Journal of
Mathematical
Inequalities
Volume 10, Number 2 (2016), 471–489
doi:10.7153/jmi-10-37

SOME BENNETT–COPSON TYPE INEQUALITIES ON TIME SCALES

S. H. SAKER, R. R. MAHMOUD AND A. PETERSON

(*Communicated by J. Pečarić*)

Abstract. In this paper, we will prove some new dynamic inequalities with two different weighted functions on a time scale. As special cases, the inequalities contain some dynamic inequalities on time scales and also involve some discrete inequalities formulated by Copson, Leindler, Bennett, Chen and Yang. The results will be proved by using Hölder's inequality and Minkowski's inequality on time scales.

1. Introduction

In 1920 Hardy [12] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^p \leqslant \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p, \quad p>1, \tag{1.1}$$

where $a_n \geqslant 0$ for $n \geqslant 1$. This inequality was discovered in his attempt to give an elementary proof of Hilbert's inequality for double series that was known at that time. In 1925 Hardy [13] proved the continuous inequality

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(x)dx, \tag{1.2}$$

by employing the calculus of variations, where $f \geqslant 0$ is integrable on any finite interval $(0,x)$ and f^p is integrable and convergent over $(0,\infty)$, where $p>1$. The constant $(p/(p-1))^p$ in the two inequalities is the best possible constant. These two inequalities have been extensively studied in the literature; we refer the reader to the books [24, 17, 18, 21] and the papers [1, 9, 16, 23, 25, 37, 38] and the references cited therein.

In 1934 Hardy et al. [14, Theorem 337] showed that the reverse of the inequality (1.2) also holds when $0<p<1$. In particular, they proved that if $f(x) \geqslant 0$, $\int_0^{\infty} f^p(x)dx < \infty$, then

$$\int_0^{\infty}\left(\frac{1}{x}\int_x^{\infty} f(t)dt\right)^p dx > \left(\frac{p}{1-p}\right)^p\int_0^{\infty}f^p(x)dx, \quad 0<p<1, \tag{1.3}$$

Mathematics subject classification (2010): 26D15, 34A40, 34N05, 39A12.
Keywords and phrases: Hardy's type inequality, Copson's inequality, Leindler's inequality, Bennett's inequality, time scales.

unless $f \equiv 0$. Again the constant $(p/(1-p))^p$ is the best possible constant.

In 1928 Copson [10] proved some new types of discrete inequalities (see also [14, Theorem 344]). In particular one of his inequalities presented is given by

$$\sum_{n=1}^{N}\left(\sum_{k=n}^{N} w_k\right)^r \geqslant r^r \sum_{k=1}^{N}(kw_k)^r, \tag{1.4}$$

where $0<r<1$ and $\{w_n\}$ is a sequence with nonnegative terms.

In 1970 Leindler [19] generalized the Hardy inequality (1.1) in such a way that the sequence $\{n^{-p}\}$ is replaced by an arbitrary sequence λ_n and did not say any thing about the integral forms of his inequalites. In particular, Leindler proved that if $p>1$, λ_n, $a_n>0$, then

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=1}^{n} a_k\right)^p \leqslant p^p \sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=n}^{\infty}\lambda_k\right)^p a_n^p, \tag{1.5}$$

and

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=n}^{\infty} a_k\right)^p \leqslant p^p \sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=1}^{n}\lambda_k\right)^p a_n^p. \tag{1.6}$$

In 1976 Copson [11] showed that the continuous counterpart of the inequality (1.4) holds. In particular, he proved that if $0<k<1$ and $g(t)$ is a non-negative function, then

$$\int_0^b\left(\int_t^b g(s)ds\right)^k dt \geqslant k^k \int_0^b (tg(t))^k dt. \tag{1.7}$$

In 1987 Bennett [4] extended Copson's inequality (1.4). One of his results states that: If $0<r<s\leqslant 1$, and

$$\sum_{k=1}^{m} v_k\left(\sum_{n=1}^{k} u_n\right)^{\frac{r}{r-s}} \leqslant \left(\sum_{n=1}^{m} u_n\right)^{\frac{1-r}{s-r}}, \tag{1.8}$$

for $m=1,2,\dots,N$, then

$$\sum_{n=1}^{N} u_n\left(\sum_{k=n}^{N} v_k w_k\right)^r \geqslant r^r\left(\sum_{k=1}^{N} v_k w_k^{\frac{r}{s}}\right)^s, \tag{1.9}$$

where u_n, v_n, w_n are nonnegative sequences for $n \geqslant 1$.

In 1988 Bennett [5, Theorem 1'] made some simple modifications to (1.9) and proved a new inequality when $0<r<s\leqslant 1$. In particular it was proved that if

$$\sum_{k=m}^{N} v_k\left(\sum_{n=k}^{N} u_n\right)^{\frac{r}{r-s}} \leqslant K\left(\sum_{n=m}^{N} u_n\right)^{\frac{1-r}{s-r}}, \tag{1.10}$$

for $m = 1, 2, \ldots, N$, then

$$\sum_{n=1}^{N} u_n \left(\sum_{k=1}^{n} v_k w_k \right)^r \geqslant K^{r-s} r^r \left(\sum_{k=1}^{N} v_k w_k^{\frac{r}{s}} \right)^s. \tag{1.11}$$

In 1990 Leindler [20] showed that the analoques of (1.5) and (1.6) are also true for $0 < p \leqslant 1$. In particular, he proved that if λ_n, $a_n > 0$ and $0 < p \leqslant 1$, then

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n} a_k \right)^p \geqslant p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{k=n}^{\infty} \lambda_k \right)^p a_n^p, \tag{1.12}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=n}^{\infty} a_k \right)^p \geqslant p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \left(\sum_{k=1}^{n} \lambda_k \right)^p a_n^p. \tag{1.13}$$

For more details of the Leindler inequalities we refer the reader to the paper [22].

In 2012 Chen and Yang [8] extended the inequality (1.4) and proved that if $0 < p \leqslant 1$, $q > 0$, $1 < \lambda \leqslant 1 + q$ and $\sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda-1}{q}\right)-1} (a_n)^p < \infty$, then

$$\left(\sum_{n=1}^{\infty} n^{\frac{p}{q}(1-\lambda)-1} \left(\sum_{m=1}^{n} a_m \right)^p \right)^{\frac{1}{p}} > \frac{q}{\lambda - 1} \left(\sum_{n=1}^{\infty} n^{p\left(1-\frac{\lambda-1}{q}\right)-1} (a_n)^p \right)^{\frac{1}{p}}, \tag{1.14}$$

where the constant factor $q/(\lambda - 1)$ is the best possible constant.

In recent years the study of dynamic inequalities on time scales has received a lot of attention and becomes a major field in pure and applied mathematics. All of these disciplines are concerned with the properties of these inequalities of various types, for more details we refer to the book [2]. The general idea is to prove a result for an inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. This idea goes back to its founder Stefan Hilger [15] which started the study of dynamic equations on time scales. The study of dynamic inequalities on time scales helps avoid proving results twice – once for differential inequality and once again for difference inequality. For example, in 2005 Řehák [26] proved the time scale version of (1.2). In 2014 Saker [27] proved the time scale versions of the Leindler inequalities (1.5), (1.6), (1.12) and (1.13) by employing a new technique that uses the Hölder inequality and a chain rule on time scales. For more details we refer the reader to some recent results [28, 29, 30, 31, 32, 33, 34, 36] of Hardy's type inequalities on time scales.

The question now arises: Is it possible to prove some new dynamic inequalities on time scales which as special cases when $\mathbb{T} = \mathbb{N}$ contain the inequalities (1.9), (1.11) and (1.14)? Our aim in this paper is to give an affirmative answer to this question.

This paper is organized as follows. In Section 2, we present some preliminaries about the theory of time scales and prove the basic lemmas that will be needed in the proofs of the main results. In Section 3, we shall state and prove the main results of this paper. In particular, in Theorems 3.1 and 3.2, we will prove the time scale versions of (1.9) and (1.11) and as a special case of Theorem 3.2 we will obtain the time scale version of (1.14). When $\mathbb{T}=\mathbb{N}$ we will obtain the inequalities (1.4), (1.12) and (1.13) that have been proved by Copson and Leindler as special cases. Also, when $\mathbb{T}=\mathbb{R}$, we will obtain the inequality (1.7) that has been proved by Copson. The main results will be proved by employing Hölder's inequality, Minkowski's inequality and an appropriate chain rule on time scales.

2. Preliminaries and basic lemmas

In the next sequel, we present some preliminaries on time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that is inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by: $\sigma(t) := \inf\{s\in\mathbb{T} : s>t\}$, $\rho(t) := \sup\{s\in\mathbb{T} : s<t\}$ respectively, where $\sup\emptyset = \inf\mathbb{T}$. A point $t\in\mathbb{T}$, is said to be left–dense if $\rho(t)=t$ and $t>\inf\mathbb{T}$, is right–dense if $\sigma(t)=t$, is left–scattered if $\rho(t)<t$ and right–scattered if $\sigma(t)>t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. Without loss of generality, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b]\cap\mathbb{T}$.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [6], [7] which summarize and organize much of the time scale calculus. In this paper, we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t)=g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t)-G(a)$. It can be shown (see [6]) that if $g\in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0\in\mathbb{T}$, and satisfies $G^{\Delta}(t)=g(t)$, $t\in\mathbb{T}$. An infinite integral is defined as $\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty}\int_a^b f(t)\Delta t$. The following simple consequence of Keller's chain rule [6, Theorem 1.90] which is needed in the proof of the main results is given by

$$(x^{\gamma}(t))^{\Delta} = \gamma\int_0^1 [hx^{\sigma}+(1-h)x]^{\gamma-1}\,dh x^{\Delta}(t), \tag{2.1}$$

and the integration by parts formula on time scales is also given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.2}$$

The Hölder's inequality, see [3, Theorem 6.2], on time scales is given by

$$\int_a^b |f(t)g(t)|\Delta t \leqslant \left[\int_a^b |f(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b |g(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{2.3}$$

where $a,\ b\in\mathbb{T},\ f,\ g\in \mathrm{C}_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$ and $\frac{1}{\gamma}+\frac{1}{\nu}=1$. This inequality is reversed if $0<\gamma<1$ and $\int_a^b |g(t)|^{\nu}\Delta t>0$.

Now we present some basic lemmas that will be used in the proofs of our main results. These lemmas are adapted from [36].

LEMMA 2.1. *Let $\mathbb{T}$ be a time scale with $a,x\in\mathbb{T}$ and $f\in \mathrm{C}_{rd}([a,\sigma(x)]_{\mathbb{T}},\mathbb{R})$. If $\alpha\geqslant 1$, then*

$$\left(\int_a^{\sigma(x)} f(t)\Delta t\right)^{\alpha}\leqslant \alpha\int_a^{\sigma(x)} f(t)\left(\int_a^{\sigma(t)} f(s)\Delta s\right)^{\alpha-1}\Delta t. \tag{2.4}$$

The inequality reverses direction if $0<\alpha<1$.

LEMMA 2.2. *Let $\mathbb{T}$ be a time scale with $x,b\in\mathbb{T}$ and $f\in \mathrm{C}_{rd}([x,b]_{\mathbb{T}},\mathbb{R})$. If $\alpha\geqslant 1$, then*

$$\left(\int_x^{b} f(t)\Delta t\right)^{\alpha}\leqslant \alpha\int_x^{b} f(t)\left(\int_t^{b} f(s)\Delta s\right)^{\alpha-1}\Delta t. \tag{2.5}$$

The inequality reverses direction if $0<\alpha\leqslant 1$.

LEMMA 2.3. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$ and $f,g\in \mathrm{C}_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$. If $m\geqslant 1$, then*

$$\left(\int_a^{b} f(t)\left(\int_a^{\sigma(t)} g(s)\Delta s\right)^{m}\Delta t\right)^{\frac{1}{m}}\leqslant \int_a^{b} g(s)\left(\int_s^{b} f(t)\Delta t\right)^{m}\Delta s. \tag{2.6}$$

In the following, we will state and prove some new lemmas which will be used in the proofs and they are also important results in their own right.

LEMMA 2.4. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$ and $f,\ g,\ h\in \mathrm{C}_{rd}([a,b]_{\mathbb{T}},\mathbb{R}^{+})$. If*

$$F(t):=\int_a^{t} f(x)\Delta x\leqslant \int_a^{t} g(x)\Delta x:=G(t), \tag{2.7}$$

then

$$\int_a^{b} f(t)H(t)\Delta t\leqslant \int_a^{b} g(t)H(t)\Delta t, \tag{2.8}$$

where $H(t):=\int_t^b h(x)\Delta x$.

Proof. Integrating by parts the left-hand side of (2.8), we have

$$\int_a^{b} f(t)H(t)\Delta t=F(t)H(t)|_a^b+\int_a^{b} F^{\sigma}(t)h(t)\Delta t.$$

By noting that $F(a)=0=H(b)$, and making use of $F^{\sigma}\leqslant G^{\sigma}$, we get that

$$\int_a^{b} f(t)H(t)\Delta t\leqslant \int_a^{b} G^{\sigma}(t)h(t)\Delta t. \tag{2.9}$$

Integrating by parts the right-hand side of the inequality (2.9), we have that

$$\int_a^b G^\sigma(t)h(t)\Delta t \leqslant G(t)H(t)|_a^b - \int_a^b g(t)H(t)\Delta t.$$

Since $G(a) = 0 = H(b)$, we have from the last inequality that

$$\int_a^b G^\sigma(t)h(t)\Delta t \leqslant \int_a^b g(t)H(t)\Delta t. \tag{2.10}$$

Combining (2.9) and (2.10), we get

$$\int_a^b f(t)H(t)\Delta t \leqslant \int_a^b g(t)H(t)\Delta t,$$

which is the desired inequality (2.8). The proof is complete. □

The following lemma is the dual of Lemma 2.4.

LEMMA 2.5. *Let* $\mathbb{T}$ *be a time scale with* $a,\ b \in \mathbb{T}$ *and* $f,\ g,\ h \in \mathrm{C}_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$. *If* k *is a positive constant such that*

$$F(t) := \int_t^b f(x)\Delta x \leqslant k\int_t^b g(x)\Delta x := kG(t), \tag{2.11}$$

then

$$\int_t^b f(t)H^\sigma(t)\Delta t \leqslant k\int_t^b g(t)H^\sigma(t)\Delta t, \tag{2.12}$$

where $H^\sigma(t) := \int_a^{\sigma(t)} h(x)\Delta x$.

Proof. Integrating by parts the left-hand side of (2.12), we have

$$\int_a^b f(t)H^\sigma(t)\Delta t = F(t)H(t)|_a^b - \int_a^b F(t)h(t)\Delta t.$$

By noting that $F(b) = 0 = H(a)$, and making use of (2.11), we obtain

$$\int_a^b f(t)H^\sigma(t)\Delta t \leqslant k\int_a^b G(t)h(t)\Delta t. \tag{2.13}$$

Again integrating by parts the right-hand side of the inequality (2.13), we have that

$$\int_a^b G(t)h(t)\Delta t = G(t)H(t)|_a^b - \int_a^b g(t)H^\sigma(t)\Delta t.$$

Since $G(b) = 0 = H(a)$, we see that

$$k\int_a^b G(t)h(t)\Delta t \leqslant k\int_a^b g(t)H^\sigma(t)\Delta t. \tag{2.14}$$

Combining (2.13) and (2.14), we get

$$\int_a^b f(t)H^\sigma(t)\Delta t \leqslant k\int_a^b g(t)H^\sigma(t)\Delta t,$$

which is the required inequality (2.12). The proof is complete. □

3. Main results

Throughout the paper, we will assume that the functions (without mentioning) are nonnegative rd-continuous functions, Δ-differentiable, locally delta integrable and the left hand side of the inequalities exits if the right hand side exists. We also assume that all the constants (without mentioning) are positive real numbers. For simplification, we define

$$A := \int_a^b u(x)\left(\int_x^b v(t)w(t)\Delta t\right)^r \Delta x, \tag{3.1}$$

$$B := \int_a^b v(t)w^r(t)\Delta t, \tag{3.2}$$

and

$$C := \int_a^b u(x)\left(\int_a^{\sigma(x)} v(t)w(t)\Delta t\right)^r \Delta x. \tag{3.3}$$

Now, we are ready to state and prove our first main result. We begin with the time scale version of Bennett's inequality (1.9).

THEOREM 3.1. *Let $\mathbb{T}$ be a time scale with $a,\ b \in \mathbb{T}$, $u,\ v,\ w$ are positive rd-continuous functions defined on $[a,b]_{\mathbb{T}}$ and $0 < r < s \leqslant 1$. If*

$$\int_a^t v(x)\left(\int_a^{\sigma(x)} u(t)\Delta t\right)^{\frac{r}{r-s}} \Delta x \leqslant \left(\int_a^t u(x)\Delta x\right)^{\frac{1-r}{s-r}}, \quad \textit{for } t \in [a,b]_{\mathbb{T}}, \tag{3.4}$$

then

$$\int_a^b u(x)\left(\int_x^b v(t)w(t)\Delta t\right)^r \Delta x \geqslant r^r\left(\int_a^b v(x)w^{\frac{r}{s}}(x)\Delta x\right)^s. \tag{3.5}$$

Proof. Actually, we will split our proof into two parts. In the first part we consider the case when $s = 1$. In this case it is enough to prove that

$$A \geqslant r^r B, \tag{3.6}$$

where A and B are defined as in (3.1) and (3.2) respectively. Now, we define

$$f = v(x)\left(\int_a^{\sigma(x)} u(t)\Delta t\right)^{\frac{r}{r-1}}, \quad g = u(x) \text{ and } H = \left(\int_x^b v(s)w(s)\Delta s\right)^r.$$

Using the functions f, g and H defined above and the condition (3.4) with $s = 1$, we see that the condition (2.7) holds. Now, by applying Lemma 2.4, we get that

$$\begin{aligned} &\int_a^b v(x)\left(\int_a^{\sigma(x)} u(t)\Delta t\right)^{r^*}\left(\int_x^b v(s)w(s)\Delta s\right)^r \Delta x \\ &\leqslant \int_a^b u(x)\left(\int_x^b v(s)w(s)\Delta s\right)^r \Delta x = A, \end{aligned} \tag{3.7}$$

where $r^{*} = r/(r-1)$. Applying the reverse of (2.5) in Lemma 2.2 on the integral $\left(\int_x^b v(t)w(t)\Delta t\right)^r$ with $f(t) = v(t)w(t)$ and $0 < \alpha = r < 1$, we get that

$$r\int_a^b u(t)\left(\int_t^b v(x)w(x)\left(\int_x^b v(s)w(s)\Delta s\right)^{r-1}\Delta x\right)\Delta t \leqslant A. \tag{3.8}$$

Applying the Minkowski inequality (2.6) on the left-hand side of (3.8) with $m=1$, we have that

$$r\int_a^b v(x)w(x)\left(\int_a^{\sigma(x)} u(t)\left(\int_t^b v(s)w(s)\Delta s\right)^{r-1}\Delta t\right)\Delta x \leqslant A. \tag{3.9}$$

From this, since $1/r + 1/r^{*} = 1$, we have that

$$r\int_a^b v^{\frac{1}{r}}(x)v^{\frac{1}{r^{*}}}(x)w(x)\left(\int_a^{\sigma(x)} u(t)\left(\int_t^b v(s)w(s)\Delta s\right)^{r-1}\Delta t\right)\Delta x \leqslant A. \tag{3.10}$$

Applying the reverse of Hölder's inequality (2.3) with $0 < \gamma = r < 1$, on the left-hand side of (3.10), we obtain that

$$rB^{\frac{1}{r}}\left(\int_a^b v(x)\left(\int_a^{\sigma(x)} u(t)\Delta t\right)^{r^{*}}\left(\int_x^b v(s)w(s)\Delta s\right)^{r}\Delta x\right)^{\frac{1}{r^{*}}} \leqslant A. \tag{3.11}$$

Substituting (3.7) into (3.11), we get that

$$A \geqslant rB^{\frac{1}{r}}A^{\frac{1}{r^{*}}},$$

which leads us directly to (3.6). The proof of the first part in the case $s=1$ is complete. Next, we prove the second part of the proof by reducing the general case $0 < r < s < 1$ to the case when $s=1$ that has been proved in the first part. Applying Hölder's inequality (2.3) on the integral

$$\int_a^t \left(v(x)w^{\frac{r}{s}}(x)\right)^{\frac{1}{\lambda}} v^{\frac{1}{\lambda^{*}}}(x)\left(\int_a^{\sigma(x)} u(t)\Delta t\right)^{r^{*}}\Delta x,$$

with exponents

$$\lambda = \frac{1-r}{1-s}, \quad \text{and } \lambda^{*} = \frac{1-r}{s-r}, \tag{3.12}$$

we get that

$$\int_a^t \left(v(x)w^{\frac{r}{s}}(x)\right)^{\frac{1}{\lambda}} v^{\frac{1}{\lambda^{*}}}(x)\left(\int_a^{\sigma(x)} u(t)\Delta t\right)^{r^{*}}\Delta x$$
$$\leqslant \beta\left(\int_a^t v(x)\left(\int_a^{\sigma(x)} u(t)\Delta t\right)^{\frac{r}{r-s}}\Delta x\right)^{\frac{1}{\lambda^{*}}},$$

where $\beta = \left(\int_a^t v(x)w^{\frac{r}{s}}(x)\Delta x\right)^{\frac{1}{\lambda}}$. Using the assumption (3.4) in the last inequality, we have that

$$\int_a^t \left(v(x)w^{\frac{r}{s}}(x)\right)^{\frac{1}{\lambda}} v^{\frac{1}{\lambda^*}}(x)\left(\int_a^{\sigma(x)} u(t)\Delta t\right)^{r^*}\Delta x \leqslant \beta\int_a^t u(x)\Delta x. \tag{3.13}$$

Now, we observe that the inequality (3.13) is just a restatement of assumption (3.4) with $s=1$ and $v(x)$ is replaced by $\beta^{-1}\left(v(x)w^{\frac{r}{s}}(x)\right)^{\frac{1}{\lambda}} v^{\frac{1}{\lambda^*}}(x)$. Applying the inequality (3.6) of the first part for the new function $\beta^{-1}\left(v(x)w^{\frac{r}{s}}(x)\right)^{\frac{1}{\lambda}} v^{\frac{1}{\lambda^*}}(x)$, we obtain that

$$\begin{aligned}&\int_a^b u(x)\left(\int_x^b \left(v(t)w^{\frac{r}{s}}(t)\right)^{\frac{1}{\lambda}} v^{\frac{1}{\lambda^*}}(t)w^{\frac{1}{s\lambda^*}}(t)\Delta t\right)^r\Delta x\\ &\geqslant \beta^{r-1}r^r\int_a^b \left(v(x)w^{\frac{r}{s}}(x)\right)^{\frac{1}{\lambda}} v^{\frac{1}{\lambda^*}}(x)w^{\frac{r}{s\lambda^*}}(x)\Delta x.\end{aligned} \tag{3.14}$$

Using (3.12) and simplifying the exponents on the inequality (3.14), we get (note that $0<r<s<1$) that

$$\int_a^b u(x)\left(\int_x^b v(t)w(t)\Delta t\right)^r\Delta x \geqslant r^r\left(\int_a^b v(x)w^{\frac{r}{s}}(x)\Delta x\right)^s, \tag{3.15}$$

which is the desired inequality (3.5). The proof is complete. □

As special cases of Theorem 3.1, we have the following results.

REMARK 3.1. If $\mathbb{T}=\mathbb{N}$, then inequality (3.5) in Theorem 3.1 reduces to Bennett's inequality (1.9).

REMARK 3.2. If $\mathbb{T}=\mathbb{R}$, then inequality (3.5) in Theorem 3.1 reduces to the following continuous inequality of Bennett-Copson type

$$\int_a^b u(x)\left(\int_x^b v(t)w(t)dt\right)^r dx \geqslant r^r\left(\int_a^b v(t)w^{\frac{r}{s}}(t)dt\right)^s, \quad 0<r<s\leqslant 1. \tag{3.16}$$

Now, we will use some different forms of the weighted functions to obtain new inequalities as special cases of Theorem 3.1. First, we assume that $c<0$, setting $0<r=q,\ s=q/p$ where $0<p<1$, and define

$$\begin{aligned}u(t) &:= \frac{\lambda(t)}{(\Lambda^\sigma(t))^{1-\frac{q}{p}(1-c)}},\ v(t) := \lambda(t)\left[(\Lambda^\sigma(t))^{p-c}\left(\frac{q}{p}(1-c)\right)^{-p}\right]^{\frac{1}{1-p}},\\ w(t) &:= (\Lambda^\sigma(t))^{\frac{c-p}{1-p}}g(t),\ \Lambda(t) = \int_a^t \lambda(s)\Delta s,\end{aligned}$$

where $\lambda(t)$ is a non-negative rd-continuous function defined on $\mathbb{T}$. So, to apply Theorem 3.1, we need to prove that the hypothesis (3.4) is satisfied for the new functions u

and v. Using the new definitions of these functions and substituting into (3.4), we get that

$$\int_a^x \lambda(t)\left[(\Lambda^\sigma(t))^{p-c}\left(\frac{q}{p}(1-c)\right)^{-p}\right]^{\frac{1}{1-p}} \tag{3.17}$$

$$\times\left(\int_a^{\sigma(t)} \frac{\lambda(s)}{(\Lambda^\sigma(s))^{1-\frac{q}{p}(1-c)}}\Delta s\right)^{\frac{q}{q-\frac{q}{p}}}\Delta t$$

$$= \int_a^x \lambda(t)(\Lambda^\sigma(t))^{\frac{p-c}{1-p}}\left(\left(\frac{q}{p}(1-c)\right)\right. \tag{3.18}$$

$$\int_a^{\sigma(t)}\lambda(s)\left(\int_a^{\sigma(s)}\lambda(\theta)\Delta\theta\right)^{\frac{q}{p}(1-c)-1}\Delta s\Big)^{\frac{p}{p-1}}\Delta t.$$

Applying Lemma 2.1 on the inner integral with $\alpha = (q/p)(1-c)$, we get that

$$\alpha\int_a^{\sigma(t)}\lambda(s)\left(\int_a^{\sigma(s)}\lambda(\theta)\Delta\theta\right)^{\alpha-1}\Delta s \geqslant \left(\int_a^{\sigma(t)}\lambda(t)\Delta t\right)^{\alpha}.$$

But since $0<p<1$, we obtain that

$$\left(\alpha\int_a^{\sigma(t)}\lambda(s)\left(\int_a^{\sigma(s)}\lambda(\theta)\Delta\theta\right)^{\alpha-1}\Delta s\right)^{\frac{p}{p-1}} \leqslant \left(\int_a^{\sigma(t)}\lambda(s)\Delta s\right)^{\alpha\frac{p}{p-1}}. \tag{3.19}$$

Substituting (3.19) into (3.17), we have that

$$\int_a^x \lambda(t)\left[(\Lambda^\sigma(t))^{p-c}\left(\frac{q}{p}(1-c)\right)^{-p}\right]^{\frac{1}{1-p}}\left(\int_a^{\sigma(t)}\frac{\lambda(s)}{(\Lambda^\sigma(s))^{1-\frac{q}{p}(1-c)}}\Delta s\right)^{\frac{q}{q-\frac{q}{p}}}\Delta t \tag{3.20}$$

$$\leqslant \int_a^x \lambda(t)(\Lambda^\sigma(t))^{\frac{p-c}{1-p}}\left((\Lambda^\sigma(t))^{\frac{q}{p}(1-c)}\right)^{\frac{p}{p-1}}\Delta t$$

$$= \int_a^x \lambda(t)(\Lambda^\sigma(t))^{\frac{p-c}{1-p}}(\Lambda^\sigma(t))^{\frac{q(1-c)}{p-1}}\Delta t$$

$$\leqslant \int_a^x \lambda(t)(\Lambda^\sigma(t))^{\frac{q}{p}(1-c)}\Delta t,$$

which proves the assumption (3.4). This and Theorem 3.1 give us the following result.

COROLLARY 3.1. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$ and $0<q\leqslant p<1$. If $c<0$, $\Lambda(t)=\int_a^t\lambda(s)\Delta s$, and $\Phi(t)=\int_t^b\lambda(s)g(s)\Delta s$ for $t\in[a,b]_{\mathbb{T}}$, then*

$$\int_a^b \lambda(t)(\Lambda^\sigma(t))^{\frac{q}{p}(1-c)-1}\Phi^q(t)\Delta t \geqslant \frac{pq^{q-1}}{(1-c)^{\frac{p(1-q)}{1-p}}}\left(\int_a^b\lambda(t)(\Lambda^\sigma(t))^{p-c}g^p(t)\Delta t\right)^{\frac{q}{p}}. \tag{3.21}$$

REMARK 3.3. If $\mathbb{T} = \mathbb{R}$, then $\Lambda^{\sigma}(t) = \Lambda(t)$ and Corollary 3.1 reduces to the following continuous inequality of Bennett-Copson type

$$\int_a^b \lambda(t)(\Lambda(t))^{\frac{q}{p}(1-c)-1}\Phi^q(t)dt \geqslant \frac{pq^{q-1}}{(1-c)^{\frac{p(1-q)}{1-p}}}\left(\int_a^b \lambda(t)(\Lambda(t))^{p-c}g^p(t)dt\right)^{\frac{q}{p}}, \tag{3.22}$$

which is essentially new and is not stated in any of Bennett's work.

REMARK 3.4. If $\mathbb{T} = \mathbb{N}$, then Corollary 3.1 reduces to the following discrete result

$$\sum_{n=1}^{N} \lambda_n \Lambda_n^{\frac{q}{p}(1-c)-1}\left(\sum_{k=n}^{N} \lambda_k x_k\right)^q \geqslant \frac{pq^{q-1}}{(1-c)^{\frac{p(1-q)}{1-p}}}\left(\sum_{n=1}^{N} \lambda_n \Lambda_n^{p-c} x_n^p\right)^{\frac{q}{p}}, \tag{3.23}$$

due to Bennett [5, Corollary 4].

If $p = q = k$, then the inequality (3.21) reduces to the following dynamic Copson-type inequality on time scales due to Saker et al. [35, Theorem 2.3].

COROLLARY 3.2. *Let $\mathbb{T}$ be a time scale with $a,\ b \in \mathbb{T}$ and $c < 0 < k < 1$. Let $\Lambda(t) = \int_a^t \lambda(s)\Delta s$, and*

$$\Phi(t) = \int_t^b \lambda(s)g(s)\Delta s, \quad t \in [a,b]_{\mathbb{T}}, \tag{3.24}$$

then

$$\int_a^b \frac{\lambda(t)}{(\Lambda^{\sigma}(t))^c}\Phi^k(t)\Delta t \geqslant \left(\frac{k}{1-c}\right)^k \int_a^b \lambda(t)(\Lambda^{\sigma}(t))^{k-c} g^k(t)\Delta t. \tag{3.25}$$

We can also obtain the Leindler type inequalities on time scales proved in [27, Theorem 2.4]. In fact by choosing

$$u(t) = \lambda(t), \quad v(t) = \lambda(t)\left(\int_a^t \lambda(s)\Delta s\right)^{\frac{p}{1-p}}, \quad w(t) = \frac{g(t)}{\lambda(t)}\left(\int_a^t \lambda(s)\Delta s\right)^{\frac{p}{p-1}}, \tag{3.26}$$

and $s = 1$, $r = p$ and substituting in (3.4), we get (since $0 < p < 1$) that

$$\begin{aligned}
&\int_a^x \lambda(t)\left(\int_a^t \lambda(s)\Delta s\right)^{\frac{p}{1-p}}\left(\int_a^{\sigma(t)} \lambda(s)\Delta s\right)^{\frac{p}{p-1}}\Delta t \\
&\leqslant \int_a^x \lambda(t)\left(\int_a^t \lambda(s)\Delta s\right)^{\frac{p}{1-p}}\left(\int_a^t \lambda(s)\Delta s\right)^{\frac{p}{p-1}}\Delta t = \int_a^x \lambda(t)\Delta t,
\end{aligned}$$

which proves that the condition (3.4) is satisfied for the new functions u and v defined in (3.26). This and Theorem 3.1 give us the following Leindler type inequality on time scales.

COROLLARY 3.3. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$ and $0 < p < 1$. If $\lambda(t) > 0$ and $g(t) \geqslant 0$, then*

$$\int_a^\infty \lambda(t) \left(\int_t^\infty g(s)\Delta s \right)^p \Delta t \geqslant p^p \int_a^\infty \lambda^{1-p}(t) \left(\int_a^{\sigma(t)} \lambda(s)\Delta s \right)^p g^p(t)\Delta t. \tag{3.27}$$

REMARK 3.5. If $\mathbb{T} = \mathbb{N}$, then Corollary 3.3 reduces to the discrete result (1.13) due to Leindler [20] and Mohapatra and Salzman [22].

In the following, we will use some simple different forms of the weighted functions to obtain new inequalities. For example, if we set $a = 0$, $s = 1$, $u = x^{-r}$, $v = (1-r)^{-r}$ and $w = f$, then inequality (3.5) in Theorem 3.1 reduces to the following inequality

$$\int_0^b \left(\frac{1}{x} \int_x^b f(t)\Delta t \right)^r \Delta x > \left(\frac{r}{1-r} \right)^r \int_0^b f^r(t)\Delta t, \tag{3.28}$$

which can be considered as the time scale version of (1.3) due to Hardy. If we set $u = t^{-r}$, $v = 1$ and $s = 1$, then the inequality (3.5) reduces to the following inequality

$$\int_a^b \left(\frac{1}{t} \int_x^b w(t)\Delta t \right)^r \Delta x \geqslant r^r \int_a^b w^r(t)\Delta t, \tag{3.29}$$

which can be considered as the time scales version of the following Copson-type inequality (see [14, Theorem 345])

$$\sum_{n=1}^N \left(\frac{1}{k} \sum_{k=n}^N w_k \right)^r \geqslant r^r \sum_{k=1}^N w_k^r.$$

If we set $a = 0$, $0 < r < 1$, $s = 1$, $u = 1$, $v = x^{\frac{-r}{r-1}}$ and $w = x^{\frac{r}{r-1}} f$, we see that the hypothesis (3.4) reads

$$\int_0^t x^{\frac{-r}{r-1}} \left(\int_0^{\sigma(x)} 1\Delta t \right)^{\frac{r}{r-1}} \Delta x = \int_0^t x^{\frac{-r}{r-1}} (\sigma(x))^{\frac{r}{r-1}} \Delta x.$$

Now, since $r/(r-1) < 0$, it follows that

$$\int_0^t x^{\frac{-r}{r-1}} \left(\int_0^{\sigma(x)} 1\Delta t \right)^{\frac{r}{r-1}} \Delta x \leqslant \int_0^t x^{\frac{-r}{r-1}} (x)^{\frac{r}{r-1}} \Delta x = \int_0^t 1\Delta x = \int_0^t u(x)\Delta x,$$

which gives the validity of hypothesis (3.4). Then the inequality (3.5) in Theorem 3.1 reduces to the following inequality

$$\int_0^b \left(\int_x^b f(t)\Delta t \right)^r \Delta x > r^r \int_a^b (tf(t))^r \Delta t, \tag{3.30}$$

which unifies the discrete inequality (1.4) and the continuous inequality (1.7) to an arbitrary time scale $\mathbb{T}$.

Next, we prove the time scale version of Bennett's inequality (1.11).

THEOREM 3.2. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$, $u,\ v,\ w$ are positive rd-continuous functions defined on $[a,b]_{\mathbb{T}}$ and $0<r<s\leqslant 1$. If*

$$\int_t^b v(x)\left(\int_x^b u(t)\Delta t\right)^{\frac{r}{r-s}}\Delta x\leqslant K\left(\int_t^b u(x)\Delta x\right)^{\frac{1-r}{s-r}},\quad \text{for } t\in[a,b]_{\mathbb{T}}, \tag{3.31}$$

then

$$\int_a^b u(x)\left(\int_a^{\sigma(x)} v(t)w(t)\Delta t\right)^r\Delta x\geqslant K^{r-s}r^r\left(\int_a^b v(x)w^{\frac{r}{s}}(x)\Delta x\right)^s, \tag{3.32}$$

where K is a positive constant depends on r and s.

Proof. Actually, we will split our proof into two parts. In the first part, we consider the case $s=1$. In this case it is enough to prove that

$$C\geqslant K^{r-1}r^rB, \tag{3.33}$$

where B and C are defined as in (3.2) and (3.3). Now, we define

$$f(x)=v(x)\left(\int_x^b u(t)\Delta t\right)^{\frac{r}{r-1}},\quad g=u(x) \text{ and } H^{\sigma}(x)=\left(\int_a^{\sigma(x)} v(t)w(t)\Delta t\right)^r.$$

Using these functions and the condition (3.31), we see that the condition (2.11) in Lemma 2.5 holds. Now applying Lemma 2.5, we get that

$$\begin{aligned}&\int_a^b v(x)\left(\int_x^b u(t)\Delta t\right)^{r^*}\left(\int_a^{\sigma(x)} v(t)w(t)\Delta t\right)^r\Delta x\\ &\leqslant K\int_a^b u(x)\left(\int_a^{\sigma(x)} v(t)w(t)\Delta t\right)^r\Delta x=KC.\end{aligned} \tag{3.34}$$

Applying Lemma 2.1 on the term C defined in (3.3), we get that

$$r\int_a^b u(t)\left(\int_a^{\sigma(t)} v(x)w(x)\left(\int_a^{\sigma(x)} v(s)w(s)\Delta s\right)^{r-1}\Delta x\right)\Delta t\leqslant C. \tag{3.35}$$

Applying the Minkowski inequality (2.6) on the left-hand side of (3.35) with $m=1$, we have that

$$r\int_a^b v(x)w(x)\left(\int_x^b u(t)\left(\int_a^{\sigma(t)} v(s)w(s)\Delta s\right)^{r-1}\Delta t\right)\Delta x\leqslant C. \tag{3.36}$$

From this we see that

$$r\int_a^b v^{\frac{1}{r}}(x)w(x)v^{\frac{1}{r^*}}(x)\left(\int_x^b u(t)\left(\int_a^{\sigma(t)} v(s)w(s)\Delta s\right)^{r-1}\Delta t\right)\Delta x\leqslant C. \tag{3.37}$$

Applying the reverse of Hölder's inequality (2.3) with $0<r<1$, on the left-hand side of (3.37), we obtain that

$$rB^{\frac{1}{r}}\left(\int_a^b v(x)\left(\int_x^b u(t)\Delta t\right)^{r^*}\left(\int_a^{\sigma(x)} v(s)w(s)\Delta s\right)^r \Delta x\right)^{\frac{1}{r^*}} \leqslant C. \tag{3.38}$$

Substituting (3.34) into (3.38), we get that

$$C \geqslant rB^{\frac{1}{r}}K^{\frac{1}{r^*}}C^{\frac{1}{r^*}},$$

which leads us directly to (3.33). The proof of the first part in the case $s=1$ is complete. Next, we prove the second part of the proof by reducing the general case $0<r<s<1$ to the proof of the first part when $s=1$. Applying Hölder's inequality (2.3) on the integral

$$\int_t^b \left(vw^{\frac{r}{s}}\right)^{\frac{1}{\lambda}}(x)v^{\frac{1}{\lambda^*}}(x)\left(\int_x^b u(t)\Delta t\right)^{r^*}\Delta x,$$

with exponents

$$\lambda = \frac{1-r}{1-s} \text{ and } \lambda^* = \frac{1-r}{s-r}, \tag{3.39}$$

we have that

$$\int_t^b \left(vw^{\frac{r}{s}}\right)^{\frac{1}{\lambda}}(x)v^{\frac{1}{\lambda^*}}(x)\left(\int_x^b u(t)\Delta t\right)^{r^*}\Delta x \leqslant h\left(\int_t^b v(x)\left(\int_x^b u(t)\Delta t\right)^{\frac{r}{r-s}}\Delta x\right)^{\frac{1}{\lambda^*}},$$

where $h=\left(\int_t^b v(x)w^{\frac{r}{s}}(x)\Delta x\right)^{\frac{1}{\lambda}}$. Using the assumption (3.31) in the last inequality, we have that

$$\int_t^b \left(vw^{\frac{r}{s}}\right)^{\frac{1}{\lambda}}(x)v^{\frac{1}{\lambda^*}}(x)\left(\int_x^b u(t)\Delta t\right)^{r^*}\Delta x \leqslant hK^{\frac{1}{\lambda^*}}\int_t^b u(x)\Delta x. \tag{3.40}$$

Now, we observe that the inequality (3.40) is just a restatement of the assumption (3.31) with $s=1$ and $v(t)$ replaced by $h^{-1}K^{\frac{-1}{\lambda^*}}\left(v(x)w^{\frac{r}{s}}(x)\right)^{\frac{1}{\lambda}}v^{\frac{1}{\lambda^*}}(x)$. So we can apply the special inequality (3.33) of the theorem for the new function

$$h^{-1}K^{\frac{-1}{\lambda^*}}\left(v(x)w^{\frac{r}{s}}(x)\right)^{\frac{1}{\lambda}}v^{\frac{1}{\lambda^*}}(x),$$

to obtain

$$\begin{aligned}&\int_a^b u(x)\left(\int_a^{\sigma(x)}\left(vw^{\frac{r}{s}}\right)^{\frac{1}{\lambda}}(t)v^{\frac{1}{\lambda^*}}(t)w^{\frac{1}{s\lambda^*}}(t)\Delta t\right)^r\Delta x \\ &\geqslant h^{r-1}r^rK^{\frac{r-1}{\lambda^*}}\int_a^b\left(vw^{\frac{r}{s}}\right)^{\frac{1}{\lambda}}(x)v^{\frac{1}{\lambda^*}}(x)w^{\frac{r}{s\lambda^*}}(x)\Delta x.\end{aligned} \tag{3.41}$$

Using (3.39) and simplifying the exponents on the above inequality, we obtain (note that $0<r<s<1$)

$$\int_a^b u(x)\left(\int_a^{\sigma(x)} v(t)w(t)\Delta t\right)^r \Delta x \geqslant K^{r-s}r^r\left(\int_a^b v(x)w^{\frac{r}{s}}(x)\Delta x\right)^s, \tag{3.42}$$

which is the desired inequality (3.32). □

As a special case of Theorem 3.2, we get the following result.

REMARK 3.6. If $\mathbb{T}=\mathbb{N}$, then the inequality (3.32) in Theorem 3.2 reduces to Bennett's inequality (1.11).

REMARK 3.7. If $\mathbb{T}=\mathbb{R}$, then inequality (3.32) in Theorem 3.2, reduces to the following continuous inequality of Bennett-Copson type

$$\int_a^b u(x)\left(\int_a^{\sigma(x)} v(t)w(t)dt\right)^r dx \geqslant K^{r-s}r^r\left(\int_a^b v(t)w^{\frac{r}{s}}(t)dt\right)^s. \tag{3.43}$$

Now, we will use some different forms of the weighted functions to obtain new inequalities as special cases from Theorem 3.2. We let

$$u(t)=\frac{\lambda(t)}{(\Omega(t))^{\frac{q}{p}(c-1)-1}},\ v(t)=\lambda(t)\left[(\Omega(t))^{p-c}\left(\frac{q}{p}(c-1)\right)^{-p}\right]^{\frac{1}{1-p}},$$
$$w(t)=(\Omega(t))^{\frac{c-p}{1-p}}g(t),\ \Omega(t)=\int_t^b\lambda(s)\Delta s,$$

$c>1$, $s=q/p$ and $r=q$. Now by following the proof of the hypothesis (3.4) in Corollary 3.1 and applying Lemma 2.2 instead of Lemma 2.1 we can prove that the condition (3.31) holds. This and Theorem 3.2 give us the following result.

COROLLARY 3.4. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$ and $0<p\leqslant q<1$. If $c>1$, $\Omega(t)=\int_t^b\lambda(s)\Delta s$, and*

$$\Phi(t)=\int_a^{\sigma(t)}\lambda(s)g(s)\Delta s,\ \textit{for } t\in[a,b]_{\mathbb{T}},$$

then

$$\int_a^b \lambda(t)\Omega^{1-\frac{q}{p}(c-1)}(t)\left(\Phi^\sigma(t)\right)^q\Delta t \geqslant K_2\left(\int_a^b\lambda(t)\Omega^{p-c}(t)g^p(t)\Delta t\right)^{\frac{q}{p}}, \tag{3.44}$$

where

$$K_2=\frac{pq^{q-1}}{(c-1)^{\frac{p(1-q)}{1-p}}}.$$

If $p=q=k$, then the reduced version of the inequality (3.44) gives the Copson-type inequality proved in [35, Theorem 2.4].

COROLLARY 3.5. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$, $c>1$ and $0<k<1$. Let $\Omega(t)=\int_t^b\lambda(s)\Delta s$, and*

$$\Phi(t)=\int_a^{\sigma(t)}\lambda(s)g(s)\Delta s,\ for\ t\in[a,b]_{\mathbb{T}},\tag{3.45}$$

then

$$\int_a^b\frac{\lambda(t)}{\Omega^c(t)}\left(\Phi^\sigma(t)\right)^k\Delta t\geqslant\left(\frac{k}{c-1}\right)^k\int_a^b\lambda(t)\Omega^{k-c}(t)g^k(t)\Delta t.\tag{3.46}$$

We can also obtain the Leindler type inequality on time scales that has been proved in [27, Theorem 2.3] by choosing

$$u(t)=\lambda(t),\ v(t)=\lambda(t)\left(\int_t^\infty\lambda(s)\Delta s\right)^{\frac{p}{1-p}},\ w(t)=\lambda^{-1}(t)\left(\int_t^\infty\lambda(s)\Delta s\right)^{\frac{p}{p-1}}g(t),\tag{3.47}$$

$s=1$ and $r=p$ as follows.

COROLLARY 3.6. *Let $\mathbb{T}$ be a time scale with $a\in\mathbb{T}$ and $0<p\leqslant 1$. If $\lambda(t)>0$ and $g(t)\geqslant 0$, then*

$$\int_a^\infty\lambda(t)\left(\int_a^{\sigma(t)}g(s)\Delta s\right)^p\Delta t\geqslant p^p\int_a^\infty\lambda^{1-p}(t)\left(\int_t^\infty\lambda(s)\Delta s\right)^p g^p(t)\Delta t.\tag{3.48}$$

Next, in the following we consider the functions

$$u(t)=\frac{\lambda-1}{q}\left[\frac{p}{q}(\lambda-1)\right]^{\frac{p}{p-1}}\sigma(t)^{\frac{p}{q}(1-\lambda)-1},\ v(t)=\sigma(t)^{\frac{p\left(1-\frac{\lambda-1}{q}\right)-1}{1-p}},$$

$$w(t)=\sigma(t)^{\frac{p\left(1-\frac{\lambda-1}{q}\right)-1}{p-1}}f(t),$$

$s=1$ and $r=p$. To apply Theorem 3.2 we need to prove that the hypothesis (3.31) is satisfied for the new functions u and v. Now, the left-hand side of (3.31) becomes

$$\left(\frac{\lambda-1}{q}\right)^{\frac{p}{p-1}}\int_x^\infty\sigma(t)^{\frac{p\left(1-\frac{\lambda-1}{q}\right)-1}{1-p}}\left(\frac{p}{q}(\lambda-1)\int_t^\infty\sigma(x)^{\frac{p}{q}(1-\lambda)-1}\Delta x\right)^{\frac{p}{p-1}}\Delta t.\tag{3.49}$$

Using the chain rule (2.1), we get that

$$\begin{aligned}\left(x^{\frac{p}{q}(1-\lambda)}\right)^\Delta&=\frac{p}{q}(1-\lambda)\int_0^1\left(h\sigma(x)+(1-h)x\right)^{\frac{p}{q}(1-\lambda)-1}dh\\&=\frac{p}{q}(\lambda-1)\int_1^0\frac{dh}{\left(h\sigma(x)+(1-h)x\right)^{\frac{p}{q}(\lambda-1)+1}}\\&\geqslant\frac{p}{q}(\lambda-1)\int_1^0\frac{dh}{\left(h\sigma(x)+(1-h)\sigma(x)\right)^{\frac{p}{q}(\lambda-1)+1}}\\&=\frac{p}{q}(1-\lambda)\left(\sigma(x)\right)^{\frac{p}{q}(1-\lambda)-1}.\end{aligned}$$

Integrating from t to ∞, we obtain that

$$\frac{p}{q}(1-\lambda)\int_t^\infty (\sigma(x))^{\frac{p}{q}(1-\lambda)-1}\Delta x \leqslant \int_t^\infty \left(x^{\frac{p}{q}(1-\lambda)}\right)^\Delta \Delta x \leqslant -t^{\frac{p}{q}(1-\lambda)},$$

which can be rewritten as

$$t^{\frac{p}{q}(1-\lambda)} \leqslant \frac{p}{q}(\lambda-1)\int_t^\infty (\sigma(x))^{\frac{p}{q}(1-\lambda)-1}\Delta x.$$

But since $1-\lambda<0$, we have that

$$\sigma(t)^{\frac{p}{q}(1-\lambda)} \leqslant \frac{p}{q}(\lambda-1)\int_t^\infty (\sigma(x))^{\frac{p}{q}(1-\lambda)-1}\Delta x.$$

By raising both sides to the power $p/(p-1)<0$, we obtain that

$$\left(\frac{p}{q}(\lambda-1)\int_t^\infty (\sigma(x))^{\frac{p}{q}(1-\lambda)-1}\Delta x\right)^{\frac{p}{p-1}} \leqslant \sigma(t)^{\frac{p}{p-1}\left(\frac{p}{q}(1-\lambda)\right)}. \tag{3.50}$$

Substituting (3.50) into (3.49), we have that

$$\begin{aligned}
&\left(\frac{\lambda-1}{q}\right)^{\frac{p}{p-1}}\int_x^\infty t^{\frac{p\left(1-\frac{\lambda-1}{q}\right)-1}{1-p}}\left(\frac{p}{q}(\lambda-1)\int_t^\infty x^{\frac{p}{q}(1-\lambda)-1}\Delta x\right)^{\frac{p}{p-1}}\Delta t\\
&\leqslant \left(\frac{\lambda-1}{q}\right)^{\frac{p}{p-1}}\int_x^\infty \sigma(t)^{\frac{p\left(1-\frac{\lambda-1}{q}\right)-1}{1-p}}\cdot\sigma(t)^{\frac{p}{p-1}\left(\frac{p}{q}(1-\lambda)\right)}\Delta t\\
&= \left(\frac{\lambda-1}{q}\right)^{\frac{p}{p-1}}\int_x^\infty \sigma(t)^{\frac{p}{q}(1-\lambda)-1}\Delta t,
\end{aligned}$$

which asserts the assumption (3.31). Now, as a special case of Theorem 3.2 by using (3.47), we obtain the time scale version of the inequality (1.14) that has been proved by Chen and Yang [8, Theorem 1] as follows.

COROLLARY 3.7. *Let $\mathbb{T}$ be a time scale with $a\in\mathbb{T}$. If $0<p<1$, $q>0$, $1<\lambda\leqslant 1+q$, then*

$$\begin{aligned}
&\left(\int_a^\infty \sigma(t)^{\frac{p}{q}(1-\lambda)-1}\left(\int_a^{\sigma(t)} f(s)\Delta s\right)^p\Delta t\right)^{\frac{1}{p}}\\
&> \frac{q}{\lambda-1}\left(\int_a^\infty (\sigma(t))^{p\left(1-\frac{\lambda-1}{q}\right)-1} f^p(t)\Delta t\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.51}$$

If we choose $q=p$ in Corollary 3.7, we obtain the following result.

COROLLARY 3.8. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. If $0 < p < 1$ and $0 < \lambda - 1 \leqslant p$, then*

$$\int_a^\infty \frac{1}{\sigma(t)^\lambda}\left(\int_a^{\sigma(t)} f(s)\Delta s\right)^p \Delta t > \left(\frac{p}{\lambda - 1}\right)^p \int_a^\infty \frac{f^p(t)}{\sigma(t)^{\lambda - p}}\Delta t. \tag{3.52}$$

If we choose $p = \lambda - 1$ in Corollary 3.8, we obtain the following result.

COROLLARY 3.9. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. If $0 < p < 1$, then*

$$\int_a^\infty \frac{1}{\sigma(t)}\left(\frac{1}{\sigma(t)}\int_a^{\sigma(t)} f(s)\Delta s\right)^p \Delta t > \int_a^\infty \frac{1}{\sigma(t)} f^p(t)\Delta t. \tag{3.53}$$

REMARK 3.8. If $\mathbb{T} = \mathbb{N}$, then (3.51), (3.52) and (3.53) reduce to the inequalities (11), (12) and (13) in [8], respectively.

REFERENCES

[1] R. P. AGARWAL, M. BOHNER AND S. H. SAKER, *Dynamic Littlewood-type inequalities*, Proc. Amer. Math. Soc. **143** (2015), 667–677.

[2] R. P. AGARWAL, D. O'REGAN AND S. H. SAKER, *Dynamic Inequalities on Time Scales*, Springer, Heid., 2014.

[3] M. ANWAR, R. BIBI, M. BOHNER AND J. PEČARIĆ, *Integral inequalities on time scales via the theory of isotonic linear functionals*, Abstr. Appl. Anal. **2011** (2011), 1–16.

[4] G. BENNETT, *Some elementary inequalities*, Quart. J. Math. Oxford (2), **38** (1987), 401–425.

[5] G. BENNETT, *Some elementary inequalities II*, Quart. J. Math. Oxford (2), **39** (1988), 385–400.

[6] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, (2001).

[7] M. BOHNER, A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[8] Q. CHEN AND B. YANG, *On a new reverse Hardy-Littlewood's type inequality*, Appl. Math. Sci. **6** (132) (2012), 6553–6561.

[9] A. ČIŽMEŠIJA, J. PEČARIĆ AND L.-E. PERSSON, *On strengthened Hardy and Pólya-Knopp's inequalities*, J. Approx. Theory **125** (2003), 74–84.

[10] E. T. COPSON, *Note on series of positive terms*, J. Lond. Math. Soc. **3** (1928), 49–51.

[11] E. T. COPSON, *Some integral inequalities*, Proc. Roy. Soc. Edin.: Section A Math. **75** (1976), 157–164.

[12] G. H. HARDY, *Notes on a theorem of Hilbert*, Math. Z. **6** (1920), 314–317.

[13] G. H. HARDY, *Notes on some points in the integral calculus*, LX. An inequality between integrals, Mess. Math. **54** (1925), 150–156.

[14] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, 2nd ED. Cambridge Univ. Press, 1934.

[15] S. HILGER, *Analysis on measure chain–a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.

[16] S. KAIJSER, L.-E. PERSSON AND A. ÖBERG, *On Carleman and Knopp's inequalities*, J. Approx. Theory **117** (2002), 140–151.

[17] A. KUFNER AND LARS-ERIK PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific Publishing (2003).

[18] A. KUFNER, L. MALIGRANDA AND L. PERSSON, *The Hardy Inequalities: About its History and Some Related Results*, Pilsen (2007).

[19] L. LEINDLER, *Generalization of inequalities of Hardy and Littlewood*, Acta Sci. Math. (Szeged), **31** (1970), 279–285.

[20] L. LEINDLER, *Further sharpening of inequalities of Hardy and Littlewood*, Acta Sci. Math. (Szeged), **54** (1990), 285–289.

[21] V. G. MAZ'JA, *Sobolev Spaces*, Springer-Verlag, Springer Series in Soviet Mathematics (1985).

[22] R. N. MOHAPATRA AND F. L. SALZMAN, *On a result of Leindler*, Math. Ineq. Appl. **5** (2002), 39–43.

[23] B. MUCKENHOUPT, *Hardy's inequality with weights*, Studia Math. **44** (1972), 31–38.

[24] B. OPIC AND A. KUFNER, *Hardy-type Inequalities*, Longman Scientific& Technical, Harlow, UK (1989).

[25] L. E. PERSSON AND V. D. STEPANOV, *Weighted integral inequalities with the geometric mean operator*, J. Ineq. Appl. **7** (2002), 727–746.

[26] P. ŘEHÁK, *Hardy inequality on time scales and its application to half-linear dynamic equations*, J. Ineq. Appl. **5** (2005), 495–507.

[27] S. H. SAKER, *Hardy-Leindler type inequalities on time scales*, Appl. Math. Inf. Sci. **8** (2014), 2975–2981.

[28] S. H. SAKER AND D. O'REGAN, *Extensions of dynamic inequalities of Hardy's type on time scales*, Math. Slovaca (in press).

[29] S. H. SAKER AND J. GRAEF, *A new class of dynamic inequalities of Hardy's type on time scales*, Dynam. Systems Appl. **23** (2014), 83–93.

[30] S. H. SAKER AND D. O'REGAN, *Hardy and Littlewood inequalities on time scales*, Bull. Malays. Math. Sci. Soc. (in press).

[31] S. H. SAKER, D. O'REGAN AND R. P. AGARWAL, *Some dynamic inequalities of Hardy's type on time scales*, Math. Ineq. Appl. **17** (2014), 1183–1199.

[32] S. H. SAKER, D. O'REGAN AND R. P. AGARWAL, *Generalized Hardy, Copson, Leindler and Bennett inequalities on time scales*, Math. Nachr. **287** (2014), 686–698.

[33] S. H. SAKER, D. O'REGAN AND R. P. AGARWAL, *Dynamic inequalities of Hardy and Copson types on time scales*, Analysis **34** (2014), 391–402.

[34] S. H. SAKER, D. O'REGAN AND R. P. AGARWAL, *Littlewood and Bennett inequalities on time scales*, Mediterr. J. Math. **8** (2014), 1–15.

[35] S. H. SAKER, D. O'REGAN AND R. P. AGARWAL, *Converses of Copson's inequalities on time scales*, Math. Ineq. Appl. **18** (2015), 241–254.

[36] S. H. SAKER, R. R. MAHMOUD AND A. PETERSON, *Weighted Hardy-type dynamic inequalities on time scales*, Mediterr. J. Math. (in press).

[37] G. TALENTI, *Sopra una diseguaglianza integrale*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **21** (3) (1967), 167–188.

[38] G. TOMASELLI, *A class of inequalities*, Boll. Unione Mat. Ital. **2** (4) (1969), 622–631.

(Received November 29, 2014)

S. H. Saker
Department of Mathematics, Faculty of Science
Mansoura University, Mansoura-Egypt
e-mail: shsaker@mans.edu.eg

R. R. Mahmoud
Department of Mathematics, Faculty of Science
Fayoum University, Fayoum-Egypt
e-mail: rrm00@fayoum.edu.eg

A. Peterson
Department of Mathematics
University of Nebraska–Lincoln
Lincoln, NE 68588-0130, U.S.A.
e-mail: apeterson1@math.unl.edu