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ON PÓLYA–SZEGÖ AND CHEBYSHEV TYPES INEQUALITIES INVOLVING THE RIEMANN–LIOUVILLE FRACTIONAL INTEGRAL OPERATORS

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Abstract. In this paper, we investigate some new Pólya-Szegö type integral inequalities involving the Riemann-Liouville fractional integral operator, and use them to prove some fractional integral inequalities of Chebyshev type, concerning the integral of the product of two functions and the product of two integrals. Certain special cases are also considered. Finally, examples for constructing the bounding functions are also given.

1. Introduction

The well-known celebrated functional was introduced by Chebyshev [3] and is defined by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \quad (1.1)$$

where f and g are two integrable functions on $[a,b]$. If f and g are synchronous, *i.e.*,

$$(f(x)-f(y))(g(x)-g(y))\geqslant 0,$$

for any $x,y\in[a,b]$, then $T(f,g)\geqslant 0$.

The functional (1.1) has attracted many researchers attention due to diverse applications in numerical quadrature, transform theory, probability and statistical problems. Among those applications, the functional (1.1) has also been employed to yield a number of integral inequalities (see, *e.g.*, [2, 5, 6, 9, 12]; for a very recent work, see also [16]).

The well known Grüss inequality [8] states

$$|T(f,g)|\leqslant\frac{(M-m)(N-n)}{4},$$

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where f and g are two integrable functions on $[a,b]$, and satisfy the following inequalities:

$$m \leqslant f(x) \leqslant M \quad \text{and} \quad n \leqslant g(y) \leqslant N,$$

for all $x, y \in [a,b]$ and for some $m, M, n, N \in \mathbb{R}$.

Pólya and Szegö [14] introduced the following inequality:

$$\frac{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}{\left(\int_a^b f(x)g(x)dx\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2.$$

Dragomir and Diamond [7] by using Pólya-Szegö inequality proved that

$$|T(f,g)| \leqslant \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{mMnN}} \int_a^b f(x)dx \int_a^b g(x)dx,$$

where f and g are two positive integrable functions on $[a,b]$ satisfying

$$0 < m \leqslant f(x) \leqslant M < \infty, \; 0 < n \leqslant g(y) \leqslant N < \infty,$$

for all $x, y \in [a,b]$ and for some $m, M, n, N \in \mathbb{R}$.

In recent years, fractional integral inequalities have proved to be one of the most powerful and far-reaching tools for the development of many branches of pure and applied mathematics. Very recently, many authors have presented some generalized inequalities involving the fractional integral operators (see, *e.g.*, [4, 11, 13, 15]; see also the very recent work [1]). Our present paper has been motivated by the above-mentioned works. The principle aim of the present paper is to establishing certain new Pólya-Szegö and Chebyshev types integral inequalities associated with Riemann-Liouville fractional integral operators.

We organize the paper as follows: in Section 3 we establish some generalized Pólya-Szegö type integral inequalities via Riemann-Liouville fractional integral operators, and use them to prove some fractional integral inequalities of Chebyshev type, concerning the integral of the product of two functions and the product of two integrals. In Section 4, as applications, we give examples for constructing the four functions for bounding the two unknown functions. Moreover, we give some estimates of Chebyshev type fractional integral inequalities of two unknown functions.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [10] and present preliminary results needed in our proofs later.

DEFINITION 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$R_{0,t}^{\alpha}\{f\}(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \tag{2.1}$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the Gamma function.

The above integral has following properties

$$R_{0,t}^{\alpha}R_{0,t}^{\beta}\{f\}(t)=R_{0,t}^{\alpha+\beta}\{f\}(t)=R_{0,t}^{\beta}R_{0,t}^{\alpha}\{f\}(t),$$

and

$$R_{0,t}^{\alpha}\{t^{\gamma}\}(t)=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma},\quad \alpha>0,\ \gamma>-1,\ t>0.$$

For $0=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=T$, we define a notation of sub-integrals of Riemann-Liouville fractional integral as

$$R_{t_j,t_{j+1}}^{\alpha}\{f\}(T)=\int_{t_j}^{t_{j+1}}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds,\quad j=0,1,\ldots,p. \tag{2.2}$$

Note that

$$\begin{aligned}R_{0,T}^{\alpha}\{f\}(T)&=\sum_{j=0}^{p}R_{t_j,t_{j+1}}^{\alpha}\{f\}(T)\\&=\int_0^{t_1}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds+\cdots+\int_{t_p}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds.\end{aligned}$$

3. Certain Pólya-Szegö and Chebyshev types inequalities involving the Riemann-Liouville fractional integral operator

In this section, we establish certain Pólya-Szegö type integral inequalities for positive integrable functions involving the Riemann-Liouville fractional integral operator (2.1).

LEMMA 3.1. *Let f and g be two positive integrable functions on $[0,\infty)$. Assume that there exist four positive integrable functions φ_1, φ_2, ψ_1 and ψ_2 on $[0,\infty)$ such that:*

(H_1) $0<\varphi_1(\tau)\leqslant f(\tau)\leqslant\varphi_2(\tau),\quad 0<\psi_1(\tau)\leqslant g(\tau)\leqslant\psi_2(\tau),\qquad (\tau\in[0,t],\,t>0)$.

Then for $t>0$ and $\alpha>0$, the following inequality holds:

$$\frac{R_{0,t}^{\alpha}\{\psi_1\psi_2f^2\}(t)R_{0,t}^{\alpha}\{\varphi_1\varphi_2g^2\}(t)}{\left(R_{0,t}^{\alpha}\{(\varphi_1\psi_1+\varphi_2\psi_2)fg\}(t)\right)^2}\leqslant\frac{1}{4}. \tag{3.1}$$

Proof. From (H_1), for $\tau\in[0,t]$, $t>0$, we have

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)}-\frac{f(\tau)}{g(\tau)}\right)\geqslant 0. \tag{3.2}$$

Analogously, we have

$$\left(\frac{f(\tau)}{g(\tau)}-\frac{\varphi_1(\tau)}{\psi_2(\tau)}\right)\geqslant 0. \tag{3.3}$$

Multiplying (3.2) and (3.3), we obtain

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)}-\frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)}-\frac{\varphi_1(\tau)}{\psi_2(\tau)}\right)\geqslant 0, \tag{3.4}$$

The inequality (3.4) can be written as

$$(\varphi_1(\tau)\psi_1(\tau)+\varphi_2(\tau)\psi_2(\tau))\,f(\tau)g(\tau)\geqslant \psi_1(\tau)\psi_2(\tau)f^2(\tau)+\varphi_1(\tau)\varphi_2(\tau)g^2(\tau). \tag{3.5}$$

Now, multiplying both sides of (3.5) by $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$ and integrating with respect to τ from 0 to t, we get

$$R^{\alpha}_{0,t}\{(\varphi_1\psi_1+\varphi_2\psi_2)\,fg\}(t)\geqslant R^{\alpha}_{0,t}\{\psi_1\psi_2 f^2\}(t)+R^{\alpha}_{0,t}\{\varphi_1\varphi_2 g^2\}(t).$$

Applying the AM-GM inequality, i.e., $a+b\geqslant 2\sqrt{ab}$, $a,b\in\mathbb{R}^+$, we have

$$R^{\alpha}_{0,t}\{(\varphi_1\psi_1+\varphi_2\psi_2)\,fg\}(t)\geqslant 2\sqrt{R^{\alpha}_{0,t}\{\psi_1\psi_2 f^2\}(t)R^{\alpha}_{0,t}\{\varphi_1\varphi_2 g^2\}(t)},$$

which leads to

$$R^{\alpha}_{0,t}\{\psi_1\psi_2 f^2\}(t)R^{\alpha}_{0,t}\{\varphi_1\varphi_2 g^2\}(t)\leqslant \frac{1}{4}\left(R^{\alpha}_{0,t}\{(\varphi_1\psi_1+\varphi_2\psi_2)\,fg\}(t)\right)^2.$$

Therefore, we obtain the inequality (3.1) as requested. □

As a special case of Lemma 3.1, we obtain the following result:

COROLLARY 3.1. Let f and g be two positive integrable functions on $[0,\infty)$ satisfying

(H_2) $0<m\leqslant f(\tau)\leqslant M<\infty,\quad 0<n\leqslant g(\tau)\leqslant N<\infty,\qquad (\tau\in[0,t],t>0).$

Then for $t>0$ and $\alpha>0$, we have

$$\frac{R^{\alpha}_{0,t}\{f^2\}(t)R^{\alpha}_{0,t}\{g^2\}(t)}{\left(R^{\alpha}_{0,t}\{fg\}(t)\right)^2}\leqslant\frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2.$$

REMARK 3.2. Corollary 3.1 is Lemma 3 of [1] whose the proof in [1] is not correct, since it is based on the inequality

$$\frac{MN}{mn}\leqslant\frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2, \tag{3.6}$$

which obviously does not holds when constants m, n, M and N satisfy (H_2) with strict inequalities. As noted above Lemma 3 of [1] is reduced as special case of our Lemma 3.1.

LEMMA 3.3. *Let all assumptions of Lemma 3.1 hold. Then for $t>0$ and $\alpha,\beta>0$, the following inequality holds:*

$$\frac{R_{0,t}^{\alpha}\{\varphi_1\varphi_2\}(t)R_{0,t}^{\beta}\{\psi_1\psi_2\}(t)R_{0,t}^{\alpha}\{f^2\}(t)R_{0,t}^{\beta}\{g^2\}(t)}{\left(R_{0,t}^{\alpha}\{\varphi_1 f\}(t)R_{0,t}^{\beta}\{\psi_1 g\}(t)+R_{0,t}^{\alpha}\{\varphi_2 f\}(t)R_{0,t}^{\beta}\{\psi_2 g\}(t)\right)^2}\leqslant\frac{1}{4}. \tag{3.7}$$

Proof. To prove (3.7), using the condition (H_1), we obtain

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\rho)}-\frac{f(\tau)}{g(\rho)}\right)\geqslant 0,$$

and

$$\left(\frac{f(\tau)}{g(\rho)}-\frac{\varphi_1(\tau)}{\psi_2(\rho)}\right)\geqslant 0,$$

which imply that

$$\left(\frac{\varphi_1(\tau)}{\psi_2(\rho)}+\frac{\varphi_2(\tau)}{\psi_1(\rho)}\right)\frac{f(\tau)}{g(\rho)}\geqslant\frac{f^2(\tau)}{g^2(\rho)}+\frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\rho)\psi_2(\rho)}. \tag{3.8}$$

Multiplying both sides of (3.8) by $\psi_1(\rho)\psi_2(\rho)g^2(\rho)$, we have

$$\varphi_1(\tau)f(\tau)\psi_1(\rho)g(\rho)+\varphi_2(\tau)f(\tau)\psi_2(\rho)g(\rho)\geqslant\psi_1(\rho)\psi_2(\rho)f^2(\tau)+\varphi_1(\tau)\varphi_2(\tau)g^2(\rho). \tag{3.9}$$

Multiplying both sides of (3.9) by $(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}/(\Gamma(\alpha)\Gamma(\beta))$ and double integrating with respect to τ and ρ from 0 to t, we have

$$\begin{aligned}&R_{0,t}^{\alpha}\{\varphi_1 f\}(t)R_{0,t}^{\beta}\{\psi_1 g\}(t)+R_{0,t}^{\alpha}\{\varphi_2 f\}(t)R_{0,t}^{\beta}\{\psi_2 g\}(t)\\&\qquad\geqslant R_{0,t}^{\alpha}\{f^2\}(t)R_{0,t}^{\beta}\{\psi_1\psi_2\}(t)+R_{0,t}^{\alpha}\{\varphi_1\varphi_2\}(t)R_{0,t}^{\beta}\{g^2\}(t).\end{aligned}$$

Applying the AM-GM inequality, we get

$$\begin{aligned}&R_{0,t}^{\alpha}\{\varphi_1 f\}(t)R_{0,t}^{\beta}\{\psi_1 g\}(t)+R_{0,t}^{\alpha}\{\varphi_2 f\}(t)R_{0,t}^{\beta}\{\psi_2 g\}(t)\\&\qquad\geqslant 2\sqrt{R_{0,t}^{\alpha}\{f^2\}(t)R_{0,t}^{\beta}\{\psi_1\psi_2\}(t)R_{0,t}^{\alpha}\{\varphi_1\varphi_2\}(t)R_{0,t}^{\beta}\{g^2\}(t)},\end{aligned}$$

which leads to the desired inequality in (3.7). The proof is completed. □

As a special case of Lemma 3.3, we get the following result:

COROLLARY 3.2. Let f and g be two positive integrable functions on $[0,\infty)$ satisfying (H_2). Then for $t>0$ and $\alpha,\beta>0$, we have

$$\frac{t^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{R_{0,t}^{\alpha}\{f^2\}(t)R_{0,t}^{\beta}\{g^2\}(t)}{\left(R_{0,t}^{\alpha}\{f\}(t)R_{0,t}^{\beta}\{g\}(t)\right)^2}\leqslant\frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2.$$

LEMMA 3.4. *Suppose that all assumptions of Lemma 3.1 are satisfied. Then for* $t>0$ *and* $\alpha,\beta>0$*, the following inequality holds:*

$$R_{0,t}^{\alpha}\{f^2\}(t)R_{0,t}^{\beta}\{g^2\}(t)\leqslant R_{0,t}^{\alpha}\{(\varphi_2 fg)/\psi_1\}(t)R_{0,t}^{\beta}\{(\psi_2 fg)/\varphi_1\}(t) \tag{3.10}$$

Proof. From condition (H_1), we have

$$\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f^2(\tau)d\tau\leqslant\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\frac{\varphi_2(\tau)}{\psi_1(\tau)}f(\tau)g(\tau)d\tau,$$

which implies

$$R_{0,t}^{\alpha}\{f^2\}(t)\leqslant R_{0,t}^{\alpha}\{(\varphi_2 fg)/\psi_1\}(t). \tag{3.11}$$

Analogously, we obtain

$$\frac{1}{\Gamma(\beta)}\int_0^t(t-\rho)^{\beta-1}g^2(\rho)d\rho\leqslant\frac{1}{\Gamma(\beta)}\int_0^t(t-\rho)^{\beta-1}\frac{\psi_2(\rho)}{\varphi_1(\rho)}f(\rho)g(\rho)d\rho,$$

from which one has

$$R_{0,t}^{\beta}\{g^2\}(t)\leqslant R_{0,t}^{\beta}\{(\psi_2 fg)/\varphi_1\}(t). \tag{3.12}$$

Multiplying (3.11) and (3.12), we get the desired inequality in (3.10). □

COROLLARY 3.3. Let f and g be two positive integrable functions on $[0,\infty)$ satisfying (H_2). Then for $t>0$ and $\alpha,\beta>0$, we have

$$\frac{R_{0,t}^{\alpha}\{f^2\}(t)R_{0,t}^{\beta}\{g^2\}(t)}{R_{0,t}^{\alpha}\{fg\}(t)R_{0,t}^{\beta}\{fg\}(t)}\leqslant\frac{MN}{mn}.$$

REMARK 3.5. The inequality (3.10) can not be reduced to the results in [1], because the proof of Lemma 6 in [1] is based on the inequality (3.6) which does not work (see also Remark 3.2).

In the sequel, we establish our main Chebyshev type integral inequalities involving the Riemann-Liouville fractional integral operator (2.1), with the help of Pólya-Szegö fractional integral inequality in Lemma 3.1 as follows.

THEOREM 3.6. *Let* f *and* g *be two positive integrable functions on* $[0,\infty)$*. Assume that there exist four positive integrable functions* φ_1*,* φ_2*,* ψ_1 *and* ψ_2 *satisfying* (H_1)*. Then for* $t>0$ *and* $\alpha,\beta>0$*, the following inequality is true*

$$\Bigg|\frac{t^{\alpha}}{\Gamma(\alpha+1)}R_{0,t}^{\beta}\{fg\}(t)+\frac{t^{\beta}}{\Gamma(\beta+1)}R_{0,t}^{\alpha}\{fg\}(t)$$
$$-R_{0,t}^{\alpha}\{f\}(t)R_{0,t}^{\beta}\{g\}(t)-R_{0,t}^{\alpha}\{g\}(t)R_{0,t}^{\beta}\{f\}(t)\Bigg|$$

$$\leqslant |G_1(f,\varphi_1,\varphi_2)(t)+G_2(f,\varphi_1,\varphi_2)(t)|^{\frac{1}{2}} \times |G_1(g,\psi_1,\psi_2)(t)+G_2(g,\psi_1,\psi_2)(t)|^{\frac{1}{2}}, \tag{3.13}$$

where

$$G_1(u,v,w)(t) = \frac{t^\beta}{4\Gamma(\beta+1)} \frac{\left(R_{0,t}^\alpha\{(v+w)u\}(t)\right)^2}{R_{0,t}^\alpha\{vw\}(t)} - R_{0,t}^\alpha\{u\}(t)R_{0,t}^\beta\{u\}(t),$$

$$G_2(u,v,w)(t) = \frac{t^\alpha}{4\Gamma(\alpha+1)} \frac{\left(R_{0,t}^\beta\{(v+w)u\}(t)\right)^2}{R_{0,t}^\beta\{vw\}(t)} - R_{0,t}^\alpha\{u\}(t)R_{0,t}^\beta\{u\}(t).$$

Proof. Let f and g be two positive integrable functions on $[0,\infty)$. For τ, $\rho \in (0,t)$ with $t>0$, we define $A(\tau,\rho)$ as

$$A(\tau,\rho) = (f(\tau)-f(\rho))(g(\tau)-g(\rho)),$$

or, equivalently,

$$A(\tau,\rho) = f(\tau)g(\tau)+f(\rho)g(\rho)-f(\tau)g(\rho)-f(\rho)g(\tau). \tag{3.14}$$

Multiplying both sides of (3.14) by $(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}/\Gamma(\alpha+1)\Gamma(\beta+1)$ and double integrating with respect to τ and ρ from 0 to t we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}A(\tau,\rho)\,d\tau\,d\rho \\
&= \frac{t^\alpha}{\Gamma(\alpha+1)}R_{0,t}^\beta\{fg\}(t) + \frac{t^\beta}{\Gamma(\beta+1)}R_{0,t}^\alpha\{fg\}(t) \\
&\quad -R_{0,t}^\alpha\{f\}(t)R_{0,t}^\beta\{g\}(t) - R_{0,t}^\beta\{f\}(t)R_{0,t}^\alpha\{g\}(t).
\end{aligned} \tag{3.15}$$

By using the Cauchy-Schwartz inequality for double integrals, we have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}A(\tau,\rho)\,d\tau\,d\rho\right| \\
&\leqslant \left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}f^2(\tau)\,d\tau\,d\rho\right. \\
&\quad +\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}f^2(\rho)\,d\tau\,d\rho \\
&\quad \left.-2\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}f(\tau)f(\rho)\,d\tau\,d\rho\right]^{\frac{1}{2}} \\
&\quad \times\left[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}g^2(\tau)\,d\tau\,d\rho\right. \\
&\quad +\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}g^2(\rho)\,d\tau\,d\rho \\
&\quad \left.-2\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}g(\tau)g(\rho)\,d\tau\,d\rho\right]^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}A(\tau,\rho)\,d\tau\,d\rho\right| \\
&\leqslant \left[\frac{t^\alpha}{\Gamma(\alpha+1)}R_{0,t}^{\beta}\left\{f^2(t)\right\}+\frac{t^\beta}{\Gamma(\beta+1)}R_{0,t}^{\alpha}\left\{f^2(t)\right\}-2R_{0,t}^{\beta}\left\{f(t)\right\}R_{0,t}^{\alpha}\left\{f(t)\right\}\right]^{\frac{1}{2}} \\
&\times\left[\frac{t^\alpha}{\Gamma(\alpha+1)}R_{0,t}^{\beta}\left\{g^2(t)\right\}+\frac{t^\beta}{\Gamma(\beta+1)}R_{0,t}^{\alpha}\left\{g^2(t)\right\}-2R_{0,t}^{\beta}\left\{g(t)\right\}R_{0,t}^{\alpha}\left\{g(t)\right\}\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.16}$$

Applying Lemma 3.1 with $\psi_1(t)=\psi_2(t)=g(t)=1$, we have

$$\frac{t^\beta}{\Gamma(\beta+1)}R_{0,t}^{\alpha}\{f^2\}(t)\leqslant\frac{t^\beta}{4\Gamma(\beta+1)}\frac{\left(R_{0,t}^{\alpha}\{(\varphi_1+\varphi_2)f\}(t)\right)^2}{R_{0,t}^{\alpha}\{\varphi_1\varphi_2\}(t)}.$$

This implies that

$$\begin{aligned}
&\frac{t^\beta}{\Gamma(\beta+1)}R_{0,t}^{\alpha}\{f^2\}(t)-R_{0,t}^{\alpha}\{f\}(t)R_{0,t}^{\beta}\{f\}(t) \\
&\qquad\leqslant\frac{t^\beta}{4\Gamma(\beta+1)}\frac{\left(R_{0,t}^{\alpha}\{(\varphi_1+\varphi_2)f\}(t)\right)^2}{R_{0,t}^{\alpha}\{\varphi_1\varphi_2\}(t)}-R_{0,t}^{\alpha}\{f\}(t)R_{0,t}^{\beta}\{f\}(t) \\
&\qquad=G_1(f,\varphi_1,\varphi_2)(t),
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\frac{t^\alpha}{\Gamma(\alpha+1)}R_{0,t}^{\beta}\{f^2\}(t)-R_{0,t}^{\alpha}\{f\}(t)R_{0,t}^{\beta}\{f\}(t) \\
&\qquad\leqslant\frac{t^\alpha}{4\Gamma(\alpha+1)}\frac{\left(R_{0,t}^{\beta}\{(\varphi_1+\varphi_2)f\}(t)\right)^2}{R_{0,t}^{\beta}\{\varphi_1\varphi_2\}(t)}-R_{0,t}^{\alpha}\{f\}(t)R_{0,t}^{\beta}\{f\}(t) \\
&\qquad=G_2(f,\varphi_1,\varphi_2)(t).
\end{aligned} \tag{3.18}$$

Also, applying the same procedure with $\phi_1(t)=\phi_2(t)=f(t)=1$, we get

$$\frac{t^\beta}{\Gamma(\beta+1)}R_{0,t}^{\alpha}\{g^2\}(t)-R_{0,t}^{\alpha}\{g\}(t)R_{0,t}^{\beta}\{g\}(t)\leqslant G_1(g,\psi_1,\psi_2)(t), \tag{3.19}$$

and

$$\frac{t^\alpha}{\Gamma(\alpha+1)}R_{0,t}^{\beta}\{g^2\}(t)-R_{0,t}^{\alpha}\{g\}(t)R_{0,t}^{\beta}\{g\}(t)\leqslant G_2(g,\psi_1,\psi_2)(t). \tag{3.20}$$

Finally, considering (3.15) to (3.20), we arrive at the desired result in (3.13). This completes the proof of Theorem 3.6. □

THEOREM 3.7. *Assume that all conditions of Theorem 3.6 are fulfilled. Then for $t>0$ and $\alpha>0$, the following inequality holds:*

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)} R^{\alpha}_{0,t}\{fg\}(t) - R^{\alpha}_{0,t}\{f\}(t) R^{\alpha}_{0,t}\{g\}(t)\right| \leqslant \left|G(f,\varphi_1,\varphi_2)(t) G(g,\psi_1,\psi_2)(t)\right|^{\frac{1}{2}}, \tag{3.21}$$

where

$$G(u,v,w)(t) = \frac{t^{\alpha}}{4\Gamma(\alpha+1)} \frac{\left(R^{\alpha}_{0,t}\{(v+w)u\}(t)\right)^2}{R^{\alpha}_{0,t}\{vw\}(t)} - \left(R^{\alpha}_{0,t}\{u\}(t)\right)^2.$$

Proof. Setting $\alpha=\beta$ in (3.13), we arrive at the desired result in (3.21). □

REMARK 3.8. If $\varphi_1 = m$, $\varphi_2 = M$, $\psi_1 = n$ and $\psi_2 = N$, then we have

$$\begin{aligned} G(f,m,M)(t) &= \frac{(M-m)^2}{4mM}\left(R^{\alpha}_{0,t}\{f\}(t)\right)^2, \\ G(g,n,N)(t) &= \frac{(N-n)^2}{4nN}\left(R^{\alpha}_{0,t}\{g\}(t)\right)^2. \end{aligned}$$

COROLLARY 3.4. Let f and g be two positive integrable functions on $[0,\infty)$ satisfying (H_2). Then for $t>0$ and $\alpha>0$, we have

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)} R^{\alpha}_{0,t}\{fg\}(t) - R^{\alpha}_{0,t}\{f\}(t) R^{\alpha}_{0,t}\{g\}(t)\right| \leqslant \frac{(M-m)(N-n)}{4\sqrt{mMnN}} R^{\alpha}_{0,t}\{f\}(t) R^{\alpha}_{0,t}\{g\}(t).$$

REMARK 3.9. Corollary 3.4 is Theorem 1 of [1] whose the proof is not correct since it again is based on inequality (3.6).

4. Applications

In this section we present a way for constructing the four bounding functions, and use them to give some estimates of Chebyshev type fractional integral inequalities of two unknown functions.

Let u be a unit step function defined by

$$u(t) = \begin{cases} 1, & t>0, \\ 0, & t\leqslant 0, \end{cases}$$

and let $u_a(t)$ be the Heaviside unit step function defined by

$$u_a(t) = u(t-a) = \begin{cases} 1, & t>a, \\ 0, & t\leqslant a. \end{cases}$$

Let φ_1 be a piecewise continuous function on $[0,T]$ defined by

$$\begin{aligned}\varphi_1(t) &= m_1(u_0(t)-u_{t_1}(t))+m_2(u_{t_1}(t)-u_{t_2}(t))+m_3(u_{t_2}(t)-u_{t_3}(t))\\ &\quad +\cdots+m_{p+1}u_{t_p}(t)\\ &= m_1u_0(t)+(m_2-m_1)u_{t_1}(t)+(m_3-m_2)u_{t_2}(t)+\cdots+(m_{p+1}-m_p)u_{t_p}(t)\\ &= \sum_{j=0}^{p}(m_{j+1}-m_j)u_{t_j}(t), \end{aligned} \tag{4.1}$$

where $m_0 \equiv 0$ and $0=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=T$.
Analogously, we define the functions φ_2, ψ_1 and ψ_2 as

$$\varphi_2(t) = \sum_{j=0}^{p}(M_{j+1}-M_j)u_{t_j}(t), \tag{4.2}$$

$$\psi_1(t) = \sum_{j=0}^{p}(n_{j+1}-n_j)u_{t_j}(t), \tag{4.3}$$

$$\psi_2(t) = \sum_{j=0}^{p}(N_{j+1}-N_j)u_{t_j}(t), \tag{4.4}$$

where constants $n_0=N_0=M_0\equiv 0$. If there is an integrable function f on $[0,T]$ satisfying condition (H_1) then we have $m_{j+1}\leqslant f(t)\leqslant M_{j+1}$ for each $t\in(t_j,t_{j+1}]$, $j=0,1,2,\dots,p$. In particular, $p=4$, the time history of f can be shown as in figure 1.

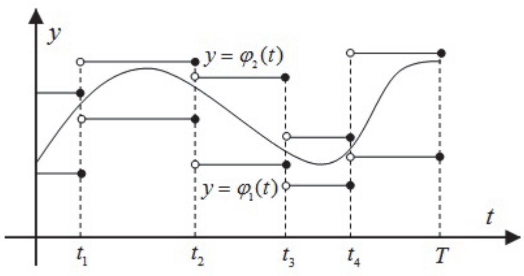

Figure 1: *Functions* f, φ_1 *and* φ_2.

PROPOSITION 4.1. *Let* f *and* g *be two positive integrable functions on* $[0,T]$. *Assume that the functions* φ_1, φ_2, ψ_1 *and* ψ_2 *defined by* (4.1), (4.2), (4.3) *and* (4.4), *respectively, satisfy* (H_1). *Then for* $\alpha>0$, *the following inequality holds:*

$$\begin{aligned}&\left(\sum_{j=0}^{p}n_{j+1}N_{j+1}R^{\alpha}_{t_j,t_{j+1}}\{f^2\}(T)\right)\left(\sum_{j=0}^{p}m_{j+1}M_{j+1}R^{\alpha}_{t_j,t_{j+1}}\{g^2\}(T)\right)\\ &\qquad\leqslant \frac{1}{4}\sum_{j=0}^{p}(m_{j+1}n_{j+1}+M_{j+1}N_{j+1})R^{\alpha}_{t_j,t_{j+1}}\{fg\}(T). \end{aligned}\tag{4.5}$$

Proof. By using the definition (2.2), we have

$$R_{0,T}^{\alpha}\{\psi_1\psi_2 f^2\}(T) = \sum_{j=0}^{p} n_{j+1}N_{j+1}R_{t_j,t_{j+1}}^{\alpha}\{f^2\}(T),$$
$$R_{0,T}^{\alpha}\{\varphi_1\varphi_2 g^2\}(T) = \sum_{j=0}^{p} m_{j+1}M_{j+1}R_{t_j,t_{j+1}}^{\alpha}\{g^2\}(T)$$

and

$$R_{0,T}^{\alpha}\{(\varphi_1\psi_1+\varphi_2\psi_2)fg\}(T) = \sum_{j=0}^{p}(m_{j+1}n_{j+1}+M_{j+1}N_{j+1})R_{t_j,t_{j+1}}^{\alpha}\{fg\}(T).$$

By applying Lemma 3.1, the desired inequality (4.5) is established. □

PROPOSITION 4.2. *Suppose that all assumptions of Proposition 4.1 are satisfied. Then for $\alpha,\beta>0$, the following inequality holds:*

$$\begin{aligned}&\Bigg|\frac{T^{\alpha}}{\Gamma(\alpha+1)}R_{0,T}^{\beta}\{fg\}(T)+\frac{T^{\beta}}{\Gamma(\beta+1)}R_{0,T}^{\alpha}\{fg\}(T)\\ &\qquad -R_{0,T}^{\alpha}\{f\}(T)R_{0,T}^{\beta}\{g\}(T)-R_{0,T}^{\alpha}\{g\}(T)R_{0,T}^{\beta}\{f\}(T)\Bigg|\\ &\leqslant \left|G_1^*(f,m_{j+1},M_{j+1})(T)+G_2^*(f,m_{j+1},M_{j+1})(T)\right|^{\frac{1}{2}}\\ &\qquad\times\left|G_1^*(g,n_{j+1},N_{j+1})(T)+G_2^*(g,n_{j+1},N_{j+1})(T)\right|^{\frac{1}{2}}\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}G_1^*(u,v,w)(t) &= \frac{T^{\beta}\Gamma(\alpha+1)}{4\Gamma(\beta+1)}\frac{\left(\sum_{j=0}^{p}(v+w)R_{t_j,t_{j+1}}^{\alpha}\{u\}(T)\right)^2}{\sum_{j=0}^{p}vw\left[(T-t_j)^{\alpha}-(T-t_{j+1})^{\alpha}\right]}\\ &\quad-\left(R_{0,T}^{\alpha}\{u\}(T)\right)\left(R_{0,T}^{\beta}\{u\}(T)\right),\\ G_2^*(u,v,w)(T) &= \frac{T^{\alpha}\Gamma(\beta+1)}{4\Gamma(\alpha+1)}\frac{\left(\sum_{j=0}^{p}(v+w)R_{t_j,t_{j+1}}^{\beta}\{g\}(T)\right)^2}{\sum_{j=0}^{p}vw\left[(T-t_j)^{\beta}-(T-t_{j+1})^{\beta}\right]}\\ &\quad-\left(R_{0,T}^{\alpha}\{u\}(T)\right)\left(R_{0,T}^{\beta}\{u\}(T)\right).\end{aligned}$$

Proof. Since

$$R^{\alpha}_{t_j,t_{j+1}}\{1\}(T) = \int_{t_j}^{t_{j+1}} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} ds = \frac{1}{\Gamma(\alpha+1)}\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right],$$

we have

$$R^{\alpha}_{0,T}\{\varphi_1\varphi_2\}(T) = \sum_{j=0}^{p} \frac{m_{j+1}M_{j+1}}{\Gamma(\alpha+1)}\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right],$$

$$R^{\alpha}_{0,T}\{\psi_1\psi_2\}(T) = \sum_{j=0}^{p} \frac{n_{j+1}N_{j+1}}{\Gamma(\alpha+1)}\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right].$$

By direct computations, we have

$$G_1(f,\varphi_1,\varphi_2)(T) = \frac{T^{\beta}\Gamma(\alpha+1)}{4\Gamma(\beta+1)} \frac{\left(\sum_{j=0}^{p}\left(m_{j+1}+M_{j+1}\right)R^{\alpha}_{t_j,t_{j+1}}\{f\}(T)\right)^2}{\sum_{j=0}^{p} m_{j+1}M_{j+1}\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right]} - \left(R^{\alpha}_{0,T}\{f\}(T)\right)\left(R^{\beta}_{0,T}\{f\}(T)\right),$$

$$G_1(g,\psi_1,\psi_2)(T) = \frac{T^{\beta}\Gamma(\alpha+1)}{4\Gamma(\beta+1)} \frac{\left(\sum_{j=0}^{p}\left(n_{j+1}+N_{j+1}\right)R^{\alpha}_{t_j,t_{j+1}}\{g\}(T)\right)^2}{\sum_{j=0}^{p} n_{j+1}N_{j+1}\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right]} - \left(R^{\alpha}_{0,T}\{g\}(T)\right)\left(R^{\beta}_{0,T}\{g\}(T)\right),$$

$$G_2(f,\varphi_1,\varphi_2)(T) = \frac{T^{\alpha}\Gamma(\beta+1)}{4\Gamma(\alpha+1)} \frac{\left(\sum_{j=0}^{p}\left(m_{j+1}+M_{j+1}\right)R^{\beta}_{t_j,t_{j+1}}\{f\}(T)\right)^2}{\sum_{j=0}^{p} m_{j+1}M_{j+1}\left[(T-t_j)^{\beta} - (T-t_{j+1})^{\beta}\right]} - \left(R^{\alpha}_{0,T}\{f\}(T)\right)\left(R^{\beta}_{0,T}\{f\}(T)\right),$$

and

$$G_2(g,\psi_1,\psi_2)(T) = \frac{T^{\alpha}\Gamma(\beta+1)}{4\Gamma(\alpha+1)} \frac{\left(\sum_{j=0}^{p}\left(n_{j+1}+N_{j+1}\right)R^{\beta}_{t_j,t_{j+1}}\{g\}(T)\right)^2}{\sum_{j=0}^{p} n_{j+1}N_{j+1}\left[(T-t_j)^{\beta} - (T-t_{j+1})^{\beta}\right]} - \left(R^{\alpha}_{0,T}\{g\}(T)\right)\left(R^{\beta}_{0,T}\{g\}(T)\right).$$

By applying Theorem 3.6, the required inequality (4.6) is established. □

COROLLARY 4.1. Let all assumptions of Proposition 4.2 be fulfilled. Then for $\alpha > 0$, the following inequality holds:

$$\left|\frac{T^{\alpha}}{\Gamma(\alpha+1)} R_{0,T}^{\alpha}\{fg\}(T) - R_{0,T}^{\alpha}\{f\}(T) R_{0,T}^{\alpha}\{g\}(T)\right| \leqslant \left|G^{*}(f, m_{j+1}, M_{j+1})(T) G^{*}(g, n_{j+1}, N_{j+1})(T)\right|^{\frac{1}{2}}, \tag{4.7}$$

where

$$G^{*}(u,v,w)(T) = \frac{T^{\alpha}}{4} \frac{\left(\sum_{j=0}^{p} (v+w) R_{t_j,t_{j+1}}^{\alpha}\{u\}(T)\right)^{2}}{\sum_{j=0}^{p} vw\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right]} - \left(R_{0,T}^{\alpha}\{u\}(T)\right)^{2}.$$

Proof. Setting $\alpha = \beta$ in (4.6), we arrive at the desired result in (4.7). □

REMARK 4.3. The accuracy of approximate bounds of inequalities (4.5), (4.7) and (4.6) is dependent on $p \in \mathbb{N}$.

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REFERENCES

[1] A. ANBER AND Z. DAHMANI, *New integral results using Pólya-Szegö inequality*, Acta Comment. Univ. Tartu. Math. **17** (2) (2013), 171–178.

[2] S. BELARBI AND Z. DAHMANI, *On some new fractional integral inequalities*, J. Inequal. Pure Appl. Math. **10** (3) (2009), Art. 86, 5 pp. (electronic).

[3] P. L. CHEBYSHEV, *Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov **2** (1882), 93–98.

[4] P. CERONE AND S. S. DRAGOMIR, *A refinement of the Grüss inequality and applications*, Tamkang J. Math. **38** (2007), 37–49.

[5] Z. DAHMANI, O. MECHOUAR AND S. BRAHAMI, *Certain inequalities related to the Chebyshev's functional involving a type Riemann-Liouville operator*, Bull. Math. Anal. Appl. **3** (4) (2011), 38–44.

[6] S. S. DRAGOMIR, *Some integral inequalities of Grüss type*, Indian J. Pure Appl. Math. **31** (4) (2000), 397–415.

[7] S. S. DRAGOMIR AND N. T. DIAMOND, *Integral inequalities of Grüss type via Pólya-Szegö and Shisha-Mond results*, East Asian Math. J. **19** (1) (2003), 27–39.

[8] G. GRÜSS, *Über das maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx$, Math. Z. **39** (1935), 215–226.

[9] S. L. KALLA AND A. RAO, *On Grüss type inequality for hypergeometric fractional integrals*, Matematiche (Catania) **66** (1) (2011), 57–64.

[10] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, 2006.

[11] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis, Mathematics and its Applications*, (East European Series) 61, Kluwer Academic Publishers Group, Dordrecht, 1993.

[12] H. ÖĞÜNMEZ AND U. M. ÖZKAN, *Fractional quantum integral inequalities*, J. Inequal. Appl. 2011, Article ID 787939, 7 pp.

[13] B. G. PACHPATTE, *On multidimensional Grüss type integral inequalities*, J. Inequal. Pure Appl. Math. **3** (2) (2002), Article 27.

[14] G. PÓLYA AND G. SZEGÖ, *Aufgaben und Lehrsatze aus der Analysis*, Band 1, Die Grundlehren der mathmatischen Wissenschaften 19, J. Springer, Berlin, 1925.

[15] W. SUDSUTAD, S. K. NTOUYAS AND J. TARIBOON, *Fractional integral inequalities via Hadamard's fractional integral*, Abst. Appl. Anal. ID **563096** (2014), 1–11.

[16] G. WANG, P. AGARWAL AND M. CHAND, *Certain Grüss type inequalities involving the generalized fractional integral operator*, J. Inequal. Appl. **(2014)** 2014:147.

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