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ON MONOTONE ĆIRIĆ QUASI–CONTRACTION MAPPINGS

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Abstract. We prove the existence of fixed points of monotone quasi-contraction mappings in metric and modular metric spaces. This is the extension of Ran and Reurings fixed point theorem for monotone contraction mappings in partially ordered metric spaces to the case of quasi-contraction mappings introduced by Ćirić. The proofs are based on Lemmas 2.1 and 3.1, which contain two crucial inequalities essential to obtain the main results.

1. Introduction

Banach's Contraction Principle [2] is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contractive condition on the mapping is simple and easy to test, because it requires only a complete metric space for its setting, and because it finds almost canonical applications in the theory of differential and integral equations. Over the years, many mathematicians tried successfully to extend this fundamental theorem. Recently a version of this theorem was given in partially ordered metric spaces [9, 12] and in metric spaces with a graph [1, 6]. In this work, we discuss the case of quasi-contractive mappings defined in partially ordered metric spaces and modular metric spaces.

For more on metric fixed point theory, the reader may consult the book [7].

2. Monotone quasi-contraction mappings in metric spaces

As a generalization to Banach Contraction Principle, Ćirić [5] introduced the concept of quasi-contraction mappings (see also [10, 11]). In this section, we investigate monotone mappings which are quasi-contraction mappings. Since in this work we discuss the fixed point theory of monotone mappings, we will need to introduce a partial order. Let (X,d) be a metric space and assume that a partial order $\leqslant$ exists in X. Throughout we assume that order intervals are closed. Recall that an order interval is any of the subsets

(i) $[a,\rightarrow) = \{x \in X; a \leqslant x\}$,

(ii) $(\leftarrow,a] = \{x \in X; x \leqslant a\}$,

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for any $a \in X$.

DEFINITION 2.1. Let $(X, d, \leqslant)$ be a partially ordered metric space as defined above. Let C be a nonempty subset of X. The mapping $T : C \to C$ is said to be:

(i) *monotone* if $T(x) \leqslant T(y)$ whenever $x \leqslant y$, for any $x, y \in C$;

(ii) *monotone quasi-contraction* if T is monotone and there exists a constant k, with $0 \leqslant k < 1$, such that for any $x, y \in C$, $x \leqslant y$, we have

$$d(T(x), T(y)) \leqslant k \max\Big(d(x,y); d(x, T(x)); d(y, T(y)); d(x, T(y)); d(y, T(x))\Big).$$

In the sequel we prove an existence fixed point theorem for such mappings. First, let T and C be as in Definition 2.1. For any $x \in C$, define the orbit $\mathscr{O}(x) = \{x, T(x), T^2(x), \cdots\}$, and its diameter by

$$\delta(x) = \sup\{d(T^n(x), T^m(x)) : n, m \in \mathbb{N}\} .$$

The following technical Lemma offers a crucial inequality to prove the main result of this section.

LEMMA 2.1. *Let $(X, d, \leqslant)$ be as above. Let C be a nonempty subset of X and $T : C \to C$ be a monotone quasi-contraction mapping. Let $x \in C$ be such that x and $T(x)$ are comparable and $\delta(x) < \infty$. Then for any $n \geqslant 1$, we have*

$$\delta(T^n(x)) \leqslant k^n \delta(x) ,$$

where $k < 1$ is the constant associated with the quasi-contraction definition of T. Moreover we have

$$d(T^n(x), T^{n+m}(x)) \leqslant k^n \delta(x) \tag{2.1}$$

for any $n, m \in \mathbb{N}$.

Proof. Since x and $T(x)$ are comparable, and T is monotone increasing, then $T^n(x)$ and $T^m(x)$ are comparable, for any $n, m \in \mathbb{N}$. Hence

$$\begin{aligned} d(T^n(x), T^{n+m}(x)) \leqslant k \max\Big(& d(T^{n-1}(x), T^{n+m-1}(x)); d(T^{n-1}(x), T^n(x)); \\ & d(T^{n+m}(x), T^{n+m-1}(x)); d(T^{n-1}(x), T^{n+m}(x)); \\ & d(T^n(x), T^{n+m-1}(x))\Big) \end{aligned}$$

for any $n, m \geqslant 1$. This obviously implies that

$$\delta(T^n(x)) \leqslant k\, \delta(T^{n-1}(x)) , \ \ n \geqslant 1.$$

Hence

$$\delta(T^n(x)) \leqslant k^n \ \delta(x) , \ \ n \geqslant 1.$$

This will imply

$$d(T^n(x),T^{n+m}(x)) \leqslant \delta(T^n(x)) \leqslant k^n\ \delta(x),$$

for any $n,m \in \mathbb{N}$. □

Using Lemma 2.1, we prove the main result of this section.

THEOREM 2.1. *Let $(X,d,\leqslant)$ be as above. Assume that (X,d) is complete. Let C be a closed nonempty subset of X and $T : C \to C$ be a monotone quasi-contraction mapping. Let $x \in C$ be such that x and $T(x)$ are comparable and $\delta(x) < \infty$. Then*

(i) $\{T^n(x)\}$ converges to $\omega \in C$ which is a fixed point of T and is comparable to x. Moreover we have

$$d(T^n(x),\omega) \leqslant k^n\ \delta(x),\ \ n \geqslant 1.$$

(ii) if $z \in C$ is a fixed point of T such that z and x are comparable, then $z = \omega$.

Proof. Let us prove (i). Assume without loss of any generality that $x \leqslant T(x)$. Lemma 2.1 implies that $\{T^n(x)\}$ is Cauchy. Since X is complete and C is closed, then there exists $\omega \in C$ such that $\{T^n(x)\}$ converges to ω. Using the inequality (2.1), and passing to the limit as $m \to \infty$, we get

$$d(T^n(x),\omega) \leqslant k^n\ \delta(x)\ ,\ \ n \geqslant 1.$$

Since T is monotone, we get $T^n(x) \leqslant T^{n+1}(x)$, for any $n \geqslant 1$. Since order intervals are closed, we conclude that $T^n(x) \leqslant \omega$, for any $n \geqslant 1$. In particular, we have $x \leqslant \omega$. In order to show that ω is a fixed point of T, note that we have

$$\begin{aligned} d(T^n(x),T(\omega)) \leqslant k\ \max\Big(&d(T^{n-1}(x),\omega);d(T^{n-1}(x),T^n(x));\\ &d(T(\omega),\omega);d(T^{n-1}(x),T(\omega));d(T^n(x),\omega)\Big) \end{aligned}$$

for any $n \geqslant 1$. If we let $n \to +\infty$, we get $d(\omega,T(\omega)) \leqslant k\ d(\omega,T(\omega))$, which forces $d(\omega,T(\omega)) = 0$ since $k < 1$. Therefore we have $T(\omega) = \omega$.

Next we show (ii). Let $z \in C$ be a fixed point of T such that x and z are comparable. Then we have

$$d(T^n(x),z) \leqslant k\ \max\Big(d(T^{n-1}(x),z);d(T^{n-1}(x),T^n(x));d(T^n(x),z)\Big),$$

for any $n \geqslant 2$. If we let $n \to +\infty$, we get

$$\limsup_{n\to\infty} d(T^n(x),z) \leqslant k\ \limsup_{n\to\infty} d(T^n(x),z).$$

Since $k < 1$, we get $\limsup\limits_{n\to\infty} d(T^n(x),z) = 0$, i.e., $\{T^n(x)\}$ converges to z. The uniqueness of the limit implies that $z = \omega$. □

In the next section, we discuss the validity of Theorem 2.1 in modular metric spaces. This is a very important class of spaces since they are similar to metric spaces in their structure but without the triangle inequality and offer a wide range of applications.

3. Monotone quasi-contraction mappings in modular metric spaces

Let X be a nonempty set. Throughout this section for a function $\omega : (0,\infty) \times X \times X \to (0,\infty)$, we will write

$$\omega_\lambda(x,y) = \omega(\lambda,x,y),$$

for all $\lambda > 0$ and $x,y \in X$.

DEFINITION 3.1. [3, 4] A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a modular metric on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x,y) = 0$, for all $\lambda > 0$;

(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$, and $x,y \in X$;

(iii) $\omega_{\lambda+\mu}(x,y) \leqslant \omega_\lambda(x,z) + \omega_\mu(z,y)$, for all $\lambda,\mu > 0$ and $x,y,z \in X$.

If instead of (i), we have only the condition (i')

$$\omega_\lambda(x,x) = 0, \text{ for all } \lambda > 0,\ x \in X,$$

then ω is said to be a pseudomodular (metric) on X. A modular metric ω on X is said to be regular if the following weaker version of (i) is satisfied

$$x = y \text{ if and only if } \omega_\lambda(x,y) = 0, \text{ for some } \lambda > 0.$$

Finally, ω is said to be convex if for $\lambda,\mu > 0$ and $x,y,z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu}(x,y) \leqslant \frac{\lambda}{\lambda+\mu}\omega_\lambda(x,z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z,y).$$

Note that for a metric pseudomodular ω on a set X, and any $x,y \in X$, the function $\lambda \to \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \leqslant \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y).$$

DEFINITION 3.2. [3, 4] Let ω be a pseudomodular on X. Fix $x_0 \in X$. The two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \ as\ \lambda \to \infty\},$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}$$

are said to be modular spaces (around x_0).

We obviously have $X_\omega \subset X_\omega^*$. In general this inclusion may be proper. It follows from [3, 4] that if ω is a modular on X, then the modular space X_ω can be equipped with a (nontrivial) metric, generated by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leqslant \lambda\},$$

for any $x,y \in X_\omega$. If ω is a convex modular on X, according to [3, 4] the two modular spaces coincide, i.e. $X^*_\omega = X_\omega$, and this common set can be endowed with the metric d^*_ω given by

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leqslant 1\},$$

for any $x,y \in X_\omega$. These distances will be called Luxemburg distances.

First attempts to generalize the classical function spaces of the Lebesgue type L^p, $1 \leqslant p < \infty$, were made in the early 1930's by Orlicz and Birnbaum in connection with orthogonal expansions. Their approach consisted in considering spaces of functions with some growth properties different from the power type growth control provided by the L^p-norms. Namely, they considered the function spaces defined as follows:

$$L^\varphi = \left\{ f : \mathbb{R} \to \mathbb{R};\ \exists \lambda > 0\ : \rho(\lambda f) = \int_{\mathbb{R}} \varphi\big(\lambda |f(x)|\big)\, dx < \infty \right\},$$

where $\varphi : [0,\infty] \to [0,\infty]$ was assumed to be a convex function increasing to infinity, i.e. the function which to some extent behaves similarly to power functions $\varphi(t) = t^p$, $1 \leqslant p < \infty$.

Modular function spaces L^φ furnishes a wonderful example of a modular metric space. Indeed define the function ω by

$$\omega_\lambda(f,g) = \rho\left(\frac{f-g}{\lambda}\right) = \int_{\mathbb{R}} \varphi\left(\frac{|f(x)-g(x)|}{\lambda}\right) dx$$

for all $\lambda > 0$, and $f,g \in L^\varphi$, then ω is a modular metric on L^φ. Moreover the distance d^*_ω is exactly the distance generated by the Luxemburg norm on L^φ.

For more examples on modular function spaces, the reader my consult the book of Kozlowski [8] and for modular metric spaces [3, 4].

DEFINITION 3.3. Let X_ω be a modular metric space.

(1) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n,x) \to 0$, as $n \to \infty$. x will be called the ω-limit of $\{x_n\}$.

(2) The sequence $\{x_n\}_{n\in N}$ in X_ω is said to be ω-Cauchy if $\omega_1(x_m,x_n) \to 0$, as $m,n \to \infty$.

(3) A subset M of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.

(4) A subset M of X_ω is said to be ω-complete if any ω-Cauchy sequence in M is a ω-convergent sequence and its ω-limit is in M.

(5) A subset M of X_ω is said to be ω-bounded if we have

$$\delta_\omega(M) = \sup\{\omega_1(x,y); x,y \in M\} < \infty.$$

(7) ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω ω-convergent to x, we have

$$\omega_1(x,y) \leqslant \liminf_{n\to\infty} \omega_1(x_n,y),$$

for any $y \in X_\omega$.

In general if $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for some $\lambda > 0$, then we may not have $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for all $\lambda > 0$. Therefore, as it is done in modular function spaces, we will say that ω satisfies Δ_2-condition if this is the case, i.e. $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for all $\lambda > 0$. In [3] and [4], one will find a discussion about the connection between ω-convergence and metric convergence with respect to the Luxemburg distances. In particular, we have

$$\lim_{n\to\infty} d_\omega(x_n,x) = 0 \text{ if and only if } \lim_{n\to\infty} \omega_\lambda(x_n,x) = 0, \text{ for all } \lambda > 0,$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$. We also have ω-convergence and d_ω convergence are equivalent if and only if the modular ω satisfies the Δ_2-condition. Moreover, if the modular ω is convex, then we know that d^*_ω and d_ω are equivalent which implies, see [3, 4],

$$\lim_{n\to\infty} d^*_\omega(x_n,x) = 0 \text{ if and only if } \lim_{n\to\infty} \omega_\lambda(x_n,x) = 0, \text{ for all } \lambda > 0,$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$.

Let (X,ω) be a modular metric space and assume that a partial order $\leqslant$ exists in X. Throughout this section, we assume that order intervals are ω-closed.

DEFINITION 3.4. Let $(X,\omega,\leqslant)$ be a partially ordered modular metric space as discussed above. Let C be a nonempty subset of X. The mapping $T : C \to C$ is said to be:

(i) *monotone* if $T(x) \leqslant T(y)$ whenever $x \leqslant y$, for any $x,y \in C$;

(ii) *monotone ω-quasi-contraction* if T is monotone and there exists $k < 1$ such that for any $x,y \in C$, $x \leqslant y$, we have

$$\omega_1(T(x),T(y)) \leqslant k \max\Big(\omega_1(x,y); \omega_1(x,T(x)); \omega_1(y,T(y)); \omega_1(x,T(y)); \omega_1(y,T(x))\Big).$$

In the sequel we prove an analogue to Theorem 2.1 in modular metric spaces. For any $x \in C$, define the orbit $\mathscr{O}(x) = \{x, T(x), T^2(x), \cdots\}$, and its diameter by

$$\delta_\omega(x) = \sup\{\omega_1(T^n(x),T^m(x)) : n,m \in \mathbb{N}\}.$$

Throughout we assume that ω is regular and satisfies the Fatou property. The following technical Lemma presents a crucial inequality to prove the main result of this section. It is the modular version of Lemma 2.1. Its proof will be omitted.

LEMMA 3.1. *Let $(X,\omega,\leqslant)$ be a partially ordered modular metric space as discussed above. Let C be a nonempty subset of X and $T:C\to C$ be a monotone ω-quasi-contraction mapping. Let $x\in C$ be such that x and $T(x)$ are comparable and $\delta_\omega(x)<\infty$. Then for any $n\geqslant 1$, we have*

$$\delta_\omega(T^n(x))\leqslant k^n\delta_\omega(x)\ ,$$

where $k<1$ is the constant associated with the ω-quasi-contraction definition of T. Moreover we have

$$\omega_1(T^n(x),T^{n+m}(x))\leqslant k^n\ \delta_\omega(x)$$

for any $n,m\in\mathbb{N}$.

Using Lemma 3.1, we prove the main result of this section.

THEOREM 3.1. *Let $(X,\omega,\leqslant)$ be a partially ordered modular metric space as discussed above. Let C be a nonempty subset of X which is ω-complete. Let $T:C\to C$ be a monotone ω-quasi-contraction mapping. Let $x\in C$ be such that x and $T(x)$ are comparable and $\delta_\omega(x)<\infty$. Then*

(i) $\{T^n(x)\}$ ω-converges to $z\in C$ which is a fixed point of T and is comparable to x, provided $\omega_1(z,T(z))<\infty$ and $\omega_1(x,T(z))<\infty$. Moreover we have

$$\omega_1(T^n(x),z)\leqslant k^n\ \delta_\omega(x),\ \ n\geqslant 1.$$

(ii) If w is a fixed point of T such that w and x are comparable and $\omega_1(T^n(x),w)<\infty$, for any $n\geqslant 1$, then $z=w$.

Proof. Let prove (i). Lemma 3.1 implies that $\{T^n(x)\}$ is ω-Cauchy. Since C is ω-complete, then there exists $z\in C$ such that $\{T^n(x)\}$ ω-converges to z. Since

$$\omega_1(T^n(x),T^{n+m}(x))\leqslant k^n\ \delta_\omega(x),$$

for any $n,m\in\mathbb{N}$, the Fatou property (once we let $m\to\infty$) will imply

$$\omega_1(T^n(x),z)\leqslant k^n\ \delta_\omega(x)\ ,\ \ n\geqslant 1.$$

Without loss of any generality, we may assume $x\leqslant T(x)$. Then we have $T^n(x)\leqslant T^{n+1}(x)$, for any $n\geqslant 1$, since T is monotone. Using the fact that order intervals are ω-closed, we get $T^n(x)\leqslant z$, for any $n\geqslant\mathbb{N}$. In particular, we have $x\leqslant z$. Next we assume $\omega_1(z,T(z))<\infty$ and $\omega_1(x,T(z))<\infty$. Let us prove that z is a fixed point of T. Then by induction, we will show that $\omega_1(T^n(x),T(z))<\infty$, and

$$\begin{aligned}\omega_1(T^n(x),T(z))\leqslant k\ \max\Big(&\omega_1(T^{n-1}(x),z);\omega_1(T^{n-1}(x),T^n(x));\\ &\omega_1(T(z),z);\omega_1(T^{n-1}(x),T(z));\omega_1(T^n(x),z)\Big)\end{aligned}$$

for any $n \geqslant 1$. If we let $n \to +\infty$, and using the Fatou property, we get $\omega_1(z, T(z)) \leqslant k\ \omega_1(z, T(z))$, which forces $\omega_1(z, T(z)) = 0$ since $k < 1$. Since ω is regular, we get $T(z) = z$.

Next we show (ii). Let $w \in C$ be a fixed point of T such that x and w are comparable and $\omega_1(T^n(x), w) < \infty$, for any $n \geqslant 1$. Then by induction, we get

$$\omega_1(T^n(x), w) \leqslant k\ \max\Big(\omega_1(T^{n-1}(x), w); \omega_1(T^{n-1}(x), T^n(x)); \omega_1(T^n(x), w)\Big),$$

for any $n \geqslant 2$. If we let $n \to +\infty$, we obtain

$$\limsup_{n\to\infty} \omega_1(T^n(x), w) \leqslant k\ \limsup_{n\to\infty} \omega_1(T^n(x), w).$$

Since $k < 1$, we get $\limsup_{n\to\infty} \omega_1(T^n(x), w) = 0$, i.e., $\{T^n(x)\}$ converges to w. The uniqueness of the limit implies that $z = w$. Indeed,we have

$$\omega_2(z, w) \leqslant \omega_1(T^n(x), z) + \omega_1(T^n(x), w),\ \ n \geqslant 1.$$

If we let $n \to +\infty$, we get $\omega_2(z, w) = 0$. Since ω is regular, we get $z = w$. □

Note that under the assumptions of Theorem 3.1, if w is another fixed point of T such that w and z are comparable and $\omega_1(z, w) < \infty$. Then we have

$$\omega_1(z, w) = \omega_1(T(z), T(w)) \leqslant k\ \omega_1(z, w),$$

which implies $z = w$, since $k < 1$.

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