

IMPROVED YOUNG AND HEINZ INEQUALITIES WITH THE KANTOROVICH CONSTANT

WENSHI LIAO AND JUNLIANG WU

(Communicated by M. Krnić)

Abstract. In this article, we study the further refinements and reverses of the Young and Heinz inequalities with the Kantorovich constant. These modified inequalities are used to establish corresponding operator inequalities on a Hilbert space and Hilbert-Schmidt norm inequalities.

1. Introduction

The well-known Young inequality for scalars is the weighted arithmetic-geometric mean inequality, which was due to William Henry Young (1863-1942). The inequality states that if $a, b > 0$ and $0 \leq v \leq 1$, then

$$(1 - v)a + vb \geq a^{1-v}b^v \tag{1}$$

with equality if and only if $a = b$. If $v = \frac{1}{2}$, it gives rise to the elementary arithmetic-geometric mean inequality $\sqrt{ab} \leq \frac{a+b}{2}$.

If $v > 1$ or $v < 0$, then the reverse of (1)

$$(1 - v)a + vb \leq a^{1-v}b^v$$

holds. For more details, the reader is referred to [1].

The Heinz mean in the parameter $0 \leq v \leq 1$, defined by

$$H_v(a, b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}, \quad a, b > 0$$

interpolates between the arithmetic mean and geometric mean, i.e.

$$\sqrt{ab} = H_{\frac{1}{2}}(a, b) \leq H_v(a, b) \leq H_1(a, b) = \frac{a+b}{2}, \quad v \in [0, 1].$$

It is easy to see that the Heinz mean is convex as a function of v on the interval $[0, 1]$, attains minimum at $v = 1/2$, and attains maximum at $v = 0$ and $v = 1$. Moreover, $H_v(a, b)$ is symmetric with respect to $v = 1/2$, that is, $H_v(a, b) = H_{1-v}(a, b)$, $v \in [0, 1]$.

Mathematics subject classification (2010): 15A15, 15A42, 15A60, 47A30.

Keywords and phrases: Young inequality, Heinz mean, Kantorovich constant, operator inequalities.

In recent years, Kittaneh and Manasrah [11, 12] improved the Young inequality (1), and obtained the following relation:

$$r(\sqrt{a} - \sqrt{b})^2 \leq (1 - v)a + vb - a^{1-v}b^v \leq R(\sqrt{a} - \sqrt{b})^2, \tag{2}$$

where $a, b > 0$, $v \in [0, 1]$, $r = \min\{v, 1 - v\}$ and $R = \max\{v, 1 - v\}$.

Later, Wu and Zhao [15] presented two improvements of (2) that

$$(1 - v)a + vb \geq r(\sqrt{a} - \sqrt{b})^2 + K(\sqrt{h}, 2)^{r_1} a^{1-v}b^v, \tag{3}$$

$$(1 - v)a + vb \leq R(\sqrt{a} - \sqrt{b})^2 + K(\sqrt{h}, 2)^{-r_1} a^{1-v}b^v,$$

where $h = \frac{b}{a}$, $K(\sqrt{h}, 2) = \frac{(\sqrt{h+1})^2}{4\sqrt{h}}$ and $r_1 = \min\{2r, 1 - 2r\}$. Note that $K(t, 2) = \frac{(t+1)^2}{4t}$ is the classical Kantorovich constant which has properties $K(1, 2) = 1$, $K(t, 2) = K(\frac{1}{t}, 2) \geq 1$ ($t > 0$) and $K(t, 2)$ is monotone increasing on $[1, \infty)$ and monotone decreasing on $(0, 1]$.

Recently, Zhao and Wu [16] obtained the refinements and reverses of Young inequality and improved inequalities (2) in the following forms:

PROPOSITION 1. [16] *Let a, b be two nonnegative real numbers and $v \in (0, 1)$.*

(I) *If $0 < v \leq \frac{1}{2}$, then*

$$(1 - v)a + vb \geq a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{a})^2, \tag{4}$$

$$(1 - v)a + vb \leq a^{1-v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 - r_1(\sqrt[4]{ab} - \sqrt{b})^2, \tag{5}$$

(II) *if $\frac{1}{2} < v < 1$, then*

$$(1 - v)a + vb \geq a^{1-v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{b})^2, \tag{6}$$

$$(1 - v)a + vb \leq a^{1-v}b^v + v(\sqrt{a} - \sqrt{b})^2 - r_1(\sqrt[4]{ab} - \sqrt{a})^2, \tag{7}$$

where $r = \min\{v, 1 - v\}$ and $r_1 = \min\{2r, 1 - 2r\}$.

Let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on a complex separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. I stands for the identity operator. $\mathcal{B}^{++}(\mathcal{H})$ denotes the cone of all positive invertible operators on \mathcal{H} . As a matter of convenience, we use the following notations to define the weighted arithmetic mean and geometric mean for operators:

$$A\nabla_v B = (1 - v)A + vB, A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

where $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and $v \in [0, 1]$. When $v = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

An operator version of the Young inequality proved in [3] says that if $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and $v \in [0, 1]$, then

$$A\nabla_v B \geq A\sharp_v B.$$

The Heinz operator mean is defined by

$$H_\nu(A, B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2}$$

for $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and $0 \leq \nu \leq 1$.

It is easy to see that the Heinz operator mean interpolates the arithmetic-geometric operator mean inequality:

$$A\sharp B \leq H_\nu(A, B) \leq A\nabla B. \tag{8}$$

Inequalities in (8) are called the Heinz operator inequalities (see [9, 10]).

The first difference-type improvement of the matrix Young inequality is due to Kittaneh and Manasrah [12] extending (2) to matrices:

$$r(A\nabla B - A\sharp B) \leq A\nabla_\nu B - A\sharp_\nu B \leq R(A\nabla B - A\sharp B) \tag{9}$$

holds for positive definite matrices A and B and $0 \leq \nu \leq 1$, where $r = \min\{\nu, 1 - \nu\}$ and $R = \max\{\nu, 1 - \nu\}$, which remain of course valid for Hilbert space operators by a standard approximation argument.

Note that Furuichi [4] independently established the first inequality in (9) for two positive operators and Kittaneh et al. [9] also proved (9) by taking a different approach. In [13], the authors provided the general refinement and reverse of the Jensen’s operator inequality and the relation (9) appears as a special case of their results.

For the ratio-type improvements of the Young inequality, the readers are referred to [4, 5, 14, 15, 17].

Zhao and Wu [16] also extended inequalities (4)–(7) to positive invertible operators and improved (9), which were shown as

PROPOSITION 2. [16] *Let $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and $\nu \in (0, 1)$.*

(I) *If $0 < \nu \leq \frac{1}{2}$, then*

$$A\nabla_\nu B \geq A\sharp_\nu B + 2\nu(A\nabla B - A\sharp B) + r_1(A\sharp B - 2A\sharp_{\frac{3}{4}} B + A), \tag{10}$$

$$A\nabla_\nu B \leq A\sharp_\nu B + 2(1 - \nu)(A\nabla B - A\sharp B) - r_1(A\sharp B - 2A\sharp_{\frac{3}{4}} B + B), \tag{11}$$

(II) *if $\frac{1}{2} < \nu < 1$, then*

$$A\nabla_\nu B \geq A\sharp_\nu B + 2(1 - \nu)(A\nabla B - A\sharp B) + r_1(A\sharp B - 2A\sharp_{\frac{3}{4}} B + B), \tag{12}$$

$$A\nabla_\nu B \leq A\sharp_\nu B + 2\nu(A\nabla B - A\sharp B) - r_1(A\sharp B - 2A\sharp_{\frac{1}{4}} B + A), \tag{13}$$

where $r = \min\{\nu, 1 - \nu\}$ and $r_1 = \min\{2r, 1 - 2r\}$.

In this paper, we are concerned with several improvements of the Young and Heinz inequalities via the Kantorovich constant. In Section 2, we present the whole series of refinements and reverses of the scalar Young inequality which will help us to derive several Heinz mean inequalities. In Section 3, we extend inequalities proved in Section 2 from the scalars setting to a Hilbert space operator setting. In Section 4, the Hilbert-Schmidt norm inequalities are established.

2. Scalar inequalities

In this section, we mainly present the direct refinements and reverses of the Young inequality for two positive numbers a, b . When $\nu = 0$ and $\nu = 1$, the Young inequality is trivial. We will study the case $\nu \in (0, 1)$.

THEOREM 1. *Let $a, b > 0$ and $\nu \in (0, 1)$.*

(I) *If $0 < \nu \leq \frac{1}{2}$, then*

$$(1 - \nu)a + \nu b \geq \nu(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{a})^2 + K(\sqrt[4]{h}, 2)^{\hat{r}_1} a^{1-\nu} b^\nu, \tag{14}$$

(II) *if $\frac{1}{2} < \nu < 1$, then*

$$(1 - \nu)a + \nu b \geq (1 - \nu)(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{b})^2 + K(\sqrt[4]{h}, 2)^{\hat{r}_1} a^{1-\nu} b^\nu, \tag{15}$$

where $h = \frac{b}{a}$, $r = \min\{\nu, 1 - \nu\}$, $r_1 = \min\{2r, 1 - 2r\}$ and $\hat{r}_1 = \min\{2r_1, 1 - 2r_1\}$.

Proof. The proof of inequality (15) is similar to that of (14). Thus, we only need to prove (14).

If $\nu = \frac{1}{4}$ and $\nu = \frac{1}{2}$, (14) becomes equality.

By the inequality (3), if $0 < \nu < \frac{1}{4}$, then we have

$$\begin{aligned} (1 - \nu)a + \nu b - \nu(\sqrt{a} - \sqrt{b})^2 &= 2\nu\sqrt{ab} + (1 - 2\nu)a \\ &\geq 2\nu(\sqrt[4]{ab} - \sqrt{a})^2 + K(\sqrt[4]{h}, 2)^{\min\{4\nu, 1-4\nu\}} a^{1-\nu} b^\nu, \end{aligned}$$

if $\frac{1}{4} < \nu < \frac{1}{2}$, then we get

$$\begin{aligned} (1 - \nu)a + \nu b - \nu(\sqrt{a} - \sqrt{b})^2 &= 2\nu\sqrt{ab} + (1 - 2\nu)a \\ &\geq (1 - 2\nu)(\sqrt[4]{ab} - \sqrt{a})^2 + K(\sqrt[4]{h}, 2)^{\min\{2-4\nu, 4\nu-1\}} a^{1-\nu} b^\nu. \end{aligned}$$

So we conclude that

$$(1 - \nu)a + \nu b \geq \nu(\sqrt{a} - \sqrt{b})^2 + r_1(\sqrt[4]{ab} - \sqrt{a})^2 + K(\sqrt[4]{h}, 2)^{\hat{r}_1} a^{1-\nu} b^\nu.$$

This completes the proof. \square

REMARK 1. By the properties of Kantorovich constant, (14) and (15) are better than (4) and (6), respectively. As a direct consequence of Theorem 1, we have the following inequality with respect to the Heinz mean:

$$\frac{a+b}{2} \geq r(\sqrt{a} - \sqrt{b})^2 + \frac{1}{2}r_1 \left[(\sqrt[4]{ab} - \sqrt{a})^2 + (\sqrt[4]{ab} - \sqrt{b})^2 \right] + K(\sqrt[4]{h}, 2)^{\hat{r}_1} H_\nu(a, b). \tag{16}$$

COROLLARY 1. Let $a, b > 0$ and $v \in (0, 1)$.

(I) If $0 < v \leq \frac{1}{2}$, then

$$((1 - v)a + vb)^2 \geq v^2(a - b)^2 + r_1(\sqrt{ab} - a)^2 + K(\sqrt{h}, 2)^{\hat{r}_1} (a^{1-v}b^v)^2. \tag{17}$$

(II) If $\frac{1}{2} < v < 1$, then

$$((1 - v)a + vb)^2 \geq (1 - v)^2(a - b)^2 + r_1(\sqrt{ab} - b)^2 + K(\sqrt{h}, 2)^{\hat{r}_1} (a^{1-v}b^v)^2. \tag{18}$$

Proof. Replacing a by a^2 and b by b^2 in (14) and (15), respectively, we have

$$(1 - v)a^2 + vb^2 \geq v(a - b)^2 + r_1(\sqrt{ab} - a)^2 + K(\sqrt{h}, 2)^{\hat{r}_1} (a^{1-v}b^v)^2$$

and

$$(1 - v)a^2 + vb^2 \geq (1 - v)(a - b)^2 + r_1(\sqrt{ab} - b)^2 + K(\sqrt{h}, 2)^{\hat{r}_1} (a^{1-v}b^v)^2. \tag{19}$$

If $0 < v \leq \frac{1}{2}$, then by the first inequality above, we obtain

$$\begin{aligned} ((1 - v)a + vb)^2 - v^2(a - b)^2 &= (1 - v)a^2 + vb^2 - v(a - b)^2 \\ &\geq r_1(\sqrt{ab} - a)^2 + K(\sqrt{h}, 2)^{\hat{r}_1} (a^{1-v}b^v)^2. \end{aligned}$$

If $\frac{1}{2} < v < 1$, then by using (19), we get

$$\begin{aligned} ((1 - v)a + vb)^2 - (1 - v)^2(a - b)^2 &= (1 - v)a^2 + vb^2 - (1 - v)(a - b)^2 \\ &\geq r_1(\sqrt{ab} - b)^2 + K(\sqrt{h}, 2)^{\hat{r}_1} (a^{1-v}b^v)^2. \quad \square \end{aligned}$$

THEOREM 2. Let $a, b > 0$ and $v \in (0, 1)$.

(I) If $0 < v \leq \frac{1}{2}$, then

$$(1 - v)a + vb \leq (1 - v)(\sqrt{a} - \sqrt{b})^2 - r_1(\sqrt[4]{ab} - \sqrt{b})^2 + K(\sqrt[4]{h}, 2)^{-\hat{r}_1} a^{1-v}b^v, \tag{20}$$

(II) If $\frac{1}{2} < v < 1$, then

$$(1 - v)a + vb \leq v(\sqrt{a} - \sqrt{b})^2 - r_1(\sqrt[4]{ab} - \sqrt{a})^2 + K(\sqrt[4]{h}, 2)^{-\hat{r}_1} a^{1-v}b^v, \tag{21}$$

where $h = \frac{b}{a}$, $r = \min\{v, 1 - v\}$, $r_1 = \min\{2r, 1 - 2r\}$ and $\hat{r}_1 = \min\{2r_1, 1 - 2r_1\}$.

Proof. The proof of inequality (21) is similar to that of (20). Thus, we only need to prove (20).

If $0 < v < \frac{1}{2}$, then by the inequality (3), we deduce

$$\begin{aligned} &K(\sqrt[4]{h}, 2)^{-\hat{r}_1} a^{1-v}b^v + (1 - v)(\sqrt{a} - \sqrt{b})^2 - (1 - v)a - vb \\ &= K(\sqrt[4]{h}, 2)^{-\hat{r}_1} a^{1-v}b^v + (1 - 2v)b + 2v\sqrt{ab} - 2\sqrt{ab} \\ &\geq K(\sqrt[4]{h}, 2)^{-\hat{r}_1} a^{1-v}b^v + K(\sqrt[4]{h}, 2)^{\hat{r}_1} a^v b^{1-v} + r_1(\sqrt[4]{ab} - \sqrt{b})^2 - 2\sqrt{ab} \\ &\geq r_1(\sqrt[4]{ab} - \sqrt{b})^2. \end{aligned}$$

If $v = \frac{1}{2}$, the inequality (20) becomes equality. \square

REMARK 2. By the properties of Kantorovich constant, (20) and (21) are better than (5) and (7), respectively. As a direct consequence of Theorem 2, we have the following inequality with respect to the Heinz mean:

$$\frac{a+b}{2} \leq R(\sqrt{a}-\sqrt{b})^2 - \frac{1}{2}r_1[(\sqrt[4]{ab}-\sqrt{a})^2 + (\sqrt[4]{ab}-\sqrt{b})^2] + K(\sqrt[4]{h}, 2)^{-\hat{r}_1}H_v(a,b), \tag{22}$$

where $R = \max\{v, 1 - v\}$.

COROLLARY 2. Let $a, b > 0$ and $v \in (0, 1)$.

(I) If $0 < v \leq \frac{1}{2}$, then

$$((1-v)a + vb)^2 \leq (1-v)^2(a-b)^2 - r_1(\sqrt{ab}-b)^2 + K(\sqrt{h}, 2)^{-\hat{r}_1}(a^{1-v}b^v)^2.$$

(II) If $\frac{1}{2} < v < 1$, then

$$((1-v)a + vb)^2 \leq v^2(a-b)^2 - r_1(\sqrt{ab}-a)^2 + K(\sqrt{h}, 2)^{-\hat{r}_1}(a^{1-v}b^v)^2.$$

Proof. Replacing a by a^2 and b by b^2 in (20) and (21), respectively, we have

$$(1-v)a^2 + vb^2 \leq (1-v)(a-b)^2 - r_1(\sqrt{ab}-b)^2 + K(\sqrt{h}, 2)^{-\hat{r}_1}(a^{1-v}b^v)^2$$

and

$$(1-v)a^2 + vb^2 \leq v(a-b)^2 - r_1(\sqrt{ab}-a)^2 + K(\sqrt{h}, 2)^{-\hat{r}_1}(a^{1-v}b^v)^2.$$

The remaining proof is similar to that of Corollary 1. \square

3. Operator inequalities

If A is a selfadjoint operator and f is a real valued continuous function on $\text{Sp}(A)$ (the spectrum of A), then $f(t) \geq 0$ for every $t \in \text{Sp}(A)$ implies that $f(A) \geq 0$, i.e., $f(A)$ is a positive operator on \mathcal{H} . Equivalently, if both f and g are real valued continuous functions on $\text{Sp}(A)$, then the following monotonic property of operator functions holds:

$$f(t) \geq g(t) \text{ for any } t \in \text{Sp}(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $\mathcal{B}(\mathcal{H})$.

THEOREM 3. Let $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and positive real numbers m, m', M, M' satisfy either $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$ or $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$.

(I) If $0 < v \leq \frac{1}{2}$, then

$$A\nabla_v B \geq 2v(A\nabla B - A\sharp B) + r_1(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A) + K(\sqrt[4]{h}, 2)^{\hat{r}_1}A\sharp_v B, \tag{23}$$

(II) if $\frac{1}{2} < v < 1$, then

$$A\nabla_v B \geq 2(1-v)(A\nabla B - A\sharp B) + r_1(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B) + K(\sqrt[4]{h}, 2)^{\hat{r}_1}A\sharp_v B, \tag{24}$$

where $h = \frac{M}{m}$, $r = \min\{v, 1 - v\}$, $r_1 = \min\{2r, 1 - 2r\}$ and $\hat{r}_1 = \min\{2r_1, 1 - 2r_1\}$. Equality holds if and only if $A = B$.

Proof. If $0 < v \leq \frac{1}{2}$, it follows from inequality (14) that for any $x > 0$,

$$(1 - v) + vx \geq v(\sqrt{x} - 1)^2 + r_1(\sqrt[4]{x} - 1)^2 + K(\sqrt[4]{x}, 2)^{\hat{r}_1} x^v. \tag{25}$$

Taking $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, under the condition $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$, we have

$$I \leq hI = \frac{M}{m}I \leq X \leq h'I = \frac{M'}{m'}I,$$

and then $\text{Sp}(X) \subset [h, h'] \subset (1, +\infty)$. Thus for positive operator X , it can be deduced from the inequality (25) and the monotonic property of operator functions that

$$(1 - v)I + vX \geq v(X - 2X^{\frac{1}{2}} + I) + r_1(X^{\frac{1}{2}} - 2X^{\frac{1}{4}} + I) + \min_{h \leq x \leq h'} K(\sqrt[4]{x}, 2)^{\hat{r}_1} X^v.$$

On the other hand, since the Kantorovich constant $K(t, 2)$ is an increasing function on $(1, +\infty)$, we get

$$\begin{aligned} (1 - v)I + vA^{-\frac{1}{2}}BA^{-\frac{1}{2}} &\geq v(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} + I) \\ &\quad + r_1((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{4}} + I) \\ &\quad + K(\sqrt[4]{h}, 2)^{\hat{r}_1} (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v, \end{aligned} \tag{26}$$

Likewise, under the condition $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$, we have $0 \leq \frac{1}{h'}I \leq X \leq \frac{1}{h}I < I$ and then $\text{Sp}(X) \subset [\frac{1}{h'}, \frac{1}{h}] \subset (0, 1)$. Thus for positive operator X , we obtain

$$(1 - v)I + vX \geq v(X - 2X^{\frac{1}{2}} + I) + r_1(X^{\frac{1}{2}} - 2X^{\frac{1}{4}} + I) + \min_{\frac{1}{h'} \leq x \leq \frac{1}{h}} K(\sqrt[4]{x}, 2)^{\hat{r}_1} X^v.$$

On the other hand, the Kantorovich constant $K(t, 2)$ is an decreasing function on $(0, 1)$ and $K(\frac{1}{t}, 2) = K(t, 2)$, we get

$$(1 - v)I + vX \geq v(X - 2X^{\frac{1}{2}} + I) + r_1(X^{\frac{1}{2}} - 2X^{\frac{1}{4}} + I) + K(\sqrt[4]{h}, 2)^{\hat{r}_1} X^v. \tag{27}$$

It is striking that we obtain two same inequalities (26) and (27) under the two different condition. Then multiplying inequality (26) or (27) by $A^{\frac{1}{2}}$ on both sides, we can deduce the required inequality (23).

If $\frac{1}{2} < v < 1$, the inequality (24) follows from inequality (15) by the similar methods. \square

The operator version of (16) can be shown as follows:

COROLLARY 3. *Under the same conditions as Theorem 3, then*

$$A \nabla B \geq 2r(A \nabla B - A \sharp B) + r_1(A \nabla B + A \sharp B - 2H_{\frac{1}{4}}(A, B)) + K(\sqrt[4]{h}, 2)^{\hat{r}_1} H_v(A, B). \tag{28}$$

THEOREM 4. *Let $A, B \in \mathcal{B}^{++}(\mathcal{H})$ and positive real numbers m, m', M, M' satisfy either $0 < m'I \leq A \leq ml < MI \leq B \leq M'I$ or $0 < m'I \leq B \leq ml < MI \leq A \leq M'I$.*

(I) *If $0 < v \leq \frac{1}{2}$, then*

$$A \nabla_v B \leq 2(1 - v)(A \nabla B - A \sharp B) - r_1(A \sharp B - 2A \sharp_{\frac{3}{4}} B + B) + K(\sqrt[4]{h}, 2)^{-\hat{r}_1} A \sharp_v B, \tag{29}$$

(II) *if $\frac{1}{2} < v < 1$, then*

$$A \nabla_v B \leq 2v(A \nabla B - A \sharp B) - r_1(A \sharp B - 2A \sharp_{\frac{1}{4}} B + A) + K(\sqrt[4]{h}, 2)^{-\hat{r}_1} A \sharp_v B, \tag{30}$$

where $h = \frac{M}{m}$, $r = \min\{v, 1 - v\}$, $r_1 = \min\{2r, 1 - 2r\}$ and $\hat{r}_1 = \min\{2r_1, 1 - 2r_1\}$. Equality holds if and only if $A = B$.

Proof. By (20) and (21), using the same ideas as in the proof of Theorem 3, we can get this theorem. \square

The operator version of (22) can be shown as

COROLLARY 4. *Under the same conditions as Theorem 4, then*

$$A \nabla B \leq 2R(A \nabla B - A \sharp B) - r_1(A \nabla B + A \sharp B - 2H_{\frac{1}{4}}(A, B)) + K(\sqrt[4]{h}, 2)^{-\hat{r}_1} H_v(A, B), \tag{31}$$

where $R = \max\{v, 1 - v\}$.

REMARK 3. (28) and (31) are sharper than (3.4) in [12].

If $0 < v \leq \frac{1}{2}$, combining (23) with (29), we have

$$\begin{aligned} 0 &\leq A \sharp_v B \\ &\leq 2v(A \nabla B - A \sharp B) + A \sharp_v B \\ &\leq 2v(A \nabla B - A \sharp B) + r_1(A \sharp B - 2A \sharp_{\frac{1}{4}} B + A) + A \sharp_v B \\ &\leq 2v(A \nabla B - A \sharp B) + r_1(A \sharp B - 2A \sharp_{\frac{1}{4}} B + A) + K(\sqrt[4]{h}, 2)^{\hat{r}_1} A \sharp_v B \\ &\leq A \nabla_v B \\ &\leq 2(1 - v)(A \nabla B - A \sharp B) - r_1(A \sharp B - 2A \sharp_{\frac{3}{4}} B + B) + K(\sqrt[4]{h}, 2)^{-\hat{r}_1} A \sharp_v B \\ &\leq 2(1 - v)(A \nabla B - A \sharp B) - r_1(A \sharp B - 2A \sharp_{\frac{3}{4}} B + B) + A \sharp_v B \\ &\leq 2(1 - v)(A \nabla B - A \sharp B) + A \sharp_v B. \end{aligned}$$

By the properties of Kantorovich constant, (23) and (24) are better than (10) and (12), respectively.

If $\frac{1}{2} < v < 1$, combining (24) with (30), we have

$$\begin{aligned} 0 &\leq A \sharp_v B \\ &\leq 2(1 - v)(A \nabla B - A \sharp B) + A \sharp_v B \\ &\leq 2(1 - v)(A \nabla B - A \sharp B) + r_1(A \sharp B - 2A \sharp_{\frac{3}{4}} B + B) + A \sharp_v B \\ &\leq 2(1 - v)(A \nabla B - A \sharp B) + r_1(A \sharp B - 2A \sharp_{\frac{3}{4}} B + B) + K(\sqrt[4]{h}, 2)^{\hat{r}_1} A \sharp_v B \end{aligned}$$

$$\begin{aligned}
 &\leq A\nabla_v B \\
 &\leq 2v(A\nabla B - A\sharp B) - r_1(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A) + K(\sqrt[4]{h}, 2)^{-\hat{r}_1} A\sharp_v B \\
 &\leq 2v(A\nabla B - A\sharp B) - r_1(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A) + A\sharp_v B \\
 &\leq 2v(A\nabla B - A\sharp B) + A\sharp_v B.
 \end{aligned}$$

(29) and (30) are better than (11) and (13), respectively.

4. Hilbert-Schmidt norm inequalities

In this section, we present the improved Young and Heinz inequalities for the Hilbert-Schmidt norm.

Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. The Hilbert-Schmidt norm of $A \in M_n(\mathbb{C})$ is denoted by $\|A\|_F^2$. It is well-known that the Hilbert-Schmidt norm is unitarily invariant in the sense that $\|UAV\|_F^2 = \|A\|_F^2$ for all unitary matrices $U, V \in M_n(\mathbb{C})$ (see [6, p. 341–342]). The spectrum of A and B are denoted by $\text{Sp}(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $\text{Sp}(B) = \{v_1, v_2, \dots, v_n\}$, respectively. The Schur product (Hadamard product) of two matrices $A, B \in M_n(\mathbb{C})$ is the entrywise product and denoted by $A \circ B$.

Hirzallah and Kittaneh [7] and Kittaneh and Manasrah [12] had showed that if $A, B, X \in M_n(\mathbb{C})$ with positive semidefinite matrices A and B , then

$$r^2 \|AX - XB\|_F^2 \leq \|(1 - v)AX + vXB\|_F^2 - \|A^{1-v}XB^v\|_F^2 \leq R^2 \|AX - XB\|_F^2, \tag{32}$$

where $v \in [0, 1]$, $r = \min\{v, 1 - v\}$ and $R = \max\{v, 1 - v\}$.

Applying Corollary 1 and 2, we derive two theorems which improve (32).

THEOREM 5. *Suppose $A, B, X \in M_n(\mathbb{C})$ such that A and B are two positive definite matrices. Let*

$$\underline{K} = \min \left\{ K((\lambda_i/v_j)^{\frac{1}{2}}, 2), i, j = 1, 2, \dots, n \right\}.$$

(I) *If $0 < v \leq \frac{1}{2}$, then*

$$\begin{aligned}
 &\|(1 - v)AX + vXB\|_F^2 - v^2 \|AX - XB\|_F^2 \\
 &\geq r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX \right\|_F^2 + \underline{K}^{\hat{r}_1} \|A^{1-v}XB^v\|_F^2,
 \end{aligned} \tag{33}$$

(II) *if $\frac{1}{2} < v < 1$, then*

$$\begin{aligned}
 &\|(1 - v)AX + vXB\|_F^2 - (1 - v)^2 \|AX - XB\|_F^2 \\
 &\geq r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB \right\|_F^2 + \underline{K}^{\hat{r}_1} \|A^{1-v}XB^v\|_F^2,
 \end{aligned} \tag{34}$$

where $r = \min\{v, 1 - v\}$, $r_1 = \min\{2r, 1 - 2r\}$ and $\hat{r}_1 = \min\{2r_1, 1 - 2r_1\}$.

Proof. Since A and B are positive definite, it follows by the spectral theorem that there exist unitary matrices $U, V \in M_n(\mathbb{C})$ such that

$$A = U\Lambda_1U^*, B = V\Lambda_2V^*,$$

where $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $\Lambda_2 = \text{diag}(v_1, v_2, \dots, v_n)$, $\lambda_i, v_i > 0$, $i = 1, 2, \dots, n$.

Let $Y = U^*XV = [y_{ij}] \in M_n(\mathbb{C})$, then

$$\begin{aligned} (1 - \nu)AX + \nu XB &= U((1 - \nu)\Lambda_1Y + \nu Y\Lambda_2)V^* \\ &= U[((1 - \nu)\lambda_i + \nu v_j) \circ Y]V^*, \\ AX - XB &= U[(\lambda_i - v_j) \circ Y]V^*, \\ A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX &= U[((\lambda_i v_j)^{\frac{1}{2}} - \lambda_i) \circ Y]V^*, \\ A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB &= U[((\lambda_i v_j)^{\frac{1}{2}} - v_j) \circ Y]V^* \end{aligned}$$

and

$$A^{1-\nu}XB^\nu = U[(\lambda_i^{1-\nu}v_j^\nu) \circ Y]V^*.$$

If $0 < \nu \leq \frac{1}{2}$, utilizing the inequality (17) and the unitary invariance of the Hilbert-Schmidt norm, we have

$$\begin{aligned} &\|(1 - \nu)AX + \nu XB\|_F^2 - \nu^2 \|AX - XB\|_F^2 \\ &= \sum_{i,j=1}^n ((1 - \nu)\lambda_i + \nu v_j)^2 |y_{ij}|^2 - \nu^2 \sum_{i,j=1}^n (\lambda_i - v_j)^2 |y_{ij}|^2 \\ &= \sum_{i,j=1}^n [((1 - \nu)\lambda_i + \nu v_j)^2 |y_{ij}|^2 - \nu^2 (\lambda_i - v_j)^2 |y_{ij}|^2] \\ &\geq \sum_{i,j=1}^n \left[r_1 ((\lambda_i v_j)^{\frac{1}{2}} - \lambda_i)^2 |y_{ij}|^2 + \mathbb{K} \left((\lambda_i / v_j)^{\frac{1}{2}}, 2 \right)^{\hat{r}_1} (\lambda_i^{1-\nu} v_j^\nu)^2 |y_{ij}|^2 \right] \\ &\geq r_1 \sum_{i,j=1}^n ((\lambda_i v_j)^{\frac{1}{2}} - \lambda_i)^2 |y_{ij}|^2 + \underline{\mathbb{K}}^{\hat{r}_1} \sum_{i,j=1}^n (\lambda_i^{1-\nu} v_j^\nu)^2 |y_{ij}|^2 \\ &= r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX \right\|_F^2 + \underline{\mathbb{K}}^{\hat{r}_1} \|A^{1-\nu}XB^\nu\|_F^2. \end{aligned}$$

Similarly, if $\frac{1}{2} < \nu < 1$, using the inequality (18), we can derive (34). \square

THEOREM 6. Suppose $A, B, X \in M_n(\mathbb{C})$ such that A and B are two positive definite matrices. Let

$$\underline{\mathbb{K}} = \min \left\{ \mathbb{K}((\lambda_i / v_j)^{\frac{1}{2}}, 2), i, j = 1, 2, \dots, n \right\}.$$

(I) If $0 < \nu \leq \frac{1}{2}$,

$$\begin{aligned} &\|(1 - \nu)AX + \nu XB\|_F^2 - (1 - \nu)^2 \|AX - XB\|_F^2 \\ &\leq \underline{\mathbb{K}}^{-\hat{r}_1} \|A^{1-\nu}XB^\nu\|_F^2 - r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB \right\|_F^2, \end{aligned} \tag{35}$$

(II) if $\frac{1}{2} < \nu < 1$, then

$$\begin{aligned} & \| (1 - \nu)AX + \nu XB \|_F^2 - \nu^2 \| AX - XB \|_F^2 \\ & \leq \underline{K}^{-\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 - r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX \right\|_F^2, \end{aligned} \tag{36}$$

where $r = \min\{\nu, 1 - \nu\}$, $r_1 = \min\{2r, 1 - 2r\}$ and $\hat{r}_1 = \min\{2r_1, 1 - 2r_1\}$.

Proof. By using the same ideas as in the prove of Theorem 5 and Corollary 2, we can obtain the required results. \square

REMARK 4. If $0 < \nu \leq \frac{1}{2}$, combining (33) and (35) with (32), we obtain

$$\begin{aligned} 0 & \leq \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq \nu^2 \| AX - XB \|_F^2 + \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq \nu^2 \| AX - XB \|_F^2 + \underline{K}^{\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq \nu^2 \| AX - XB \|_F^2 + r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX \right\|_F^2 + \underline{K}^{\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq \| (1 - \nu)AX + \nu XB \|_F^2 \\ & \leq (1 - \nu)^2 \| AX - XB \|_F^2 - r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB \right\|_F^2 + \underline{K}^{-\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq (1 - \nu)^2 \| AX - XB \|_F^2 + \underline{K}^{-\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq (1 - \nu)^2 \| AX - XB \|_F^2 + \| A^{1-\nu}XB^\nu \|_F^2, \end{aligned}$$

if $\frac{1}{2} < \nu < 1$, combining (34) and (36) with (32), we can obtain similar results,

$$\begin{aligned} 0 & \leq \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq (1 - \nu)^2 \| AX - XB \|_F^2 + \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq (1 - \nu)^2 \| AX - XB \|_F^2 + \underline{K}^{\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq (1 - \nu)^2 \| AX - XB \|_F^2 + r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - XB \right\|_F^2 + \underline{K}^{\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq \| (1 - \nu)AX + \nu XB \|_F^2 \\ & \leq \nu^2 \| AX - XB \|_F^2 - r_1 \left\| A^{\frac{1}{2}}XB^{\frac{1}{2}} - AX \right\|_F^2 + \underline{K}^{-\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq \nu^2 \| AX - XB \|_F^2 + \underline{K}^{-\hat{r}_1} \| A^{1-\nu}XB^\nu \|_F^2 \\ & \leq \nu^2 \| AX - XB \|_F^2 + \| A^{1-\nu}XB^\nu \|_F^2, \end{aligned}$$

which are improvements of $\| A^{1-\nu}XB^\nu \|_F^2 \leq \| (1 - \nu)AX + \nu XB \|_F^2$ (see [2, 8]).

Acknowledgement. The authors thank the handling editor and referees for careful reading of the manuscript and useful comments.

REFERENCES

- [1] M. BAKHERAD AND M. S. MOSLEHIAN, *Reverses and variations of Heinz inequality*, Linear Multilinear Algebra **63**, 10 (2015), 1972–1980.
- [2] R. BHATIA AND K. R. PARTHASARATHY, *Positive definite functions and operator inequalities*, Bull. London Math. Soc. **32**, 2 (2000), 214–228.
- [3] T. FURUTA, J. MIČIĆ HOT, J. PEČARIĆ AND Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, Element, Zagreb, 2005.
- [4] S. FURUICHI, *On refined Young inequalities and reverse inequalities*, J. Math. Inequal, **5**, 1 (2011), 21–31.
- [5] S. FURUICHI, *Refined Young inequalities with Specht's ratio*, J. Egypt. Math. Soc. **20**, 1 (2012), 46–49.
- [6] R. A. HORN AND C. R. JOHNSON, *Matrix analysis, 2nd ed*, Cambridge University Press, New York, 2013.
- [7] O. HIRZALLAH AND F. KITTANEH, *Matrix Young inequalities for the Hilbert-Schmidt norm*, Linear Algebra Appl. **308** (2000), 77–84.
- [8] H. KOSAKI, *Arithmetic-geometric mean and related inequalities for operators*, J. Funct. Anal. **156**, 2 (1998), 429–451.
- [9] F. KITTANEH, M. KRNIĆ, N. LOVRIČEVIĆ AND J. PEČARIĆ, *Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators*, Publ. Math. Debrecen. **80**, 3-4 (2012), 465–478.
- [10] F. KITTANEH AND M. KRNIĆ, *Refined Heinz operator inequalities*, Linear Multilinear Algebra. **61**, 8 (2013), 1148–1157.
- [11] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361**, 1 (2010), 262–269.
- [12] F. KITTANEH AND Y. MANASRAH, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra. **59**, 9 (2011), 1031–1037.
- [13] M. KRNIĆ, N. LOVRIČEVIĆ, J. PEČARIĆ, *Jensen's operator and applications to mean inequalities for operators in Hilbert space*, Bull. Malays. Math. Sci. Soc. **35**, 1 (2012), 1–14.
- [14] M. TOMINAGA, *Specht's ratio in the Young inequality*, Sci. Math. Japon. **55**, 3 (2002), 583–588.
- [15] J. L. WU AND J. G. ZHAO, *Operator inequalities and reverse inequalities related to the Kittaneh-Manasrah inequalities*, Linear Multilinear Algebra. **62**, 7 (2014), 884–894.
- [16] J. G. ZHAO AND J. L. WU, *Operator inequalities involving improved Young and its reverse inequalities*, J. Math. Anal. Appl. **421**, 2 (2015), 1779–1789.
- [17] H. L. ZUO, G. H. SHI AND M. FUJII, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal. **5**, 4 (2011), 551–556.

(Received April 29, 2015)

Wenshi Liao
 College of Mathematics and Statistics
 Chongqing University
 Chongqing 401331, P. R. China
 e-mail: liaowenshi@gmail.com

Junliang Wu
 College of Mathematics and Statistics
 Chongqing University
 Chongqing 401331, P. R. China
 e-mail: jlwu678@163.com