen's inequality in measure theoretic settings. We also explain natural transition from the Riemann integral in Steffensen's inequality to an integral with measure.

1. Introduction and preliminary results

In 1918 Steffensen proved the following inequality, [15]:

THEOREM 1.1. *Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \leqslant g \leqslant 1$ and*

$$\lambda = \int_a^b g(t)dt. \tag{1.1}$$

Then we have

$$\int_{b-\lambda}^{b} f(t)dt \leqslant \int_a^b f(t)g(t)dt \leqslant \int_a^{a+\lambda} f(t)dt. \tag{1.2}$$

Measure theoretic aspect of Steffensen's inequality has long and interesting history. In order to generalize result from [10] for higher order convexity version of Steffensen's inequality, Fink introduced in [3] an auxiliary classes of functions, and gave necessary and sufficient conditions for Steffensen's inequality in measure theoretic environment. In his paper, he replaced the middle integral in (1.2) with integral with respect to regular Borel measure, and expressed edge integrals in (1.2) via Lebesgue measure. Evard and Gauchman in [2, 4] and [5] extended Steffensen's inequality to a more general version replacing line segment $[a,b]$ with a general set X, with measure concentrated on it, and replacing $[b-\lambda,b]$ and $[a,a+\lambda]$ with measurable, upper and lower separating, subsets U, V of X, giving sufficient conditions for Steffensen's inequality in that setting.

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Our starting point for generalization of Steffensen's inequality is one observation in [11] (see also [14, p. 15]) where it is pointed out that (1.2) follows from the following two identities (assuming that (1.1) holds):

$$\begin{aligned}&\int_a^{a+\lambda} f(t)dt-\int_a^b f(t)g(t)dt\\&=\int_a^{a+\lambda}[f(t)-f(a+\lambda)][1-g(t)]dt+\int_{a+\lambda}^b[f(a+\lambda)-f(t)]g(t)dt\end{aligned}\tag{1.3}$$

and

$$\begin{aligned}&\int_a^b f(t)g(t)dt-\int_{b-\lambda}^b f(t)dt\\&=\int_a^{b-\lambda}[f(t)-f(b-\lambda)]g(t)dt+\int_{b-\lambda}^b[f(b-\lambda)-f(t)][1-g(t)]dt.\end{aligned}\tag{1.4}$$

The full impact of (1.3) and (1.4) on characterization of Steffensen's inequality can be found in [13, p. 183–186] and summarized in the following three theorems.

THEOREM 1.2. *Let f and g be integrable functions on $[a,b]$ and let $\lambda=\int_a^b g(t)dt$.*

(a) The second inequality in (1.2) holds for every nonincreasing function f if and only if

$$\int_a^x g(t)dt\leqslant x-a \text{ and } \int_x^b g(t)dt\geqslant 0,\quad \text{for every } x\in[a,b].$$

(b) The first inequality in (1.2) holds for every nonincreasing function f if and only if

$$\int_x^b g(t)dt\leqslant b-x \text{ and } \int_a^x g(t)dt\geqslant 0,\quad \text{for every } x\in[a,b].$$

Using identities (1.3) and (1.4) and integration by parts, Pečarić in [12] also proved converse results.

THEOREM 1.3. *Let $f:I\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ ($[a,b]\subseteq I$ where I is an interval in $\mathbb{R}$) be integrable functions, and $a+\lambda\in I$ where λ is given by (1.1). Then*

$$\int_a^{a+\lambda} f(t)dt\leqslant\int_a^b f(t)g(t)dt$$

holds for every nonincreasing function f *if and only if*

$$\int_a^x g(t)dt \geqslant x-a, \text{ for } x \in [a,a+\lambda] \quad \text{and} \quad \int_x^b g(t)dt \leqslant 0, \text{ for } x \in (a+\lambda,b],$$

and $0 \leqslant \lambda \leqslant b-a$;
or

$$\int_a^x g(t)dt \geqslant x-a, \quad \text{for } x \in [a,b],$$

and $\lambda > b-a$;
or

$$\int_x^b g(t)dt \leqslant 0, \quad \text{for } x \in [a,b]$$

and $\lambda < 0$.

THEOREM 1.4. *Let* $f : I \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ *(* $[a,b] \subseteq I$ *where* I *is an interval in* $\mathbb{R}$*) be integrable functions, and* $b-\lambda \in I$ *where* λ *is given by* (1.1)*. Then*

$$\int_{b-\lambda}^b f(t)dt \geqslant \int_a^b f(t)g(t)dt$$

holds for every nonincreasing function f *if and only if*

$$\int_a^x g(t)dt \leqslant 0, \text{ for } x \in [a,b-\lambda] \quad \text{and} \quad \int_x^b g(t)dt \geqslant b-x, \quad \text{for } x \in (b-\lambda,b],$$

and $0 \leqslant \lambda \leqslant b-a$;
or

$$\int_x^b g(t)dt \geqslant b-x, \quad \text{for } x \in [a,b],$$

and $\lambda > b-a$;
or

$$\int_a^x g(t)dt \leqslant 0, \quad \text{for } x \in [a,b]$$

and $\lambda < 0$.

The next two theorems from [8], with sufficient conditions, we will need in the main section.

THEOREM 1.5. *Let μ be a positive, finite, measure on $\mathscr{B}([a,b])$ and let f and g be measurable functions such that f is nonincreasing and $0 \leqslant g \leqslant 1$. If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t), \tag{1.5}$$

then

$$\int_{[a,a+\lambda]} f(t)d\mu(t) \geqslant \int_{[a,b]} f(t)g(t)d\mu(t). \tag{1.6}$$

THEOREM 1.6. *Let μ be a positive, finite, measure on $\mathscr{B}([a,b])$ and let f and g be measurable functions such that f is nonincreasing and $0 \leqslant g \leqslant 1$. If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\mu((b-\lambda,b]) = \int_{[a,b]} g(t)d\mu(t),$$

then

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \leqslant \int_{[a,b]} f(t)g(t)d\mu(t).$$

2. Main results

The aim of this section is to find analogous results of Theorems 1.2, 1.3 and 1.4 in measure theoretic settings. Integration by parts, used there, is what we want to avoid in our measure theoretic approach so our assumptions on the function f are just slightly changed.

2.1. Sufficient and necessary conditions for Steffensen's inequality

THEOREM 2.1. *Let μ be a finite, positive measure on $\mathscr{B}([a,b])$, let $g:[a,b] \to \mathbb{R}$ be a μ-integrable function.*

(a) Let λ be a positive constant such that $\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t)$. The inequality

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leqslant \int_{[a,a+\lambda]} f(t)d\mu(t) \tag{2.1}$$

holds for every nonincreasing, right-continuous function $f:[a,b] \to \mathbb{R}$ if and only if

$$\int_{[a,x)} g(t)d\mu(t) \leqslant \mu([a,x)) \quad \text{and} \quad \int_{[x,b]} g(t)d\mu(t) \geqslant 0, \quad \text{for every } x \in [a,b]. \tag{2.2}$$

(b) Let λ *be a positive constant such that* $\mu((b-\lambda,b]) = \int_{[a,b]} g(t)d\mu(t)$. *The inequality*

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \leqslant \int_{[a,b]} f(t)g(t)d\mu(t)$$

holds for every nonincreasing, right-continuous function $f:[a,b]\to\mathbb{R}$ *if and only if*

$$\int_{[x,b]} g(t)d\mu(t) \leqslant \mu([x,b]) \quad \text{and} \quad \int_{[a,x)} g(t)d\mu(t) \geqslant 0, \quad \text{for every } x \in [a,b].$$

Proof.
(a) For the sufficiency part we use the identity

$$\begin{aligned}&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t)\\&= \int_{[a,a+\lambda]} [f(t)-f(a+\lambda)][1-g(t)]d\mu(t) + \int_{(a+\lambda,b]} [f(a+\lambda)-f(t)]g(t)d\mu(t)\end{aligned} \tag{2.3}$$

similar to (1.3). We define a new measure ν on $\sigma-$algebra $\mathscr{B}((a,b])$ such that, on an algebra of finite disjoint unions of half open intervals, we set $\nu((c,d]) = f(c)-f(d)$, for $a<c<d\leqslant b$, and then we pass to $\mathscr{B}((a,b])$ in a unique way (for details see, for example, [1, p. 21]).

Now, using Fubini, we have

$$\begin{aligned}&\int_{[a,a+\lambda]} [f(t)-f(a+\lambda)][1-g(t)]d\mu(t)\\&= \int_{[a,a+\lambda]} \left[\int_{(t,a+\lambda]} d\nu(x)\right][1-g(t)]d\mu(t) = \int_{(a,a+\lambda]} \left[\int_{[a,x)} (1-g(t))d\mu(t)\right] d\nu(x).\end{aligned}$$

Similarly,

$$\int_{(a+\lambda,b]} [f(a+\lambda)-f(t)]g(t)\mu(t) = \int_{(a+\lambda,b]} \left[\int_{[x,b]} g(t)d\mu(t)\right] d\nu(x).$$

This means that (2.3) is in fact

$$\begin{aligned}&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t)\\&= \int_{(a,a+\lambda]} \left[\int_{[a,x)} (1-g(t))d\mu(t)\right] d\nu(x) + \int_{(a+\lambda,b]} \left[\int_{[x,b]} g(t)d\mu(t)\right] d\nu(x),\end{aligned} \tag{2.4}$$

concluding (2.1) under assumptions (2.2).

The previous conditions are also necessary. In fact, if x is any element of $[a,b]$, then let f be the function defined by

$$f(t)=\begin{cases}1, & t<x;\\ 0, & t\geqslant x.\end{cases}$$

Using inequality (1.6) from Theorem 1.5 we obtain

$$\begin{aligned}\int_{[a,x)} g(t)d\mu(t) &= \int_{[a,b]} f(t)g(t)d\mu(t) \leqslant \int_{[a,a+\lambda]} f(t)d\mu(t)\\ &= \begin{cases}\mu([a,x)), & x\in[a,a+\lambda];\\ \mu([a,a+\lambda]), & x\in(a+\lambda,b].\end{cases}\end{aligned} \tag{2.5}$$

If $x\in(a+\lambda,b]$ then $\mu([a,x))\geqslant\mu([a,a+\lambda])$, from (2.5), we have

$$\int_{[a,x)} g(t)d\mu(t)\leqslant\mu([a,x)),\quad \text{for every } x\in[a,b].$$

Also, if $x\in(a+\lambda,b]$, from (2.5) we have $\int_{[a,x)} g(t)d\mu(t)\leqslant\mu([a,a+\lambda])=\int_{[a,b]} g(t)d\mu(t)$, concluding

$$\int_{[x,b]} g(t)d\mu(t)\geqslant 0,\quad \text{for every } x\in(a+\lambda,b].$$

Finally, if $x\in[a,a+\lambda]$, then

$$\begin{aligned}\int_{[x,b]} g(t)d\mu(t) &= \int_{[a,b]} g(t)d\mu(t)-\int_{[a,x)} g(t)d\mu(t)\\ &\geqslant \mu([a,a+\lambda])-\mu([a,x))=\mu([x,a+\lambda])\geqslant 0,\end{aligned}$$

concluding

$$\int_{[x,b]} g(t)d\mu(t)\geqslant 0,\quad \text{for every } x\in[a,b].$$

(b) The proof of this part is similar to the proof of (a)-part so we omit the details. □

2.2. Sufficient and necessary conditions for reversed Steffensen's inequality

THEOREM 2.2. *Let μ be a finite, positive measure on $\mathscr{B}(I)$, $g:[a,b]\to\mathbb{R}$ ($[a,b]\subseteq I$, I is an interval in $\mathbb{R}$) be a μ-integrable function, and $a+\lambda\in I$ where*

$$\mu([a,a+\lambda])=\int_{[a,b]} g(t)d\mu(t), \tag{2.6}$$

for $\lambda\geqslant 0$, and

$$-\mu([a+\lambda,a])=\int_{[a,b]} g(t)d\mu(t),$$

for $\lambda < 0$.

Then, for $\lambda \geqslant 0$,

$$\int_{[a,a+\lambda]} f(t)d\mu(t) \leqslant \int_{[a,b]} f(t)g(t)d\mu(t); \tag{2.7}$$

and for $\lambda < 0$,

$$-\int_{[a+\lambda,a]} f(t)d\mu(t) \leqslant \int_{[a,b]} f(t)g(t)d\mu(t); \tag{2.8}$$

for every nonincreasing, right continuous function $f: I \to \mathbb{R}$ *if and only if either*

$$\int_{[a,x)} g(t)d\mu(t) \geqslant \mu([a,x)),\ \textit{for}\ x \in [a,a+\lambda]\ \textit{and} \int_{[x,b]} g(t)d\mu(t) \leqslant 0,\ \textit{for}\ x \in (a+\lambda,b], \tag{2.9}$$

where $0 \leqslant \lambda \leqslant b-a$;
or

$$\int_{[a,x)} g(t)d\mu(t) \geqslant \mu([a,x)), \quad \textit{for}\ x \in [a,b], \tag{2.10}$$

where $\lambda > b-a$;
or

$$\int_{[x,b]} g(t)d\mu(t) \leqslant 0, \quad \textit{for}\ x \in [a,b], \tag{2.11}$$

where $\lambda < 0$.

Proof. Necessity part. Putting

$$f(t) = \begin{cases} 1, & t < x; \\ 0, & t \geqslant x \end{cases}$$

in (2.7) we get (2.9), (2.10) and (2.11) for choices $\lambda \in [0,b-a]$, $\lambda \in (b-a,+\infty)$, and $\lambda \in (-\infty,0)$, respectively.

Sufficiency part. Let $\lambda \in [0,b-a]$. Then from (2.4) we have

$$\begin{aligned} &\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\ &= \int_{(a,a+\lambda]} \left[\int_{[a,x)} (1-g(t))d\mu(t)\right] d\nu(x) + \int_{(a+\lambda,b]} \left[\int_{[x,b]} g(t)d\mu(t)\right] d\nu(x) \leqslant 0, \end{aligned}$$

where ν is a measure defined on $\mathscr{B}((a,b])$, with $\nu((c,d]) = f(c) - f(d)$, for $c < d$, $c,d \in I$.

If $\lambda \in (b-a,+\infty)$ then

$$\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) = \int_{[a,b]} f(t)(1-g(t))d\mu(t) + \int_{(b,a+\lambda]} f(t)d\mu(t)$$
$$= \int_{[a,b]} (f(t)-f(b))(1-g(t))d\mu(t) + \int_{(b,a+\lambda]} (f(t)-f(b))d\mu(t)$$
$$= \int_{[a,b]} \int_{[a,x)} (1-g(t))d\mu(t)d\nu(x) - \int_{(b,a+\lambda]} \int_{[x,a+\lambda]} d\mu(t)d\nu(x)$$
$$= \int_{[a,b]} \left(\mu([a,x)) - \int_{[a,x)} g(t)d\mu(t)\right) d\nu(x) - \int_{(b,a+\lambda]} \mu([x,a+\lambda])d\nu(x) \leqslant 0.$$

If $\lambda \in (-\infty,0)$ then

$$-\int_{[a+\lambda,a]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t)$$
$$= \int_{[a,b]} (f(a)-f(t))g(t)d\mu(t) + \int_{[a+\lambda,a]} (f(a)-f(t))d\mu(t)$$
$$= \int_{[a,b]} \int_{[x,b]} g(t)d\mu(t)d\nu(x) - \int_{[a+\lambda,a]} \mu([a+\lambda,x])d\nu(x) \leqslant 0. \quad \square$$

THEOREM 2.3. *Let μ be a finite, positive measure on $\mathscr{B}(I)$, $g:[a,b]\to\mathbb{R}$ ($[a,b]\subseteq I$, I is an interval in $\mathbb{R}$) be a μ-integrable function, and $b-\lambda \in I$ where*

$$\mu((b-\lambda,b]) = \int_{[a,b]} g(t)d\mu(t),$$

for $\lambda \geqslant 0$, and

$$-\mu([b,b-\lambda)) = \int_{[a,b]} g(t)d\mu(t),$$

for $\lambda < 0$.

Then, for $\lambda \geqslant 0$

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \geqslant \int_{[a,b]} f(t)g(t)d\mu(t); \tag{2.12}$$

and for $\lambda < 0$,

$$-\int_{[b,b-\lambda)} f(t)d\mu(t) \geqslant \int_{[a,b]} f(t)g(t)d\mu(t); \tag{2.13}$$

for every nonincreasing, right continuous function $f:I\to\mathbb{R}$ if and only if either

$$\int_{[a,x)} g(t)d\mu(t) \leqslant 0, \text{ for } x \in [a,b-\lambda] \text{ and } \int_{[x,b]} g(t)d\mu(t) \geqslant \mu([x,b]), \text{ for } x \in (b-\lambda,b],$$

where $0 \leqslant \lambda \leqslant b-a$;
or

$$\int_{[x,b]} g(t)d\mu(t) \geqslant \mu([x,b]), \quad \textit{for } x \in [a,b]$$

where $\lambda > b-a$;
or

$$\int_{[a,x)} g(t)d\mu(t) \leqslant 0, \quad \textit{for } x \in [a,b]$$

where $\lambda < 0$.

Proof. Similar to the proof of Theorem 2.2. □

THEOREM 2.4. *Let* μ *be a finite, positive measure on* $\mathscr{B}(I)$, $g:[a,b] \to \mathbb{R}$ *be a* μ*-integrable function for which there exists* $c \in [a,b]$ *such that* $g(x) \geqslant 1$ *for* $x \in [a,c]$ *and* $g(x) \leqslant 0$ *for* $x \in (c,b]$. *Then* (2.7) *(resp.* (2.8)*) for* $\lambda \geqslant 0$ *(resp.* $\lambda < 0$*) is valid for every nonincreasing function* $f: I \to \mathbb{R}$ *provided that* $[a,b] \subseteq I$ *and* $a+\lambda \in I$.

Proof. Let $\lambda \in [0,b-a]$. Suppose that $c \leqslant a+\lambda$. Then it is obvious that

$$\int_{[a,x)} g(t)d\mu(t) \geqslant \mu([a,x)), \text{for } x \in [a,c] \quad \text{and} \quad \int_{[x,b]} g(t)d\mu(t) \leqslant 0, \text{for } x \in (a+\lambda,b].$$

Suppose that for some $x_0 \in (c,a+\lambda]$ we have $\int_{[a,x_0)} g(t)d\mu(t) < \mu([a,x_0))$. Since $\int_{[x_0,b]} g(t)d\mu(t) \leqslant 0$, it follows $\mu([a,a+\lambda]) = \int_{[a,b]} g(t)d\mu(t) < \mu([a,x_0))$, hence $a+\lambda < x_0$, what is, evidently, a contradiction. Analogously, in the case $c > a+\lambda$ we can also prove that (2.9) holds.

Let $\lambda \in (b-a,\infty)$. Then $\int_{[a,x)} g(t)d\mu(t) \geqslant \mu([a,x))$ for $x \in [a,c]$. For $x \in (c,b]$ we have

$$\begin{aligned}\int_{[a,x)} g(t)d\mu(t) &= \int_{[a,b]} g(t)d\mu(t) - \int_{[x,b]} g(t)d\mu(t)\\ &\geqslant \int_{[a,b]} g(t)d\mu(t) = \mu([a,a+\lambda]) \geqslant \mu([a,x)),\end{aligned}$$

and condition (2.10) is fulfilled.

If $\lambda \in (-\infty,0)$ then if $x \in [a,c]$

$$\int_{[x,b]} g(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t) - \int_{[a,x)} g(t)d\mu(t) = -\mu([a+\lambda,a]) - \mu([a,x)) \leqslant 0;$$

if $x \in (c,b]$ then $\int_{[x,b]} g(t)d\mu(t) \leqslant 0$ so (2.11) is again valid. □

Similar to Theorem 2.4 we can prove the next theorem.

THEOREM 2.5. *Let* μ *be a finite, positive measure on* $\mathscr{B}(I)$, $g:[a,b] \to \mathbb{R}$ *be a* μ*-integrable function for which there exists* $c \in [a,b]$ *such that* $g(x) \leqslant 0$ *for* $x \in [a,c]$ *and* $g(x) \geqslant 1$ *for* $x \in (c,b]$. *Then* (2.12) *(resp.* (2.13)*) for* $\lambda \geqslant 0$ *(resp.* $\lambda < 0$*) is valid for every nonincreasing function* $f: I \to \mathbb{R}$ *provided that* $[a,b] \subseteq I$ *and* $b-\lambda \in I$.

3. Some applications and concluding remarks

REMARK 3.1. Under assumption (1.5) and from (1.6) it follows that

$$\mathfrak{L}(f) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,a+\lambda]} f(t)d\mu(t)$$

is a linear functional, acting on μ-integrable functions, positive for nondecreasing functions f on $[a,b]$. This fact, combined with results from paper [7], is used in paper [8] in order to produce exponentially convex functions and even further, some new Cauchy means. The same construction can be applied here, but under modified conditions on the function g and measure μ. For example, if (2.6) and (2.11) are valid then

$$\mathfrak{M}(f) = \int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t).$$

is a linear functional which is positive for nondecreasing functions f on $[a,b]$. Using similar construction as in [7] we could produce exponential convexity and Cauchy means.

REMARK 3.2. It is obvious that the choice of measure μ in the main section covers some known results: Lebesgue measure gives us the classic Steffensen inequality, counting measure gives us Jensen-Steffensen's inequality (even with relaxed conditions, see [6]), and Lebesgue-Stieltjes measure gives us results for Steffensen's inequality from [9].

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