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THE LOG–CONVEXITY OF GENOCCHI NUMBERS AND THE MONOTONICITY OF SOME SEQUENCES RELATED TO GENOCCHI NUMBERS

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Abstract. In this paper, we investigate the properties of Genocchi number $\{G_n\}_{n\geqslant 1}$. We prove that the sequence $\{|G_{2n}|\}_{n\geqslant 1}$ is log-convex. In addition, we discuss the monotonicity of some sequences related to $\{G_n\}_{n\geqslant 1}$. In particular, we show that $\{\sqrt[n]{|G_{2n}|}\}_{n\geqslant 1}$ is strictly increasing and $\{\sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ is strictly decreasing.

1. Introduction

For $n \geqslant 1$, let G_n denote the n^{th} term of the Genocchi numbers. The sequence $\{G_n\}_{n\geqslant 1}$ is a sequence of integers, which is defined by

$$\frac{2t}{e^t+1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi.$$

Genocchi numbers G_n satisfy $G_3 = G_5 = G_7 = \cdots = 0$. Some initial values of $\{G_n\}$ are as follows:

n	1	2	4	6	8	10	12	14	16	18	20
G_n	1	-1	1	-3	17	-155	2073	-38227	929569	-28820619	1109652905

For $n \geqslant 0$, let $\{B_n\}$ denote the Bernoulli numbers. It is well known that

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|,$$

where

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad (Re(x) > 1)$$

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is the Riemann zeta function. It is clear that

$$\zeta(2n) = 1 + \eta_n,$$

where

$$\frac{1}{2^{2n}} \leqslant \eta_n \leqslant \frac{3}{2^{2n}}.$$

For $n \geqslant 1$, let $\{A_{2n-1}\}$ denote tangent numbers. Tangent numbers $\{A_{2n-1}\}$ are defined by

$$\sum_{n=1}^{\infty} A_{2n-1} \frac{t^{2n-1}}{(2n-1)!} = \tan t, \quad |t| < \frac{\pi}{2}.$$

Genocchi numbers are related to Bernoulli numbers and tangent numbers, and there two formula correlating them, which are (see [6])

$$G_{2n} = 2(1-2^{2n})B_{2n}, \quad n \geqslant 1, \tag{1}$$

$$A_{2n-1} = 4^{n-1}|G_{2n}|/n, \quad n \geqslant 1. \tag{2}$$

Genocchi numbers have been studied in many subjects such as elementary number theory, complex analytic number theory, theory of modular forms, p-adic analytic number theory and quantum physics. They have drawn much attention. See for instance [1, 2, 3, 4, 8, 9, 10]. In this paper, we focus on the log-behavior of Genocchi numbers and the monotonicity of some sequences related to Genocchi numbers.

A sequence $\{z_n\}_{n\geqslant 0}$ of positive real numbers is said to be log-convex (log-concave) if $z_n^2 \leqslant z_{n-1}z_{n+1}$ ($z_n^2 \geqslant z_{n-1}z_{n+1}$) for all $n \geqslant 1$. Log-convexity and log-concavity are important properties of combinatorial sequences and they are also fertile sources of inequalities. It is clear that a sequence $\{z_n\}_{n\geqslant 0}$ is log-convex (log-concave) if and only if the sequence $\{z_{n+1}/z_n\}_{n\geqslant 0}$ is nondecreasing (nonincreasing). The log-behavior of $\{|B_{2n}|\}_{n\geqslant 1}$ has been studied in [5]. It seems that the log-behavior of $\{|G_{2n}|\}_{n\geqslant 1}$ has not been investigated. In this paper, we discuss the log-convexity of $\{|G_{2n}|\}_{n\geqslant 1}$. We will prove that $\{|G_{2n}|\}_{n\geqslant 1}$, $\{|G_{2n}|/n!\}_{n\geqslant 1}$ and $\{n|G_{2n}|\}_{n\geqslant 1}$ are log-convex. In addition, we also consider the monotonicity of some sequences involving $\{G_n\}_{n\geqslant 1}$. In [12], Sun posed a series of conjectures on monotonicity of sequences as the types $\{\sqrt[n]{z_n}\}$ and $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$, where $\{z_n\}_{n\geqslant 0}$ is a combinatorial sequence of positive integers. In [5, 7, 11, 13, 14], many conjectures of [12] are confirmed. In particular, Wang and Zhu [13] show that the monotonicity of $\{\sqrt[n]{z_n}\}$ is related to the log-convexity (log-concavity) of $\{z_n\}$. In this paper, we also show that $\{\sqrt[n]{|G_{2n}|}\}_{n\geqslant 1}$ is strictly increasing and $\{\sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ is strictly decreasing.

2. Main results for $\{G_n\}$

In this section, we state and prove the main results of this paper. The following result given by Chen, Guo and Wang in [5] will be useful:

LEMMA 2.1. *For* $n > 1$,

$$\frac{|B_{2n-2}||B_{2n+2}|}{|B_{2n}|^2} \geqslant \frac{(n+1)(2n+1)}{n(2n-1)}. \tag{3}$$

Now we discuss the log-convexity of some sequences related to $\{G_n\}_{n\geqslant 1}$.

THEOREM 2.1. *The sequences* $\{|G_{2n}|\}_{n\geqslant 1}$, $\{|G_{2n}|/n!\}_{n\geqslant 1}$ *and* $\{n|G_{2n}|\}_{n\geqslant 1}$ *are log-convex.*

Proof. For $n \geqslant 1$, let

$$s_{2n} = \frac{|G_{2n+2}|}{|G_{2n}|}, \quad t_{2n} = \frac{|G_{2n+2}|}{(n+1)|G_{2n}|}, \quad u_{2n} = \frac{(n+1)|G_{2n+2}|}{n|G_{2n}|}.$$

By applying (1) and (3), we have

$$\begin{aligned}
\frac{s_{2n+2}}{s_{2n}} &= \frac{(2^{2n+4}-1)(2^{2n}-1)|B_{2n+4}B_{2n}|}{(2^{2n+2}-1)^2|B_{2n+2}|^2} \\
&> \frac{(2^{4n+4}-2^{2n+4}-2^{2n}+1)(n+2)(2n+3)}{(2^{4n+4}-2^{2n+3}+1)(n+1)(2n+1)}, \\
\frac{t_{2n+2}}{t_{2n}} &> \frac{(2^{4n+4}-2^{2n+4}-2^{2n}+1)(2n+3)}{(2^{4n+4}-2^{2n+3}+1)(2n+1)}, \\
\frac{u_{2n+2}}{u_{2n}} &> \frac{(2^{4n+4}-2^{2n+4}-2^{2n}+1)(n+2)^2(2n+3)n}{(2^{4n+4}-2^{2n+3}+1)(n+1)^3(2n+1)}.
\end{aligned}$$

For $n \geqslant 1$, let

$$\begin{aligned}
f_1(n) &= (2^{4n+4}-2^{2n+4}-2^{2n}+1)(n+2)(2n+3)-(2^{4n+4}-2^{2n+3}+1)(n+1)(2n+1), \\
f_2(n) &= (2^{4n+4}-2^{2n+4}-2^{2n}+1)(2n+3)-(2^{4n+4}-2^{2n+3}+1)(2n+1), \\
f_3(n) &= (2^{4n+4}-2^{2n+4}-2^{2n}+1)(n+2)^2(2n+3)n \\
&\quad -(2^{4n+4}-2^{2n+3}+1)(n+1)^3(2n+1).
\end{aligned}$$

By straightforward calculation, we have

$$\begin{aligned}
f_1(n) &= (4n+5)(2^{4n+4}+1)-2^{2n}(18n^2+95n+94) \\
&> 2^{4n+4}(4n+5)-2^{2n}(18n^2+95n+94), \\
f_2(n) &= 2(2^{4n+4}+1)-2^{2n}(18n+43), \\
f_3(n) &= (2^{4n+4}+1)(4n^3+11n^2+7n-1)-2^{2n}(18n^4+131n^3+268n^2+164n-8).
\end{aligned}$$

For $n \geqslant 1$, we can prove by induction that

$$2^{2n+4} > 9(n+3). \tag{4}$$

By means of the inequality (4), we have

$$\begin{aligned} f_1(n) &> 2^{2n}[9(n+3)(4n+5) - 18n^2 - 95n - 94] \\ &> 0, \\ f_2(n) &> 0, \\ f_3(n) &> 0. \end{aligned}$$

Then

$$s_{2n+2}/s_{2n} > 1, \quad t_{2n+2}/t_{2n} > 1, \quad u_{2n+2}/u_{2n} > 1,$$

and the sequences $\{s_{2n}\}_{n\geqslant 1}$, $\{t_{2n}\}_{n\geqslant 1}$ and $\{u_{2n}\}_{n\geqslant 1}$ are strictly increasing. Hence the sequences $\{|G_{2n}|\}_{n\geqslant 1}$, $\{|G_{2n}|/n!\}_{n\geqslant 1}$ and $\{n|G_{2n}|\}_{n\geqslant 1}$ are log-convex. □

COROLLARY 2.1. *The sequence* $\{A_{2n-1}\}_{n\geqslant 1}$ *is log-convex.*

Proof. It is obvious that the sequence $\{4^{n-1}/n\}_{n\geqslant 1}$ is log-convex. It follows from (2) that the sequence $\{A_{2n-1}\}_{n\geqslant 1}$ is log-convex. □

Sun [12] presented the following conjecture related to Bernoulli numbers:

(C1) The sequence $\{\sqrt[n]{|B_{2n}|}\}_{n\geqslant 1}$ is strictly increasing.

(C2) The sequence $\{\sqrt[n+1]{|B_{2n+2}|}/\sqrt[n]{|B_{2n}|}\}_{n\geqslant 2}$ is strictly decreasing.

The answers to (C1) and (C2) are both positive. Recently, the monotonicities of $\{\sqrt[n]{|B_{2n}|}\}_{n\geqslant 1}$ and $\{\sqrt[n+1]{|B_{2n+2}|}/\sqrt[n]{|B_{2n}|}\}_{n\geqslant 2}$ have been verified. See [5, 11]. In the rest of this section, we investigate the monotonicity of some sequences involving $\{G_n\}_{n\geqslant 1}$.

THEOREM 2.2. *The sequences* $\{\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$, $\{\sqrt[n]{|G_{2n}|/n!}\}_{n\geqslant 2}$ *and* $\{\sqrt[n]{n|G_{2n}|}\}_{n\geqslant 2}$ *are strictly increasing, and*

$$\sqrt[n]{|G_{2n}|} \sim \frac{4n^2}{(e\pi)^2}, \quad (n \to +\infty). \tag{5}$$

Proof. For $n \geqslant 2$, let $s_{2n} = |G_{2n+2}|/|G_{2n}|$. Since

$$\begin{aligned} \sqrt[n]{|G_{2n}|} < \sqrt[n+1]{|G_{2n+2}|} \quad &\Longleftrightarrow \quad (n+1)\ln|G_{2n}| - n\ln|G_{2n+2}| < 0 \\ &\Longleftrightarrow \quad \ln|G_{2n}| - n\ln s_{2n} < 0, \end{aligned}$$

we show that $\ln|G_{2n}| - n\ln s_{2n} < 0$ for $n \geqslant 2$. In fact,

$$\ln|G_{2n}| - n\ln s_{2n} = \ln s_{2n-2} + \ln s_{2n-4} + \cdots + \ln s_2 - n\ln s_{2n}.$$

Since $\{s_{2n}\}_{n\geqslant 2}$ is strictly increasing and $s_4 = 3$, $\ln|G_{2n}| - n\ln s_{2n} < 0$.

Hence $\{\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ is strictly increasing.

Using the similar method, we can prove that the sequences $\{\sqrt[n]{|G_{2n}|/n!}\}_{n\geqslant 2}$ and $\{\sqrt[n]{n|G_{2n}|}\}_{n\geqslant 2}$ are strictly increasing.

From (1),

$$|B_{2n}| \sim (2n)!/(2\pi)^{2n}, \quad (n \to +\infty), \tag{6}$$

and

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\theta_n}, \quad (n \to +\infty), \quad \text{where} \quad \frac{1}{12n+1} < \theta_n < \frac{1}{12n} \quad \text{for all} \quad n \geqslant 1,$$

we obtain

$$|G_{2n}| \sim 2^{2n+2}\left(\frac{n}{e\pi}\right)^{2n}\sqrt{\pi n}, \quad (n \to +\infty). \tag{7}$$

Therefore we have (5). □

COROLLARY 2.2. *The sequence* $\{\sqrt[n]{A_{2n-1}}\}_{n\geqslant 1}$ *is strictly increasing and*

$$\sqrt[n]{A_{2n-1}} \sim \frac{16n^2}{(e\pi)^2}, \quad (n \to +\infty). \tag{8}$$

Proof. Since the sequences $\{\sqrt[n]{4^{n-1}/n}\}_{n\geqslant 2}$ and $\{\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ are both strictly increasing, $\{\sqrt[n]{A_{2n-1}}\}_{n\geqslant 2}$ is strictly increasing. Noting that $A_1 < \sqrt{A_3}$, the sequence $\{\sqrt[n]{A_{2n-1}}\}_{n\geqslant 1}$ is strictly increasing.

By using (2) and (5), we have (8). □

In fact, the monotonic increasing property of $\{\sqrt[n]{A_{2n-1}}\}_{n\geqslant 1}$ has been proved in [10].

THEOREM 2.3. *The sequence* $\{\sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ *is strictly decreasing.*

Proof. For $n \geqslant 1$,

$$\begin{aligned} &\sqrt[n+2]{|G_{2n+4}|}/\sqrt[n+1]{|G_{2n+2}|} < \sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|} \\ \Longleftrightarrow\ &\frac{\ln|G_{2n+4}|}{n+2} + \frac{\ln|G_{2n}|}{n} - \frac{2\ln|G_{2n+2}|}{n+1} < 0 \end{aligned}$$

By using (1), we have

$$\begin{aligned} &\frac{\ln|G_{2n+4}|}{n+2} + \frac{\ln|G_{2n}|}{n} - \frac{2\ln|G_{2n+2}|}{n+1} \\ = &\frac{2\ln 2}{n(n+1)(n+2)} + \frac{1}{n+2}\ln\left(1 - \frac{1}{2^{2n+4}}\right) + \frac{1}{n}\ln\left(1 - \frac{1}{2^{2n}}\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{2}{n+1}\ln\left(1-\frac{1}{2^{2n+2}}\right)+\frac{1}{n+2}\ln|B_{2n+4}|+\frac{1}{n}\ln|B_{2n}|-\frac{2}{n+1}\ln|B_{2n+2}|\\
=&\frac{2\ln 2}{n(n+1)(n+2)}+\frac{1}{n+2}\ln\left(1-\frac{1}{2^{2n+4}}\right)+\frac{1}{n}\ln\left(1-\frac{1}{2^{2n}}\right)\\
&+\frac{2}{n+1}\ln\left(1+\frac{1}{2^{2n+2}-1}\right)+\frac{1}{n+2}\ln|B_{2n+4}|+\frac{1}{n}\ln|B_{2n}|-\frac{2}{n+1}\ln|B_{2n+2}|.
\end{aligned}
$$

By means of the following inequality

$$\ln(1+x)\leqslant x,\quad x>-1,$$

we get

$$
\begin{aligned}
&\frac{\ln|G_{2n+4}|}{n+2}+\frac{\ln|G_{2n}|}{n}-\frac{2\ln|G_{2n+2}|}{n+1}\\
\leqslant&\frac{2\ln 2}{n(n+1)(n+2)}-\frac{1}{2^{2n+4}(n+2)}-\frac{1}{2^{2n}n}+\frac{2}{(2^{2n+2}-1)(n+1)}\\
&+\frac{1}{n+2}\ln|B_{2n+4}|+\frac{1}{n}\ln|B_{2n}|-\frac{2}{n+1}\ln|B_{2n+2}|.
\end{aligned}
$$

Noting that (see [10])

$$
\begin{aligned}
\frac{1}{n+2}\ln|B_{2n+4}|+\frac{1}{n}\ln|B_{2n}|-\frac{2}{n+1}\ln|B_{2n+2}|<&-\frac{2}{(n+1)^2}+\frac{\ln n+2+\ln(16\pi)}{n(n+1)(n+2)}\\
&+\frac{1}{6n^2}+\frac{12}{2^{2n}n},\quad n\geqslant 3,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{\ln|G_{2n+4}|}{n+2}+\frac{\ln|G_{2n}|}{n}-\frac{2\ln|G_{2n+2}|}{n+1}\\
\leqslant&-\frac{1}{2^{2n+4}(n+2)}-\frac{1}{2^{2n}n}+\frac{2}{(2^{2n+2}-1)(n+1)}-\frac{2}{(n+1)^2}+\frac{\ln n+2+\ln(64\pi)}{n(n+1)(n+2)}\\
&+\frac{1}{6n^2}+\frac{12}{2^{2n}n}\\
<&-\frac{1}{2^{2n+4}(n+2)}-\frac{2}{(n+1)^2}\left[1-\frac{\ln n+2+\ln(64\pi)}{2n(n+1)(n+2)}-\frac{(n+1)^2}{12n^2}-\frac{6(n+1)^2}{2^{2n}n}\right].
\end{aligned}
$$

For $n\geqslant 4$, we can verify that

$$1-\frac{\ln n+2+\ln(64\pi)}{2n(n+1)(n+2)}-\frac{(n+1)^2}{12n^2}-\frac{6(n+1)^2}{2^{2n}n}>0.$$

Then

$$\frac{\ln|G_{2n+4}|}{n+2}+\frac{\ln|G_{2n}|}{n}-\frac{2\ln|G_{2n+2}|}{n+1}<0,\quad n\geqslant 4,$$

and the sequence $\{\sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}\}_{n\geqslant 4}$ is strictly decreasing. On the other hand,

$$\sqrt[n+2]{|G_{2n+4}|}/\sqrt[n+1]{|G_{2n+2}|} < \sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}, \quad (n=2,3).$$

Hence the sequence $\{\sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ is strictly decreasing. □

Since the sequence $\{\sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ is strictly decreasing, $\{\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ is log-concave.

COROLLARY 2.3. *The sequence* $\{\sqrt[n+1]{(n+1)|G_{2n+2}|}/\sqrt[n]{n|G_{2n}|}\}_{n\geqslant 1}$ *is strictly decreasing.*

Proof. We first prove that the sequence $\{\sqrt[n+1]{n+1}/\sqrt[n]{n}\}_{n\geqslant 2}$ is strictly decreasing. It is clear that

$$\sqrt[n+1]{n+1}/\sqrt[n]{n} > \sqrt[n+2]{n+2}/\sqrt[n+1]{n+1}$$
$$\iff 2n(n+2)\ln(n+1)-(n+1)(n+2)\ln n - n(n+1)\ln(n+2) > 0.$$

Now we show that

$$2n(n+2)\ln(n+1)-(n+1)(n+2)\ln n - n(n+1)\ln(n+2) > 0, \quad n\geqslant 2.$$

By computation, we have

$$\begin{aligned}
&2n(n+2)\ln(n+1)-(n+1)(n+2)\ln n - n(n+1)\ln(n+2)\\
&=(n^2+n)\ln\left(1+\frac{1}{n}\right)-2\ln n+2n\ln\left(1+\frac{1}{n}\right)-(n^2+n)\ln\left(1+\frac{1}{n+1}\right)\\
&>(n^2+n)\ln\left(1+\frac{1}{n+1}\right)-2\ln n+2n\ln\left(1+\frac{1}{n}\right).
\end{aligned}$$

By using the following inequality

$$\frac{x}{1+x} < \ln(1+x), \quad x>0,$$

we obtain

$$\begin{aligned}
&2n(n+2)\ln(n+1)-(n+1)(n+2)\ln n - n(n+1)\ln(n+2)\\
&>\frac{n^2+n}{n+2}-2\ln n+2n\ln\left(1+\frac{1}{n}\right)\\
&=\frac{n^2+n-2(n+2)\ln n}{n+2}+2n\ln\left(1+\frac{1}{n}\right)\\
&>0, \quad n\geqslant 2.
\end{aligned}$$

Then the sequence $\{\sqrt[n+1]{n+1}/\sqrt[n]{n}\}_{n\geqslant 2}$ is strictly decreasing. It follows from Theorem 2.3 that the sequence $\{\sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}\}_{n\geqslant 2}$ is strictly decreasing. Hence the sequence $\{\sqrt[n+1]{(n+1)|G_{2n+2}|}/\sqrt[n]{n|G_{2n}|}\}_{n\geqslant 2}$ is also strictly decreasing. On the other hand, we can verify that $\sqrt{2|G_4|}/|G_2| > \sqrt[3]{3|G_6|}/\sqrt{2|G_4|}$.

Therefore, $\{\sqrt[n+1]{(n+1)|G_{2n+2}|}/\sqrt[n]{n|G_{2n}|}\}_{n\geqslant 1}$ is strictly decreasing. □

THEOREM 2.4. *There exist a positive integer M such that the sequence*

$$\{\sqrt[n]{|G_{2n+2}|/|G_{2n}|}\}$$

is strictly decreasing when $n \geqslant M$, and

$$\lim_{n\to+\infty} \sqrt[n]{|G_{2n+2}|/|G_{2n}|} = 1.$$

Proof. Since $\sqrt[n]{|G_{2n+2}|/|G_{2n}|} = \sqrt[n]{(2^{2n+2}-1)|B_{2n+2}|/[(2^{2n}-1)|B_{2n}|]}$, we first discuss the monotonicity of the sequences

$$\{\sqrt[n]{(2^{2n+2}-1)/(2^{2n}-1)}\} \quad \text{and} \quad \{\sqrt[n]{|B_{2n+2}|/|B_{2n}|}\}.$$

For $n \geqslant 1$, since

$$\begin{aligned}(n+1)\ln\frac{2^{2n+2}-1}{2^{2n}-2} - n\ln\frac{2^{2n+4}-1}{2^{2n+2}-1} &= \ln\frac{2^{2n+2}-1}{2^{2n}-1} + n\ln\frac{(2^{2n+2}-1)^2}{(2^{2n}-1)(2^{2n+4}-1)} \\ &> n\ln\frac{(2^{2n+2}-1)^2}{(2^{2n}-1)(2^{2n+4}-1)} \\ &> 0,\end{aligned}$$

the sequence $\{\sqrt[n]{(2^{2n+2}-1)/(2^{2n}-1)}\}$ is strictly decreasing.

For $n \geqslant 1$,

$$\begin{aligned}&(n+1)\ln\frac{|B_{2n+2}|}{|B_{2n}|} - n\ln\frac{|B_{2n+4}|}{|B_{2n+2}|} \\ &= \ln\frac{|B_{2n+2}|^{2n+1}}{|B_{2n}|^{n+1}|B_{2n+4}|^n} \\ &= \ln|B_{2n+2}| - \ln|B_{2n}| + n\Big(2\ln|B_{2n+2}| - \ln|B_{2n}| - \ln|B_{2n+4}|\Big).\end{aligned}$$

There is an identity for $B_{2n}|$ (see [10])

$$\ln|B_{2n}| = 2n\ln n + cn + \frac{\ln n}{2} + \frac{\ln(16\pi)}{2} + \theta_{2n} + \ln(1+\eta_n), \tag{9}$$

where $c = -2 - 2\ln\pi$. By using (9) and straightford calculation, we derive

$$\begin{aligned}\ln|B_{2n+2}| - \ln|B_{2n}| = &\left(2n - \frac{1}{2}\right)\ln\left(1 + \frac{1}{n}\right) + 2\ln(1+n) + c + \theta_{2n+2} \\ &-\theta_{2n} + \ln(1+\eta_{n+1}) - \ln(1+\eta_n),\end{aligned}$$

$$\begin{aligned}&n(2\ln|B_{2n+2}| - \ln|B_{2n}| - \ln|B_{2n+4}|) \\ &= n\left[(4n+5)\ln(n+1) - \left(2n + \frac{1}{2}\right)\ln n - \left(2n + \frac{9}{2}\right)\ln(n+2)\right.\end{aligned}$$

$$+(2\theta_{2n+2}-\theta_{2n}-\theta_{2n+4})+2\ln(1+\eta_{n+1})-\ln(1+\eta_n)-\ln(1+\eta_{n+2})\Big]$$

$$=n\Big[\left(2n+\frac{1}{2}\right)\ln(n+1)-\left(2n+\frac{1}{2}\right)\ln n-\left(2n+\frac{9}{2}\right)\ln\left(1+\frac{1}{n+1}\right)$$
$$+(2\theta_{2n+2}-\theta_{2n}-\theta_{2n+4})+2\ln(1+\eta_{n+1})-\ln(1+\eta_n)-\ln(1+\eta_{n+2})\Big]$$
$$=2n^2\ln\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}+\frac{n}{2}\ln\left(1+\frac{1}{n}\right)-\frac{9n}{2}\ln\left(1+\frac{1}{n+1}\right)+n(2\theta_{2n+2}-\theta_{2n}$$
$$-\theta_{2n+4})+2n\ln(1+\eta_{n+1})-n\ln(1+\eta_n)-n\ln(1+\eta_{n+2}).$$

Noting that

$$\frac{1}{2^{2n}}<\eta_n<\frac{3}{2^{2n}},\qquad \frac{1}{12n+1}<\theta_n<\frac{1}{12n},$$

we get

$$\lim_{n\to\infty}n(2\ln|B_{2n+2}|-\ln|B_{2n}|-\ln|B_{2n+4}|)$$
$$=\lim_{n\to+\infty}\left[2n^2\ln\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}+\frac{n}{2}\ln\left(1+\frac{1}{n}\right)-\frac{9n}{2}\ln\left(1+\frac{1}{n+1}\right)\right]$$
$$=-2$$

and $\lim_{n\to\infty}(\ln|B_{2n+2}|-\ln|B_{2n}|)=+\infty$. Then we obtain

$$\lim_{n\to\infty}\left[(n+1)\ln\frac{|B_{2n+2}|}{|B_{2n}|}-n\ln\frac{|B_{2n+4}|}{|B_{2n+2}|}\right]=+\infty,$$

and there exists a positive integer M such that

$$(n+1)\ln\frac{|B_{2n+2}|}{|B_{2n}|}-n\ln\frac{|B_{2n+4}|}{|B_{2n+2}|}>0,\quad (n\geqslant M).$$

Then the sequence $\{\sqrt[n]{|B_{2n+2}|/|B_{2n}|}\}_{n\geqslant M}$ is strictly decreasing. Hence the sequence $\{\sqrt[n]{|G_{2n+2}|/|G_{2n}|}\}_{n\geqslant M}$ is strictly decreasing.

By means of (7), we derive

$$\lim_{n\to+\infty}\sqrt[n]{|G_{2n+2}|/|G_{2n}|}=1.\quad\square$$

3. Conclusions

In this paper, we discuss the properties of Genocchi numbers $\{G_n\}_{n\geqslant 1}$. We have derived some results for $\{G_n\}_{n\geqslant 1}$ for the log-convexity of some sequences including $\{|G_{2n}|\}_{n\geqslant 1}$, $\{|G_{2n}|/n!\}_{n\geqslant 1}$ and $\{n|G_{2n}|\}_{n\geqslant 1}$. We have also investigated the monotonicity of some sequences related to Genocchi numbers. In particular, we discuss the monotonicity of $\{\sqrt[n]{|G_{2n}|}\}$ and $\{\sqrt[n+1]{|G_{2n+2}|}/\sqrt[n]{|G_{2n}|}\}$. In the future, we will discuss the properties of some sequences such as tangent numbers $T(n,k)$, arctangent numbers $t(n,k)$ and Salié integers S_{2n}, which are defined by

$$\frac{(\tan t)^k}{k!}=\sum_{n=k}^{\infty}T(n,k)\frac{t^n}{n!},\quad \frac{(\arctan t)^k}{k!}=\sum_{n=k}^{\infty}t(n,k)\frac{t^n}{n!},\quad \frac{\cosh t}{\cos t}=\sum_{n=0}^{\infty}S_{2n}\frac{t^{2n}}{(2n)!}.$$

For more details of $T(n,k)$, $t(n,k)$ and S_{2n}, see [6]. The investigation for the log-behavior of the above sequences will be the future work.

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