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SQUARING OPERATOR α–GEOMETRIC MEAN INEQUALITY

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Abstract. In this paper, we square operator α-geometric mean inequality as follows: If $0 < m_1^2 \leqslant A \leqslant M_1^2$ and $0 < m_2^2 \leqslant B \leqslant M_2^2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$, then for every unital positive linear map Φ and $\alpha \in [0,1]$, the following inequality holds:

$$\{\Phi(A)\sharp_\alpha\Phi(B)\}^2 \leqslant \frac{K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)^{-2}(G+g)^2}{4Gg}\Phi^2(A\sharp_\alpha B)$$

where the generalized Kantorovich constant $K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)$ is defined by

$$K(m,M,\alpha) = \frac{mM^\alpha - Mm^\alpha}{(\alpha-1)(M-m)}\left(\frac{\alpha-1}{\alpha}\frac{M^\alpha - m^\alpha}{mM^\alpha - Mm^\alpha}\right)^\alpha$$

and $G = M_1(M_1^{-1}M_2)^{2\alpha}M_1$, $g = m_1(m_1^{-1}m_2)^{2\alpha}m_1$.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$ with the identity I. Throughout the paper, a capital letter means an operator in $\mathbb{B}(\mathscr{H})$. An operator A is called positive (in symbol: $A \geqslant 0$) if $\langle Ax,x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. We write $A > 0$ if it is a positive invertible operator. For self-adjoint operators $A,B \in \mathbb{B}(\mathscr{H})$, we say $B \geqslant A$ if $B - A \geqslant 0$, i.e., $\langle Ax,x\rangle \geqslant \langle Bx,x\rangle$ for all $x \in \mathscr{H}$. A linear map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ is called positive if $A \geqslant 0$ implies $\Phi(A) \geqslant 0$. If this implication holds for $>$ instead of $\geqslant$, we say that Φ is strictly positive. Φ is said to be unital if $\Phi(I) = I$. For $A,B > 0$ and $\alpha \in [0,1]$, the α-geometric mean $A\sharp_\alpha B$ is defined by $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$.

Seo [7, Theorem 3] gave α-geometric mean inequality as follows:

THEOREM 1.1. *Let $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ be a unital positive linear map and let A and B be positive operators such that $0 < m_1^2 \leqslant A \leqslant M_1^2$ and $0 < m_2^2 \leqslant B \leqslant M_2^2$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$. Then for $\alpha \in [0,1]$*

$$\Phi(A)\sharp_\alpha\Phi(B) \leqslant K(m,M,\alpha)^{-1}\Phi(A\sharp_\alpha B) \tag{1.1}$$

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where we suppose $(\frac{m_2}{M_1})^2 = m$, $(\frac{M_2}{m_1})^2 = M$ *and the generalized Kantorovich constant* $K(m,M,\alpha)$ [2, Definition 2.2] *is defined by*

$$K(m,M,\alpha) = \frac{mM^\alpha - Mm^\alpha}{(\alpha-1)(M-m)}\left(\frac{\alpha-1}{\alpha}\frac{M^\alpha - m^\alpha}{mM^\alpha - Mm^\alpha}\right)^\alpha$$

for any real number $\alpha \in \mathbb{R}$.

Squaring operator inequalities has been an active area of study in the past several years; see for example, [5, 6, 3, 8]. The most successful one is that operator Kantorovich inequality can be squared [6]. In this paper, we mainly follow this line of study, our main result is a relation between $(\Phi(A)\sharp_\alpha\Phi(B))^2$ and $\Phi^2(A\sharp_\alpha B)$.

2. Main results

The following is attributed to Ando which is a converse of (1.1).

LEMMA 2.1. [4] *Let* Φ *be a unital positive linear map and* A, B *be positive operators. Then for* $\alpha \in [0,1]$

$$\Phi(A\sharp_\alpha B) \leqslant \Phi(A)\sharp_\alpha\Phi(B). \tag{2.1}$$

The next Lemma shows that t^2 is order preserving in a certain sense:

LEMMA 2.2. [1, Theorem 6] *Let* $0 < m \leqslant A \leqslant M$ *and* $A \leqslant B$. *Then*

$$A^2 \leqslant \frac{(M+m)^2}{4Mm}B^2.$$

Now we give our main result.

THEOREM 2.3. *Let* $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ *be a positive linear map and let* A *and* B *be positive operators such that* $0 < m_1^2 \leqslant A \leqslant M_1^2$ *and* $0 < m_2^2 \leqslant B \leqslant M_2^2$ *for some positive real numbers* $m_1 < M_1$ *and* $m_2 < M_2$. *Then for* $\alpha \in [0,1]$

$$(\Phi(A)\sharp_\alpha\Phi(B))^2 \leqslant \beta\Phi^2(A\sharp_\alpha B) \tag{2.2}$$

where

$$\beta := \begin{cases} \dfrac{K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)^{-2}(G+g)^2}{4Gg} & \text{if} \quad g \leqslant t_0 \\[2ex] \dfrac{K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)^{-1}(G+g)-G}{g} & \text{if} \quad g \geqslant t_0, \end{cases} \tag{2.3}$$

$G = M_1(M_1^{-1}M_2)^{2\alpha}M_1$, $g = m_1(m_1^{-1}m_2)^{2\alpha}m_1$ *and the generalized Kantorovich constant* $K(m,M,\alpha)$ [2, Definition 2.2] *is defined by*

$$K(m,M,\alpha) = \frac{mM^\alpha - Mm^\alpha}{(\alpha-1)(M-m)}\left(\frac{\alpha-1}{\alpha}\frac{M^\alpha - m^\alpha}{mM^\alpha - Mm^\alpha}\right)^\alpha$$

for any real number $\alpha \in \mathbb{R}$.

Proof. Let

$$G := M_1(M_1^{-1}M_2)^{2\alpha}M_1, \quad g := m_1(m_1^{-1}m_2)^{2\alpha}m_1.$$

It follows from the order-preserving property of the operator α-geometric mean and

$$m_1^2 \leqslant A \leqslant M_1^2 \quad \text{and} \quad m_2^2 \leqslant B \leqslant M_2^2$$

that

$$g \leqslant A\sharp_\alpha B \leqslant G \tag{2.4}$$

whence

$$g \leqslant \Phi(A\sharp_\alpha B) \leqslant G, \tag{2.5}$$

$$g \leqslant \Phi(A)\sharp_\alpha \Phi(B) \leqslant G. \tag{2.6}$$

Theorem 1.1 and inequality (2.1) yield that

$$\Phi(A\sharp_\alpha B) \leqslant \Phi(A)\sharp_\alpha \Phi(B) \leqslant K(m,M,\alpha)^{-1}\Phi(A\sharp_\alpha B). \tag{2.7}$$

By (2.5) we have

$$(G-\Phi(A\sharp_\alpha B))(\Phi(A\sharp_\alpha B)-g) \geqslant 0$$

Hence,

$$(\Phi(A\sharp_\alpha B))^2 \leqslant (G+g)\Phi(A\sharp_\alpha B) - Gg. \tag{2.8}$$

In the same way, by (2.6) we also have

$$(\Phi(A)\sharp_\alpha \Phi(B))^2 \leqslant (G+g)(\Phi(A)\sharp_\alpha \Phi(B)) - Gg. \tag{2.9}$$

Employing (2.7) and (2.9) we have

$$\begin{aligned} 0 &\leqslant (\Phi(A\sharp_\alpha B))^{-1}(\Phi(A)\sharp_\alpha \Phi(B))^2(\Phi(A\sharp_\alpha B))^{-1} \\ &\leqslant \Phi(A\sharp_\alpha B)^{-1}\{(G+g)(\Phi(A)\sharp_\alpha \Phi(B)) - Gg\}\Phi(A\sharp_\alpha B)^{-1} \\ &\leqslant \{K(m,M,\alpha)^{-1}(G+g)\Phi(A\sharp_\alpha B) - Gg\}\Phi(A\sharp_\alpha B)^{-2}. \end{aligned} \tag{2.10}$$

Consider the real function $f(t)$ on $(0,\infty)$ defined as

$$f(t) := \frac{K(m,M,\alpha)^{-1}(G+g)t - Gg}{t^2}. \tag{2.11}$$

Then we can conclude from (2.8), (2.9), (2.10) and (2.11) that

$$(\Phi(A\sharp_\alpha B))^{-1}(\Phi(A\sharp_\alpha B))^2(\Phi(A\sharp_\alpha B))^{-1} \leqslant \max_{g\leqslant t\leqslant G} f(t). \tag{2.12}$$

Notice that

$$f(g) \geqslant f(G)$$

and

$$f'(t) = \frac{2Gg - K(m,M,\alpha)^{-1}(G+g)t}{t^3}. \tag{2.13}$$

The function $f(t)$ has only one stationary (= maximum) point at

$$t_0 := \frac{2Gg}{K(m,M,\alpha)^{-1}(g+G)} \tag{2.14}$$

with the maximum value

$$f(t_0) = \frac{K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)^{-2}(G+g)^2}{4Gg}. \tag{2.15}$$

Therefore we can conclude that

$$\max_{g\leqslant t\leqslant G} f(t) \leqslant \begin{cases} f(t_0) & \text{if } g \leqslant t_0 \\ f(g) & \text{if } g \geqslant t_0. \end{cases}$$

Clearly

$$f(g) = \frac{K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)^{-1}(G+g) - G}{g}. \tag{2.16}$$

This completes the proof of Theorem 2.3 □

An immediate consequence of Theorem 2.3 reads as follows. It can be also deduced from Lemma 2.1 and Lemma 2.2.

COROLLARY 2.4. *Under the same conditions as in Theorem 2.3,*

$$\{\Phi(A)\sharp_\alpha\Phi(B)\}^2 \leqslant \frac{K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)^{-2}(G+g)^2}{4Gg}\Phi^2(A\sharp_\alpha B) \tag{2.17}$$

REMARK 2.5. It is easy to know that the coefficient

$$\frac{K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)^{-1}(G+g)}{2\sqrt{Gg}}$$

in (2.17) is larger than

$$K\left((\frac{m_2}{M_1})^2,(\frac{M_2}{m_1})^2,\alpha\right)^{-1}$$

in (1.1). However, we give the relation between $(\Phi(A)\sharp_\alpha\Phi(B))^2$ and $\Phi^2(A\sharp_\alpha B)$.

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