Journal of Mathematical Inequalities
Volume 10, Number 2 (2016), 551–558 doi:10.7153/jmi-10-43

ON THE JAMES TYPE CONSTANT AND VON NEUMANN–JORDAN CONSTANT FOR A CLASS OF BANAŚ–FRĄCZIECK TYPE SPACES

CHANGSEN YANG AND XIANGZHAO YANG

(*Communicated by J. Pečarić*)

Abstract. As a generalization of Banaś-Frączieck space, the space $X_{\lambda,p}$ that denotes $\mathbb{R}^2$ endowed with the norm
$$\|x\|_{\lambda,p} = \max\{\lambda|x_1|, \|x\|_p\}$$
for $\lambda > 1$, $p \geqslant 1$ and $x = (x_1,x_2) \in \mathbb{R}^2$ is well defined. In this note, the exact value of the the James type constants $J_{X_{\lambda,p},t}(1)$ and von Neumann-Jordan constant $C_{NJ}(X_{\lambda,p})$ about this space for $p \geqslant 2$ are investigated.

1. Introduction and preliminaries

Let X be a non-trivial Banach space, and S_X denote the unit ball sphere of X. It can be recalled that the modulus of convexity of a Banach space X is defined for $\varepsilon \in [0,2]$ in [1] as

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2}, x,y \in S_X, \|x-y\| = \varepsilon\right\}.$$

The function $\delta_X(\varepsilon)$ is continuous on $[0,2)$ and strictly increasing on $[\varepsilon_0(X),2]$, where $\varepsilon_0(X) = \sup\{\varepsilon \in [0,2], \delta_X(\varepsilon) = 0\}$ is the characteristic of convexity of X.

Many geometric properties for a Banach space X is closely related with its modulus of convexity. For example, an important relationship between the James constant which was defined by (see [2])

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|), x,y \in S_X\}$$

and the modulus of convexity is the following formula (see [3])

$$J(X) = \sup\left\{\varepsilon \in (0,2) : \delta_X(\varepsilon) \leqslant 1 - \frac{\varepsilon}{2}\right\}.$$

Mathematics subject classification (2010): Primary 46B20; Secondary 46B99.
Keywords and phrases: Jordan-von Neumann constant, modulus of convexity, Banaś-Frączieck type space, James type constant.

Paper JMI-10-43

Recently, Takahashi [4] introduced the James type constant

$$J_{X,t}(\tau) = \sup\{\mu_t(\|x+\tau y\|, \|x-\tau y\|) : x, y \in S_X\},$$

where $\tau \geqslant 0$, $-\infty \leqslant t < +\infty$. Here, we denote $\mu_t(a,b) = (\frac{a^t+b^t}{2})^{\frac{1}{t}}$ $(t \neq 0)$ and $\mu_0(a,b) = \lim_{t\to 0} \mu_t(a,b) = \sqrt{ab}$ for two positive numbers a and b, respectively. Here $\mu_t(a,b)$ is nondecreasing in t and $\mu_{-\infty}(a,b) = \lim_{t\to-\infty} \mu_t(a,b) = \min(a,b)$. It is well known that, $J(X) = J_{X,-\infty}(1)$ and Alonso-Llorens-Fuster's constant $T(X)$ (see [5]) is equal to $J_{X,0}(1)$.

Let $\lambda > 1$, and $X_{\lambda,p}$ denote $\mathbb{R}^2$ endowed with the norm

$$\|x\|_{\lambda,p} = \max\{\lambda|x_1|, \|x\|_p\}$$

for $\lambda > 1$, $p \geqslant 1$ and $x = (x_1, x_2) \in \mathbb{R}^2$. We may say these Banach spaces to be Banaś-Frączieck type spaces because it can be reduced to a Banaś-Frączieck space (see [6, 7]) by letting $p = 2$.

Another important constant of a Banach space X is the von Neumann-Jordan constant (hereafter referred to as the NJ constant) that was introduced by Clarkson [2] as the smallest constant C for which

$$\frac{1}{C} \leqslant \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leqslant C$$

holds for all $x, y \in X$ with $(x,y) \neq (0,0)$. An equivalent definition of the NJ constant is found in [6, 10] as the following form:

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X \right\}.$$

In this note, the exact value of the James type constants $J_{X_{\lambda,p},t}(1)$ for $p \geqslant 2$ and the von Neumann-Jordan constant for $p \geqslant 2$ and $\lambda > 1$ such that $(\lambda^p - 1)^{p-2}(\lambda^2 - 1)^p \geqslant 1$ about this space are investigated.

2. James type constants $J_{X_{\lambda,p},t}(1)$

Firstly, we give the exact value of the modulus of convexity for the space $X_{\lambda,p}$ with $p \geqslant 2$.

LEMMA 2.1. *If $\lambda > 1$ and $p \geqslant 2$, then*

$$\delta_{X_{\lambda,p}}(\varepsilon) = \begin{cases} 0, & 0 \leqslant \varepsilon \leqslant 2[1 - \frac{1}{\lambda^p}]^{\frac{1}{p}}; \\ 1 - \lambda[1 - \frac{\varepsilon^p}{2^p}]^{\frac{1}{p}}, & 2[1 - \frac{1}{\lambda^p}]^{\frac{1}{p}} \leqslant \varepsilon \leqslant \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}}; \\ 1 - [1 - \frac{\varepsilon^p}{2^p\lambda^p}]^{\frac{1}{p}}, & \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}} \leqslant \varepsilon \leqslant 2. \end{cases}$$

Proof. (i) Let $0 \leqslant \varepsilon \leqslant 2[1-\frac{1}{\lambda^p}]^{\frac{1}{p}}$.

Taking $x=\left(\frac{1}{\lambda},[1-\frac{1}{\lambda^p}]^{\frac{1}{p}}\right), y=\left(\frac{1}{\lambda},-[1-\frac{1}{\lambda^p}]^{\frac{1}{p}}\right)$, we have $\|x\|_{\lambda,p}=\|y\|_{\lambda,p}=1$, $\|x-y\|_{\lambda,p} \geqslant \varepsilon$, and $\|\frac{x+y}{2}\|_{\lambda,p}=1$, Hence $\delta_{X_{\lambda,p}}(\varepsilon)=0$.

(ii) Let $2[1-\frac{1}{\lambda^p}]^{\frac{1}{p}} \leqslant \varepsilon \leqslant \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}}$.

Taking $x=\left([1-\frac{\varepsilon^p}{2^p}]^{\frac{1}{p}},\frac{\varepsilon}{2}\right), y=\left([1-\frac{\varepsilon^p}{2^p}]^{\frac{1}{p}},-\frac{\varepsilon}{2}\right)$, we have $\|x\|_{\lambda,p}=\|y\|_{\lambda,p}=1$, $\|x-y\|_{\lambda,p}=\varepsilon$, and $\|\frac{x+y}{2}\|_{\lambda,p}=\lambda[1-\frac{\varepsilon^p}{2^p}]^{\frac{1}{p}}$. So $\delta_{X_{\lambda,p}}(\varepsilon) \leqslant 1-\lambda[1-\frac{\varepsilon^p}{2^p}]^{\frac{1}{p}}$. On the other hand, for any $x,y \in S_{X_{\lambda,p}}$ such that $\|x-y\|_{\lambda,p}=\varepsilon$, we have

(a) If $\|x-y\|_p=\varepsilon$, then

$$\left\|\frac{x+y}{2}\right\|_{\lambda,p} \leqslant \lambda\left\|\frac{x+y}{2}\right\|_p \leqslant \lambda\left[1-\frac{\varepsilon^p}{2^p}\right]^{\frac{1}{p}}$$

holds by the following Clarkson's inequality

$$\|x+y\|_p^p+\|x-y\|_p^p \leqslant 2^{p-1}(\|x\|_p^p+\|y\|_p^p) \leqslant 2^p.$$

(b) If $\lambda|x_1-y_1|=\varepsilon$, then

$$|x_1+y_1|^p+\varepsilon^p=|x_1+y_1|^p+\lambda^p|x_1-y_1|^p \leqslant 2^{p-1}\lambda^p(|x_1|^p+|y_1|^p) \leqslant 2^p,$$

and

$$\|x+y\|_p^p+\lambda^p\varepsilon^p \leqslant 2^p-\|x-y\|_p^p+\lambda^p\varepsilon^p \leqslant 2^p-\frac{\varepsilon^p}{\lambda^p}+\lambda^p\varepsilon^p \leqslant 2^p\lambda^p.$$

Hence

$$\left\|\frac{x+y}{2}\right\|_{\lambda,p}=\max\left\{\lambda\left|\frac{x_1+y_1}{2}\right|,\left\|\frac{x+y}{2}\right\|_p\right\} \leqslant \lambda\left[1-\frac{\varepsilon^p}{2^p}\right]^{\frac{1}{p}}.$$

Therefore

$$\delta_X(\varepsilon) \geqslant 1-\lambda\left[1-\frac{\varepsilon^p}{2^p}\right]^{\frac{1}{p}}.$$

(iii) Let $\frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}} \leqslant \varepsilon \leqslant 2$.

For any $x,y \in S_{X_{\lambda,p}}$ with $\|x-y\|_{\lambda,p}=\varepsilon$, we have

$$\|x+y\|_p^2+\frac{\varepsilon^p}{\lambda^p} \leqslant \|x+y\|_p^p+\|x-y\|_p^p \leqslant 2^p,$$

and

(a) If $\|x-y\|_p=\varepsilon$,

$$\lambda^p|x_1+y_1|^p+\frac{\varepsilon^p}{\lambda^p} \leqslant \lambda^p\|x+y\|_p^p+\frac{\varepsilon^p}{\lambda^p} \leqslant \lambda^p(2^p-\varepsilon^p)+\frac{\varepsilon^p}{\lambda^p} \leqslant 2^p;$$

(b) If $\lambda|x_1-y_1|=\varepsilon$,

$$\lambda^p|x_1+y_1|^p+\frac{\varepsilon^p}{\lambda^p}=\lambda^p|x_1+y_1|^p+|x_1-y_1|^p \leqslant (\lambda^p-1)\frac{2^p}{\lambda^p}+2^{p-1}(|x_1|^p+|y_1|^p) \leqslant 2^p.$$

So, we have

$$\delta_X(\varepsilon) \geqslant 1 - \left[1 - \frac{\varepsilon^p}{2^p\lambda^p}\right]^{\frac{1}{p}}.$$

On the other hand, By taking $x = \left(\frac{\varepsilon}{2\lambda}, [1 - \frac{\varepsilon^p}{2^p\lambda^p}]^{\frac{1}{p}}\right)$ and $y = \left(-\frac{\varepsilon}{2\lambda}, [1 - \frac{\varepsilon^p}{2^p\lambda^p}]^{\frac{1}{p}}\right)$, we also have $\|x\|_{\lambda,p} = \|y\|_{\lambda,p} = 1$, $\|x-y\|_{\lambda,p} = \varepsilon$ and $\|\frac{x+y}{2}\|_{\lambda,p} = [1 - \frac{\varepsilon^p}{2^p\lambda^p}]^{\frac{1}{p}}$, Therefore

$$\delta_X(\varepsilon) \leqslant 1 - \left[1 - \frac{\varepsilon^p}{2^p\lambda^p}\right]^{\frac{1}{p}}. \quad \square$$

To give the exact value of the James type constant $J_{X_{\lambda,p},t}(1)$, we need the following Lemma.

LEMMA 2.2. *For any Banach space X, we have (see [8])*

$$J_{X,t}(1) = \sup\left\{\left(\frac{\varepsilon^t + 2^t(1-\delta_X(\varepsilon))^t}{2}\right)^{\frac{1}{t}}, 0 \leqslant \varepsilon \leqslant 2\right\},$$

where $-\infty < t < +\infty$, $t \neq 0$.

THEOREM 2.3. *Let $p \geqslant 2$, $\lambda > 1$ and $X_{\lambda,p}$ be the Banaś-Frączieck type spaces.*
(a) If $t \geqslant p$, then

$$J_{X_{\lambda,p},t}(1) = 2^{1-\frac{1}{t}}\left(1 + \left(1 - \frac{1}{\lambda^p}\right)^{\frac{t}{p}}\right)^{\frac{1}{t}}; \tag{2.1}$$

(b) If $t < p$ and $\lambda^p \leqslant 1 + \lambda^{\frac{tp}{t-p}}$, then

$$J_{X_{\lambda,p},t}(1) = 2^{1-\frac{1}{t}}\lambda\left(1 + \lambda^{\frac{tp}{t-p}}\right)^{\frac{1}{t}-\frac{1}{p}}; \tag{2.2}$$

(c) If $t < p$ and $\lambda^p \geqslant 1 + \lambda^{\frac{tp}{t-p}}$, then (2.1) is also valid.

Proof. Let $f_1(\varepsilon) = \frac{\varepsilon^t + 2^t}{2}$, $f_2(\varepsilon) = \frac{\varepsilon^t + \lambda^t(2^p - \varepsilon^p)^{\frac{t}{p}}}{2}$ and $f_3(\varepsilon) = \frac{\varepsilon^t + (2^p - \frac{\varepsilon^p}{\lambda^p})^{\frac{t}{p}}}{2}$. Then by applying Lemma 2.1 and Lemma 2.2, we have

$$J_{X_{\lambda,p},t}(1) = \max\{\alpha, \beta, \gamma\},$$

where $\alpha = \max\left\{f_1(t)^{\frac{1}{t}} : 0 \leqslant \varepsilon \leqslant 2[1 - \frac{1}{\lambda^p}]^{\frac{1}{p}}\right\}$, $\beta = \max\left\{f_2(t)^{\frac{1}{t}} : 2[1 - \frac{1}{\lambda^p}]^{\frac{1}{p}} \leqslant \varepsilon \leqslant \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}}\right\}$, and $\gamma = \max\left\{f_3(t)^{\frac{1}{t}} : \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}} \leqslant \varepsilon \leqslant 2\right\}$. Since (2.1) is obvious for $t = p$, so we only need consider the following cases.

Case I. If $t > p$. Then we can get $\frac{2\lambda^{\frac{t}{t-p}}}{(1+\lambda^{\frac{tp}{t-p}})^{\frac{1}{p}}} \geqslant \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}}$. From $f_2'(\varepsilon) = \frac{1}{2}t\varepsilon^{p-1}[\varepsilon^{t-p} - \lambda^t(2^p - \varepsilon^p)^{\frac{t-p}{p}}] > 0$ if and only if $\varepsilon \geqslant \frac{2\lambda^{\frac{t}{t-p}}}{(1+\lambda^{\frac{tp}{t-p}})^{\frac{1}{p}}}$, we have $f_2'(\varepsilon) < 0$ on $2[1 -$

$\frac{1}{\lambda^p}]^{\frac{1}{p}} \leqslant \varepsilon \leqslant \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}}$. Similarly, we also have $f_3'(\varepsilon) \geqslant 0$ on $\frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}} \leqslant \varepsilon \leqslant 2$ by applying $f_3'(\varepsilon) = \frac{1}{2}t\varepsilon^{p-1}[\varepsilon^{t-p} - \lambda^{-p}(2^p - \frac{\varepsilon^p}{\lambda^p})^{\frac{t-p}{p}}]$ and $\frac{2\lambda^{\frac{-p}{t-p}}}{(1+\lambda^{\frac{-tp}{t-p}})^{\frac{1}{p}}} \leqslant \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}}$. Hence, we can get $\alpha = \beta = \gamma = 2^{1-\frac{1}{t}}\left(1+(1-\frac{1}{\lambda^p})^{\frac{t}{p}}\right)^{\frac{1}{t}}$. Therefore, (2.1) is valid.

Case II. If $0<t<p$ and $\lambda^p \leqslant 1+\lambda^{\frac{tp}{t-p}}$, then $\frac{2\lambda^{\frac{t}{t-p}}}{(1+\lambda^{\frac{tp}{t-p}})^{\frac{1}{p}}} \in \left[2[1-\frac{1}{\lambda^p}]^{\frac{1}{p}}, \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}}\right]$ and f_2 gets its maximum at $\frac{2\lambda^{\frac{t}{t-p}}}{(1+\lambda^{\frac{tp}{t-p}})^{\frac{1}{p}}}$. So $\beta = 2^{1-\frac{1}{t}}\lambda\left(1+\lambda^{\frac{tp}{t-p}}\right)^{\frac{1}{t}-\frac{1}{p}} \geqslant f_2(2[1-\frac{1}{\lambda^p}]^{\frac{1}{p}})^{\frac{1}{t}} = \alpha$. Similarly, we also have $\gamma = 2^{1-\frac{1}{t}}\lambda\left(1+\lambda^{\frac{tp}{t-p}}\right)^{\frac{1}{t}-\frac{1}{p}}$. Therefore, (2.2) is valid.

Case III. If $0<t<p$ and $\lambda^p > 1+\lambda^{\frac{tp}{t-p}}$, then f_2 and f_3 get its maximum at $2(1-\frac{1}{\lambda^p})^{\frac{1}{p}}$ and 2, respectively. Hence, we have $\alpha = \beta = \gamma = 2^{1-\frac{1}{t}}\left(1+(1-\frac{1}{\lambda^p})^{\frac{t}{p}}\right)^{\frac{1}{t}}$ by $t>0$. So (2.1) is also valid.

Case IV. If $t<0$ and $\lambda^p \leqslant 1+\lambda^{\frac{tp}{t-p}}$, then $\frac{2\lambda^{\frac{t}{t-p}}}{(1+\lambda^{\frac{tp}{t-p}})^{\frac{1}{p}}} \in \left[2[1-\frac{1}{\lambda^p}]^{\frac{1}{p}}, \frac{2\lambda}{[1+\lambda^p]^{\frac{1}{p}}}\right]$ and f_2 gets its minimum at $\frac{2\lambda^{\frac{t}{t-p}}}{(1+\lambda^{\frac{tp}{t-p}})^{\frac{1}{p}}}$. So $\beta = 2^{1-\frac{1}{t}}\lambda\left(1+\lambda^{\frac{tp}{t-p}}\right)^{\frac{1}{t}-\frac{1}{p}} \geqslant f_2(2[1-\frac{1}{\lambda^p}]^{\frac{1}{p}})^{\frac{1}{t}} = \alpha$. Similarly, we also have $\gamma = 2^{1-\frac{1}{t}}\lambda\left(1+\lambda^{\frac{tp}{t-p}}\right)^{\frac{1}{t}-\frac{1}{p}}$.

Case V. If $t<0$ and $\lambda^p > 1+\lambda^{\frac{tp}{t-p}}$, then f_2 and f_3 get its minimum at $2(1-\frac{1}{\lambda^p})^{\frac{1}{p}}$ and 2, respectively. Hence, we also have $\alpha = \beta = \gamma = 2^{1-\frac{1}{t}}\left(1+(1-\frac{1}{\lambda^p})^{\frac{t}{p}}\right)^{\frac{1}{t}}$ by $t<0$. $\square$

REMARK 2.4. By applying the following formula (see [8])

$$J_{X,0}(1) = \sup\{\sqrt{2\varepsilon(1-\delta_X(\varepsilon)}: 0 \leqslant \varepsilon \leqslant 2\}$$

and Lemma 2.1, we also have

COROLLARY 2.5. *Let* $p \geqslant 2$, $\lambda > 1$.
(1) If $\lambda > 2^{\frac{1}{p}}$, *then* $J_{X_{\lambda,p},0}(1) = 2(1-\frac{1}{\lambda^p})^{\frac{1}{2p}}$.
(2) If $\lambda \leqslant 2^{\frac{1}{p}}$, *then* $J_{X_{\lambda,p},0}(1) = 2^{1-\frac{1}{p}}\sqrt{\lambda}$.

3. Von Neumann-Jordan constant

In [7], we have the following result

THEOREM 3.1. *Let* $\lambda \geqslant 1$ *and* $\mathbb{R}^2_\lambda$ *is the Banaś-Frączieck space. Then,*

$$C_{NJ}(\mathbb{R}^2_\lambda) = 2 - \frac{1}{\lambda^2}.$$

Now we give a generalization as follow

THEOREM 3.2. *Let $p \geqslant 2$ and $\lambda > 1$ such that $(\lambda^p - 1)^{p-2}(\lambda^2 - 1)^p \geqslant 1$, then*

$$C_{NJ}(X_{\lambda,p}) = 1 + \left(1 - \frac{1}{\lambda^p}\right)^{\frac{2}{p}}.$$

In order to prove this theorem, first we give the following Lemma.

LEMMA 3.1. *If $\lambda \geqslant 1$, $p \geqslant 2$ such that $(\lambda^p - 1)^{p-2}(\lambda^2 - 1)^p \geqslant 1$, and $|x_1| \leqslant \frac{1}{\lambda}$, $|y_1| \leqslant \frac{1}{\lambda}$, then $\lambda^p \geqslant 2$ and*

$$(\lambda^2 - 1)|x_1 y_1| + (1 - |x_1|^p)^{\frac{1}{p}}(1 - |y_1|^p)^{\frac{1}{p}} \leqslant \frac{\lambda^2 - 1}{\lambda^2} + \left(1 - \frac{1}{\lambda^p}\right)^{\frac{2}{p}}. \qquad (3.1)$$

Proof. From $(\lambda^p - 1)^{p-2}(\lambda^2 - 1)^p \geqslant 1$, we have $\lambda^p \geqslant 2$. Now we may assume that $0 \leqslant x_1 \leqslant \frac{1}{\lambda}$ and $0 \leqslant y_1 \leqslant \frac{1}{\lambda}$. Taking $h(x) = (\lambda^2 - 1)x^2 + (1 - x^p)^{\frac{2}{p}}$ for $x \in [0, \frac{1}{\lambda}]$, we have

$$h'(x) = 2x^{p-1}[(\lambda^2 - 1)x^{2-p} - (1 - x^p)^{\frac{2-p}{p}}] \geqslant 0$$

by $x^p \leqslant \frac{1}{\lambda^p}$ and $(\lambda^p - 1)^{p-2}(\lambda^2 - 1)^p \geqslant 1$. Hence for $x \in [0, \frac{1}{\lambda}]$, we have

$$h(x) \leqslant \frac{\lambda^2 - 1}{\lambda^2} + \left(1 - \frac{1}{\lambda^p}\right)^{\frac{2}{p}}.$$

Let $F(x,y) = (\lambda^2 - 1)xy + (1 - x^p)^{\frac{1}{p}}(1 - y^p)^{\frac{1}{p}}$. If $(x_0, y_0) \in (0, \frac{1}{\lambda}) \times (0, \frac{1}{\lambda})$ such that $F_x(x_0, y_0) = F_y(x_0, y_0) = 0$, then $x_0 = y_0$. So

$$F(x_0, y_0) = h(x_0) \leqslant \frac{\lambda^2 - 1}{\lambda^2} + \left(1 - \frac{1}{\lambda^p}\right)^{\frac{2}{p}}. \qquad (3.2)$$

Next, letting $g(x) = \frac{\lambda^2 - 1}{\lambda}x + (1 - \frac{1}{\lambda^p})^{\frac{1}{p}}(1 - x^p)^{\frac{1}{p}}$, then for $x \in [0, \frac{1}{\lambda}]$ we have

$$\begin{aligned} g'(x) &= \frac{\lambda^2 - 1}{\lambda} - \left[1 - \frac{1}{\lambda^p}\right]^{\frac{1}{p}}(x^{-p} - 1)^{\frac{1}{p} - 1} \\ &\geqslant \frac{\lambda^2 - 1}{\lambda} - \left(1 - \frac{1}{\lambda^p}\right)^{\frac{1}{p}}(\lambda^p - 1)^{\frac{1}{p} - 1} \\ &= \frac{\lambda^2 - 1}{\lambda} - \frac{1}{\lambda}(\lambda^p - 1)^{\frac{2}{p} - 1} \geqslant 0 \end{aligned}$$

by $\lambda^p \geqslant 2$. Hence

$$g(x_1) \leqslant g\left(\frac{1}{\lambda}\right) = \frac{\lambda^2 - 1}{\lambda^2} + \left(1 - \frac{1}{\lambda^p}\right)^{\frac{2}{p}}.$$

Therefore for $x, y \in [0, \frac{1}{\lambda}]$,

$$\max\{F(x,0), F(0,y)\} \leqslant 1 \leqslant \frac{\lambda^2 - 1}{\lambda^2} + \left(1 - \frac{1}{\lambda^p}\right)^{\frac{2}{p}}, \qquad (3.3)$$

and

$$\max\left\{F\left(x,\frac{1}{\lambda}\right),F\left(\frac{1}{\lambda},y\right)\right\}\leqslant\frac{\lambda^2-1}{\lambda^2}+\left(1-\frac{1}{\lambda^p}\right)^{\frac{2}{p}}. \tag{3.4}$$

From (3.2)–(3.4), we get $F(x,y)\leqslant\frac{\lambda^2-1}{\lambda^2}+(1-\frac{1}{\lambda^p})^{\frac{2}{p}}$ on $[0,\frac{1}{\lambda}]\times[0,\frac{1}{\lambda}]$, and hence (3.1) is valid. □

Proof of Theorem 3.2. Assume that $\lambda>1$ such that $(\lambda^p-1)^{p-2}(\lambda^2-1)^p\geqslant 1$. Note that $ex(B_X)=\{(z_1,z_2):|z_1|^p+|z_2|^p=1,|z_1|\leqslant\frac{1}{\lambda}\}$.

Now we prove that

$$\frac{\|x+\tau y\|^2+\|x-\tau y\|^2}{2(1+\tau^2)}\leqslant 1+\left(1-\frac{1}{\lambda^p}\right)^{\frac{2}{p}}. \tag{3.5}$$

holds for any $x,y\in ex(B_X)$ and any $\tau\in[0,1]$.

Letting $x=(x_1,x_2)$, $y=(y_1,y_2)$, then we have the following three cases.

I a). If $\|x+\tau y\|_p\leqslant|\lambda(x_1+\tau y_1)|$ and $\|x-\tau y\|_p\leqslant|\lambda(x_1-\tau y_1)|$, then

$$\begin{aligned}\|x+\tau y\|^2+\|x-\tau y\|^2&=\lambda^2[(x_1+\tau y_1)^2+(x_1-\tau y_1)^2]\\&=2\lambda^2(x_1^2+\tau^2y_1^2)\leqslant 2(1+\tau^2).\end{aligned} \tag{3.6}$$

I b). If $\|x+\tau y\|_p>|\lambda(x_1+\tau y_1)|$ and $\|x-\tau y\|_p>|\lambda(x_1-\tau y_1)|$, then

$$\begin{aligned}\|x+\tau y\|^2+\|x-\tau y\|^2&=\|x+\tau y\|_p^2+\|x-\tau y\|_p^2\\&\leqslant 2^{1-\frac{2}{p}}[(1+\tau)^p+(1-\tau)^p)]^{\frac{2}{p}}\\&\leqslant 2^{2-\frac{2}{p}}(1+\tau^2)\end{aligned} \tag{3.7}$$

by Hölder and Hanner's inequality.

I c). If $\|x+\tau y\|_p\leqslant|\lambda(x_1+\tau y_1)|$ and $\|x-\tau y\|_p>|\lambda(x_1-\tau y_1)|$, or $\|x+\tau y\|_p>|\lambda(x_1+\tau y_1)|$ and $\|x-\tau y\|_p\leqslant|\lambda(x_1-\tau y_1)|$, then

$$\begin{aligned}\|x+\tau y\|^2+\|x-\tau y\|^2&\leqslant\lambda^2(x_1\pm\tau y_1)^2+(x_1\mp\tau y_1)^2+(x_2\mp\tau y_2)^2\\&\leqslant 1+\tau^2+x_1^2+x_2^2+\tau^2(y_1^2+y_2^2)+2\tau(\lambda^2-1)|x_1y_1|\\&\quad+2\tau[1-|x_1|^p]^{\frac{1}{p}}[1-|y_1|^p]^{\frac{1}{p}}\\&\leqslant\left(1+\frac{1}{\lambda^2}+\left(1-\frac{1}{\lambda^p}\right)^{\frac{2}{p}}\right)(1+\tau^2)\\&\quad+2\tau\left(\frac{\lambda^2-1}{\lambda^2}+\left(1-\frac{1}{\lambda^p}\right)^{\frac{2}{p}}\right),\end{aligned} \tag{3.8}$$

holds by Lemma 3.1 and $x_1^2+(1-|x_1|^p)^{\frac{2}{p}}\leqslant\frac{1}{\lambda^2}+(1-\frac{1}{\lambda^p})^{\frac{2}{p}}$ for $0\leqslant|x_1|\leqslant\frac{1}{\lambda}$. Hence, (3.6)–(3.8) imply (3.5). Hence we have $C_{NJ}(X_{\lambda,p})\leqslant 1+(1-\frac{1}{\lambda^p})^{\frac{2}{p}}$. On the other hand, if taking $x=(\frac{1}{\lambda},\ (1-\frac{1}{\lambda^p})^{\frac{1}{p}}),y=(\frac{1}{\lambda},-(1-\frac{1}{\lambda^p})^{\frac{1}{p}})$, we have

$$C_{NJ}(X_{\lambda,p})\geqslant\frac{\|x+y\|^2+\|x-y\|^2}{4}=1+\left(1-\frac{1}{\lambda^p}\right)^{\frac{2}{p}}.$$

which completes the proof of Theorem 3.2. □

Acknowledgement. We express our cordial thanks to the referees. This work is supported by the National Natural Science Foundation of China (Nos. 11271112; 11201127) and Technology and the Innovation Team in Henan Province (No. 14IRT-STHN023).

REFERENCES

[1] J. A. CLARKSON, *The von Neumann-Jordan constant for the Lebesgue space*, Ann. of Math., **38** (1937), 114–115.

[2] R. C. JAMES, *Uniformly nonsquare spaces*, Ann. of Math., **80** 1964, 542–550.

[3] J. GAO, K. S. LAU, *On two classes of Banach spaces with uniform normal structure*, Studia Math. **99**(1991), 41–56.

[4] Y. TAKAHASHI, *Some geometric constants of Banach spaces – a unified approach*, Banach and function spaces II, Yokohama Publ., Yokohama, (2008), 191–220.

[5] J. ALONSO, E. LLORENS-FUSTER, *Geometric mean and triangles inscribed in a semicircle in Banach spaces*, J. Math. Anal. Appl. **340** (2008), 1271–1283.

[6] J. BANAŚ AND K. FRĄCZEK, *Deformation of Banach spaces*, Comment. Math. Univ. Carolinae **34** (1993), no. 1, 47–53.

[7] C. YANG, *Jordan-von Neumann constant for Banaś-Frączek space*, Banach J. Math. Anal. **8** (2014), no. 2, 185–192.

[8] C. YANG, Y. WANG, *Some properties of James type constant*, Appl. Math. Lett. **25** (2012), 538–544.

(Received April 22, 2015)

Changsen Yang
Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control
College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, Henan, P. R. China
e-mail: yangchangsen0991@sina.com

Xiangzhao Yang
Shiyuan college of Guangxi teachers education university
Nianning, 530001, China
e-mail: 175904994@qq.com